

GROUPS WITH MANY SELF-CENTRALIZING OR SELF-NORMALIZING SUBGROUPS

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ABSTRACT. The purpose of this paper is to present a comprehensive overview of known and new results concerning the structure of groups in which all subgroups, except those having a given property, are either self-centralizing or self-normalizing.

1. INTRODUCTION

Let G be any group, and H be a subgroup of G . The centralizer of H in G is the subgroup

$$C_G(H) = \{g \in G \mid g^{-1}hg = h, \text{ for all } h \in H\}.$$

Clearly, $H \leq C_G(H)$ if and only if H is abelian. The normalizer of H in G is the subgroup

$$N_G(H) = \{g \in G \mid H^g = H\}.$$

Of course, $H \leq N_G(H)$. Moreover, $C_G(H)$ is always a normal subgroup of $N_G(H)$, and the quotient group $N_G(H)/C_G(H)$ is isomorphic to a subgroup of $\text{Aut}(H)$, the group of all automorphisms of H .

The subgroup H is called self-centralizing (in G) if $C_G(H) \leq H$. Using Zorn's Lemma, it is easy to prove that a group does not contain proper self-centralizing subgroups if and only if it is abelian. Every maximal abelian subgroup of an arbitrary group is self-centralizing, as well as the Fitting subgroup of any soluble group. Furthermore, the generalized Fitting subgroup $F^*(G)$ of G , that is the subgroup of G generated by all subnormal nilpotent or quasisimple subgroups of G , is known to always be self-centralizing. Clearly, an abelian subgroup A of G is self-centralizing if and only if $C_G(A) = A$. In particular, the trivial subgroup of G is self-centralizing if and only if G is trivial. Let C_{tr} denote the class of groups in which all non-trivial subgroups are self-centralizing. It is quite easy to prove that a locally graded group (i.e., a group in which every non-trivial finitely generated subgroup has a

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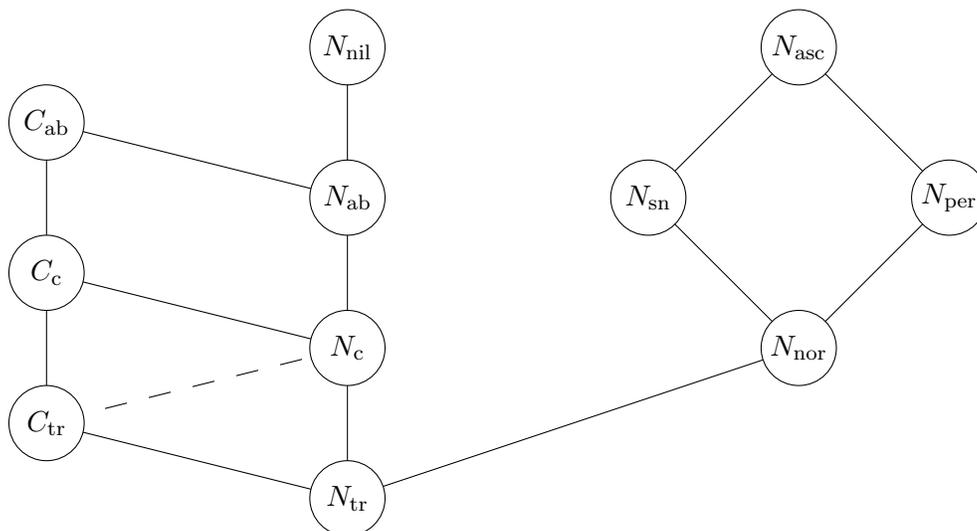
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non-trivial finite image) lying in C_{tr} has to be finite. Therefore it has to be either cyclic of prime order or non-abelian of order being the product of two different primes (see [5]). In Sections 2 and 3 we present known results regarding the classes C_c and C_{ab} of groups in which all non-cyclic (respectively: non-abelian) subgroups are self-centralizing. Apart from that, new results concerning C_c -groups which are infinite and residually finite are proved in Section 2.

The subgroup H is called self-normalizing (in G) if $N_G(H) = H$. Obviously, every self-normalizing subgroup is self-centralizing. For a finite group G , the converse holds if and only if G is abelian [11, Theorem 3.1]. It is an easy consequence of Sylow’s theorems that if P is any p -Sylow subgroup of a finite group G then $N_G(P)$ is a self-normalizing subgroup of G . It is also well known that every finite soluble group has Carter subgroups, which are nilpotent self-normalizing subgroups. Another example of self-normalizing subgroup is the complement of any Frobenius group. Let N_{tr} denote the class of groups in which all non-trivial subgroups are self-normalizing. As pointed out in [4], N_{tr} -groups are periodic and simple. Furthermore, in the locally finite case they are trivial or of prime order. Infinite examples of N_{tr} -groups are the Tarski p -groups, which are infinite simple groups whose proper non-trivial subgroups have prime order. In Sections 4–10 we survey known results concerning the structure of groups in which all subgroups are self-normalizing, except those having a certain embedding or structural property. We denote by N_{nor} , N_{sn} , N_{asc} , N_{per} the classes of groups in which all non-normal (respectively: non-subnormal, non-ascendant, non-permutable) subgroups are self-normalizing. Furthermore, we denote by N_c , N_{ab} , N_{nil} the classes of groups in which all non-cyclic (respectively: non-abelian, non-nilpotent) subgroups are self-normalizing. All results included in Section 9, dealing with the class N_c , are new.

The diagram below shows those inclusions, between the classes of groups considered in this paper, which immediately follow from the definitions. As usual, if in the diagram the class \mathcal{Y} lies below the class \mathcal{X} and there is a continuous line connecting \mathcal{Y} and \mathcal{X} , then the inclusion $\mathcal{Y} \subseteq \mathcal{X}$ holds.



All the inclusions from the diagram are proper. The symmetric group $\text{Sym}(3)$ belongs to the class C_{tr} (since each of its proper non-trivial subgroups is maximal abelian) but not to the class N_{tr} (since it has

a subgroup of index 2). The dihedral group D_8 belongs to the class C_c (since each of its proper non-cyclic subgroups is isomorphic to $C_2 \times C_2$ and hence it is self-centralizing, as the center has order 2) but not to the class N_c (since the above subgroups have index 2). The group $\text{Sym}(4)$ belongs to the class C_{ab} (in fact, it belongs to C_c by (3.1.1) of Theorem 2.1) but not to the class N_{ab} (as the non-abelian subgroup $\text{Alt}(4)$ has index 2). Obviously, the cyclic group C_6 belongs to the class C_c but not to the class C_{tr} . The wreath product $C_3 \wr C_3 = (C_3 \times C_3 \times C_3) \rtimes C_3$ is the Sylow 3-subgroup of $\text{Sym}(9)$. It is a group of order 3^4 and maximal class, so it belongs to the class C_{ab} by (i) of Theorem 3.3. On the other side, it does not belong to the class C_c by (1) of Theorem 2.1. For any non-prime integer n , the cyclic group C_n belongs to the class N_c but not to the class N_{tr} , since it has a non-trivial proper (normal) subgroup. The elementary abelian 2-group $C_2 \times C_2 \times C_2$ belongs to the class N_{ab} but not to the class N_c , since it has a non-cyclic subgroup of index 2. The special linear group $\text{SL}_2(5)$ belongs to the class N_{nil} by Theorem 10.4, but not to the class N_{ab} since its Sylow 2-subgroup is isomorphic to the quaternion group Q_8 and has normalizer of order 24. The group $\text{Sym}(3)$ belongs to the class N_{nor} but not to the class N_{tr} , as noticed above. The group D_8 belongs to the class N_{sn} , since each of its subgroups is subnormal, but it does not belong to the class N_{per} , since each of its non-central subgroups of order 2 is neither permutable nor self-normalizing (see Section 7 for the definition and basic properties of a permutable subgroup). It follows that the inclusions $N_{nor} \subset N_{sn}$ and $N_{per} \subset N_{asc}$ are proper (see Section 6 for the definition and basic properties of an ascendant subgroup). For finitely generated groups, the inclusion $N_{per} \subseteq N_{sn}$ holds (see, for instance, [14, 13.2.5]). In general, this is not the case. Indeed, let consider the group $G = T \rtimes A$, where $T = \langle t \rangle$ is infinite cyclic, $A = \mathbb{Z}_{p^\infty}$ (the Prüfer p -group) for some odd prime p , and $a^t = a^{p+1}$ for all $a \in A$. Then every subgroup of G is permutable (see [14, Exercise 13.2.3]), hence G belongs to the class N_{per} . On the other side, not every subgroup of G is subnormal. It follows from Theorem 5.4 that the group G does not belong to the class N_{sn} . Therefore also the inclusions $N_{nor} \subset N_{per}$ and $N_{sn} \subset N_{asc}$ are proper.

Since Tarski groups are in each of the above classes, in order to obtain a satisfactory description it is convenient to restrict our investigation to locally graded groups. The inclusion which is represented in the diagram by a dashed line holds for locally graded groups. This is an easy consequence of the above mentioned results in [5] and Proposition 9.6. Also this inclusion is proper, as the cyclic group C_6 shows. So far, it is not known whether the inclusion $C_{tr} \subset N_c$ is true in general.

2. GROUPS IN WHICH ALL NON-CYCLIC SUBGROUPS ARE SELF-CENTRALIZING

This class of groups, that we denote by C_c , has been studied in [5] and [7]. Finite C_c -groups are completely classified. It turns out that a non-cyclic finite p -group in the class C_c is either an extraspecial p -group of order p^3 for some odd prime p , or a 2-group of maximal class.

Theorem 2.1. [7, Theorem 2.1] *Let G be a finite group. Then G is a C_c -group if and only if one of the following holds:*

- (1) *if G is nilpotent, then either*

- (1.1) G is cyclic;
- (1.2) G is elementary abelian of order p^2 for some prime p ;
- (1.3) G is an extraspecial p -group of order p^3 for some odd prime p ; or
- (1.4) G is a dihedral, semidihedral or quaternion 2-group.
- (2) if G is supersoluble but not nilpotent, then, letting p denote the largest prime divisor of $|G|$ and $P \in \text{Syl}_p(G)$, we have that P is a normal subgroup of G and one of the following holds:
- (2.1) P is cyclic and either
- (2.1.1) $G \cong D \rtimes C$, where C is cyclic, D is cyclic and every non-trivial element of D acts fixed point freely on C (so G is a Frobenius group);
- (2.1.2) $G \cong D \rtimes C$, where C is a cyclic group of odd order, D is a quaternion group, and $C_G(C) = C \times D_0$ where D_0 is a cyclic subgroup of index 2 in D with G/D_0 a dihedral group; or
- (2.1.3) $G \cong D \rtimes C$, where D is a cyclic q -group, C is a cyclic q' -group (here q denotes the smallest prime dividing the order of G), $\{1\} < Z(G) < D$ and $G/Z(G)$ is a Frobenius group;
- (2.2) P is extraspecial and G is a Frobenius group with cyclic Frobenius complement of odd order dividing $p - 1$.
- (3) if G is not supersoluble and $F^*(G)$ is nilpotent, then either (3.1) or (3.2) below holds.
- (3.1) $F^*(G)$ is elementary abelian of order p^2 , $F^*(G)$ is a minimal normal subgroup of G and one of the following holds:
- (3.1.1) $p = 2$ and $G \cong \text{Sym}(4)$ or $G \cong \text{Alt}(4)$; or
- (3.1.2) p is odd and $G = G_0 \rtimes N$ is a Frobenius group with Frobenius kernel N and Frobenius complement G_0 which is itself an C_c -group. Furthermore, either
- (3.1.2.1) G_0 is cyclic of order dividing $p^2 - 1$ but not dividing $p - 1$;
- (3.1.2.2) G_0 is quaternion;
- (3.1.2.3) G_0 is supersoluble as in (2.1.2) with $|C|$ dividing $p - \epsilon$ where $p \equiv \epsilon \pmod{4}$;
- (3.1.2.4) G_0 is supersoluble as in (2.1.3) with D a 2-group, $C_D(C)$ a non-trivial maximal subgroup of D and $|C|$ odd dividing $p - 1$ or $p + 1$;
- (3.1.2.5) $G_0 \cong \text{SL}_2(3)$;
- (3.1.2.6) $G_0 \cong \text{SL}_2(3) \cdot 2$ and $p \equiv \pm 1 \pmod{8}$; or
- (3.1.2.7) $G_0 \cong \text{SL}_2(5)$ and 60 divides $p^2 - 1$.
- (3.2) $F^*(G)$ is extraspecial of order p^3 and one of the following holds:
- (3.2.1) $G \cong \text{SL}_2(3)$ or $G \cong \text{SL}_2(3) \cdot 2$ (with quaternion Sylow 2-subgroups of order 16); or
- (3.2.2) $G = K \rtimes N$ where N is extraspecial of order p^3 and exponent p with p an odd prime, K centralizes $Z(N)$ and is cyclic of odd order dividing $p + 1$. Furthermore, $G/Z(N)$ is a Frobenius group.
- (4) if $F^*(G)$ is not nilpotent, then either
- (4.1) $F^*(G) \cong \text{SL}_2(p)$ where p is a Fermat prime, $|G/F^*(G)| \leq 2$ and G has quaternion Sylow 2-subgroups; or

(4.2) $G \cong \text{PSL}_2(9)$, $\text{Mat}(10)$ or $\text{PSL}_2(p)$ where p is a Fermat or Mersenne prime.

The structure of locally finite C_c -groups is also completely described.

Theorem 2.2. [7, Theorem 3.7] *An infinite locally finite group G lies in the class C_c if and only if one of the following holds:*

- (i) $G \cong \mathbb{Z}_{p^\infty}$ for some prime p ;
- (ii) $G = A\langle y \rangle$ where $A \cong \mathbb{Z}_{2^\infty}$ and $\langle y \rangle$ has order 2 or 4, with $y^2 \in A$ and $a^y = a^{-1}$, for all $a \in A$;
- (iii) $G \cong A \rtimes D$, where $A \cong \mathbb{Z}_{p^\infty}$ and $\{1\} \neq D \leq C_{p-1}$ for some odd prime p .

An infinite soluble group lies in the class C_c if and only if it is either locally finite, or cyclic, or dihedral [7, Theorem 3.6]. Nilpotent C_c -groups are either abelian or finite p -groups.

Theorem 2.3. [7, Corollary 3.8] *An infinite locally nilpotent group G lies in the class C_c if and only if one of the following holds:*

- (i) G is cyclic;
- (ii) $G \cong \mathbb{Z}_{p^\infty}$ for some prime p ;
- (iii) $G = A\langle y \rangle$ where $A \cong \mathbb{Z}_{2^\infty}$ and $\langle y \rangle$ has order 2 or 4, with $y^2 \in A$ and $a^y = a^{-1}$, for all $a \in A$.

If G is an infinite C_c -group then every non-cyclic normal subgroup of G is infinite. Indeed, if N is such a subgroup, then $C_G(N) \leq N$, and $G/C_G(N)$ embeds in $\text{Aut}(N)$. If N is finite it follows that $G/C_G(N)$ is finite, and so G/N is finite too, giving the contradiction that G is finite.

We want to close this section by discussing the existence of infinite residually finite groups in the class C_c . Free groups are easy examples of torsion-free residually finite C_c -groups. Besides, the infinite dihedral group is an example of an infinite residually finite C_c -group having periodic elements [7, Lemma 3.1].

A residually finite C_c -group which is locally finite has to be finite. If not, by [7, Theorem 3.7] its Fitting subgroup is a Prüfer p -group, that is not possible since these groups are not residually finite.

We have no examples of infinite periodic residually finite C_c -groups. The following result points out considerable structural restrictions for such groups.

Theorem 2.4. *Let G be an infinite periodic residually finite C_c -group. Then:*

- (i) the centralizer $C_G(a)$ is finite for all non-trivial elements $a \in G$;
- (ii) every non-trivial normal subgroup of G is infinite;
- (iii) the center $Z(G)$ is trivial;
- (iv) the Fitting subgroup of G is trivial;
- (v) the centralizer $C_G(H)$ is trivial, for all infinite subgroups H of G ;
- (vi) G has no elements of order 2.

Proof. Assume there exists an element $a \in G$ with $C = C_G(a)$ infinite. Let N be a normal subgroup of C with C/N finite and $a \notin N$. Thus $a \in C_C(N) \setminus N$, so $C_C(N) \not\leq N$. Hence N is cyclic, a contradiction since N is infinite. This proves (i).

By part (i) we know that $C_G(N)$ is finite. If N is finite it follows that $G/C_G(N)$ is finite, and so G is finite, a contradiction. This proves (ii).

Assume $Z(G) \neq \{1\}$. Then $Z(G)$ is infinite by part (ii). Thus $Z(G)$ is not cyclic, and so $G = C_G(Z(G)) \leq Z(G)$. It follows that G is infinite abelian, so it is a Prüfer group, a contradiction since G is residually finite. Thus (iii) is proved.

If the Fitting subgroup of G is non-trivial, then there exists a nontrivial normal subgroup N of G . Thus N is infinite abelian, a contradiction since G is periodic and residually finite. This proves (iv).

Let H be an infinite subgroup of G , and suppose that $C_G(H)$ contains a non-trivial element a . Then $H \leq C_G(a)$, a contradiction by part (i). This proves (v).

Let a be an element of G having order 2. Then $C_G(a)$ is finite by part (i). By a result due to Šunkov (see, for instance, [14, 14.3.8]), G has a non-trivial element with infinite centralizer, a contradiction. Thus (vi) is proved. \square

3. GROUPS IN WHICH ALL NON-ABELIAN SUBGROUPS ARE SELF-CENTRALIZING

This class of groups, here denoted by C_{ab} , has been studied in [6]. Clearly, the center of a C_{ab} -group is contained in every non-abelian subgroup.

Theorem 3.1. [6, Theorem 3.1 and Proposition 3.2] *Every nilpotent group in C_{ab} is either abelian or a finite p -group.*

Therefore, in order to study nilpotent C_{ab} -groups it suffices to deal with finite p -groups. However, to give a complete classification of finite p -groups in C_{ab} is a long standing open problem posed by Y. Berkovich [1, Problem 9]. So far, we have enough information on the structure of these groups only in the cases when they are either metacyclic, or of maximal class, or of exponent p .

Theorem 3.2. [6, Proposition 6.8] *Let G be any finite metacyclic p -group. Then G is a C_{ab} -group.*

Theorem 3.3. [6, Theorem 6.14] *Let G be a p -group of maximal class of order p^n .*

- (i) *If $p \in \{2, 3\}$ or $n \leq 3$, then G is a C_{ab} -group.*
- (ii) *If $p \geq 5$ and $n \geq 4$, then G is a C_{ab} -group if and only if the 2-step centralizer $C_G(\gamma_2(G)/\gamma_4(G))$ is abelian.*

Theorem 3.4. [6, Theorem 6.15] *Let G be a finite p -group of exponent p lying in the class C_{ab} . If $|G| > p^p$, then G is elementary abelian. Otherwise, either G is elementary abelian, or G has maximal class and an elementary abelian subgroup of index p .*

On the other hand, the structure of infinite supersoluble C_{ab} -groups is completely described.

Theorem 3.5. [6, Theorem 4.1 and Theorem 4.2] *Let G be an infinite supersoluble group.*

- (i) *If G has no elements of even order and G is a C_{ab} -group, then G is abelian.*

- (ii) If G is not abelian, then G is a C_{ab} -group if and only if $G = A\langle x \rangle$ where A is abelian, $a^x = a^{-1}$ for all $a \in A$, $|x| = 2^n$, $A = \langle a_1 \rangle \times \cdots \times \langle a_t \rangle \times \langle d \rangle$, a_i of infinite or odd order for all $i \in \{1, \dots, t\}$, $|d| = 2^h$, $d^{2^{h-1}} = x^{2^{n-1}}$ if $h > 0$.

4. GROUPS IN WHICH ALL NON-NORMAL SUBGROUPS ARE SELF-NORMALIZING

This class of groups, here denoted by N_{nor} , has been studied in [10]. All N_{nor} -groups are T -groups (i.e., groups in which normality is a transitive relation in the set of all subgroups). Moreover they are periodic or abelian. Periodic N_{nor} -groups are intractable in general. So the author investigates locally finite N_{nor} -groups. It comes out that such groups are soluble. Furthermore, in the nilpotent case they are Dedekind groups (i.e., groups in which all subgroups are normal). The following result completes the description of the structure of locally finite N_{nor} -groups.

Theorem 4.1. [10] *Let G be a locally finite non-Dedekind group. Then G is an N_{nor} -group if and only if G is a soluble T -group, G/G' is a cyclic group of prime-power order p^n , the center $Z(G)$ has order p^{n-1} , the derived group G' has no involutions, and the orders of G' and of G/G' are coprime.*

5. GROUPS IN WHICH ALL NON-SUBNORMAL SUBGROUPS ARE SELF-NORMALIZING

This class, that we denote by N_{sn} , has been studied in [12]. The following results give a full description of locally graded N_{sn} -groups.

Theorem 5.1. [12, Theorem A] *Let G be a periodic locally nilpotent N_{sn} -group. Then every subgroup of G is subnormal.*

Groups in which all subgroups are subnormal have been extensively studied (see, for instance, [2]).

Theorem 5.2. [12, Theorem B] *Let G be a locally finite group that is not locally nilpotent. Then G is an N_{sn} -group if and only if $G = A \rtimes P$, where $P = \langle g \rangle$ is a cyclic p -subgroup for some prime p , A is a nilpotent p' -subgroup such that $C_P(A) = \langle g^p \rangle$, $G' = A$ and $C_G(P) = P$.*

Theorem 5.3. [12, Theorem C] *Every periodic locally graded N_{sn} -group is locally finite.*

Theorem 5.4. [12, Theorem D] *Let G be a non-periodic N_{sn} -group. Then every subgroup of G is subnormal; in particular, if G is torsion-free then G is nilpotent.*

6. GROUPS IN WHICH ALL NON-ASCENDANT SUBGROUPS ARE SELF-NORMALIZING

This class, here denoted by N_{asc} , has been studied in [13]. Recall that a subgroup H of a group G is said to be ascendant (in G) if there exists an ascending series from H to G , that is, a chain of subgroups well-ordered by inclusion and indexed by the corresponding ordinal numbers,

$$H = H_0 \triangleleft H_1 \triangleleft \cdots \triangleleft H_\alpha \triangleleft H_{\alpha+1} \triangleleft \cdots \triangleleft H_\beta = G,$$

with the additional stipulation that for each limit ordinal λ the subgroup H_λ is the union of all subgroups H_γ with $\gamma < \lambda$. A Gruenberg group is a group in which every cyclic subgroup is ascendant. Every Gruenberg group is locally nilpotent. Conversely, locally nilpotent periodic N_{asc} -groups are Gruenberg groups ([13, Corollary 2.2]).

Theorem 6.1. [13, Corollary 2.4] *Let G be a locally finite group that is not locally nilpotent. Then G is an N_{asc} -group if and only if $G = A \rtimes P$, where $P = \langle g \rangle$ is a cyclic p -subgroup for some prime p , A is a normal nilpotent p' -subgroup such that $C_P(A) = \langle g^p \rangle$, $G' = A$ and $C_G(P) = P$.*

Theorem 6.2. [13, Theorem 2.5] *Every non-periodic N_{asc} -group is a Gruenberg group.*

Remember that a group is called hyperabelian if it has an ascending series of normal subgroups whose factors are abelian.

Theorem 6.3. [13, Theorem 3.8] *Let G be a hyperabelian N_{asc} -group. If G is locally nilpotent, then every subgroup of G is in fact ascendant.*

7. GROUPS IN WHICH ALL NON-PERMUTABLE SUBGROUPS ARE SELF-NORMALIZING

This class, that we denote by N_{per} , has been studied in [3] and [13]. Recall that a subgroup H of a group G is said to be permutable (in G) if $HK = KH$ for every subgroup K of G (see, for instance, [15] for a survey of the properties of permutable subgroups). Obviously, normal subgroups are permutable; on the other hand, it is well known that permutable subgroups are ascendant [16].

Theorem 7.1. [3], [13, Corollary 4.2] *A periodic locally graded group G that is not locally nilpotent is an N_{per} -group if and only if there exists a normal subgroup A of G such that $G = A \rtimes P$, where $P = \langle g \rangle$ is a cyclic p -subgroup for some prime p , A is an abelian p' -subgroup with $C_P(A) = \langle g^p \rangle$, $G' = A$, $C_G(P) = P$ and every subgroup of A is G -invariant.*

Theorem 7.2. [13, Corollary 4.4] *Let G be an N_{per} -group. If G is not periodic, then every subgroup of G is permutable.*

8. GROUPS IN WHICH ALL NON-ABELIAN SUBGROUPS ARE SELF-NORMALIZING

This class of groups, here denoted by N_{ab} , has been studied in [4] and [8]. Examples of N_{ab} -groups are all minimal non-abelian groups, that are non-abelian groups in which every proper subgroup is abelian. The structure of finite minimal non-abelian p -groups is well known ([1, Exercise 8a, p. 29]). All nilpotent N_{ab} -groups having nilpotency class ≥ 2 are finite.

Theorem 8.1. [8, Proposition 2.6] *Every nilpotent N_{ab} -group is either abelian or a finite minimal non-abelian p -group for some prime p .*

The structure of all finite N_{ab} -groups is completely described.

Theorem 8.2. [8, Theorem 2.14] *Every finite N_{ab} -group is either soluble or simple.*

Theorem 8.3. [8, Theorem 2.13] *A finite soluble non-nilpotent group G is an N_{ab} -group if and only if G splits as $G = \langle x \rangle \rtimes A$, where $\langle x \rangle$ is p -group for some prime p , A is an abelian p' -group, x^p is central and x acts fixed point freely on A .*

Theorem 8.4. [8, Theorem 2.17] *A finite non-abelian simple group G is an N_{ab} -group if and only if G is isomorphic to $\text{Alt}(5)$ or $\text{PSL}_2(2^n)$, where $2^n - 1$ is a prime.*

The structure of all infinite soluble N_{ab} -groups is fully determined by the following two results. The first one is an easy consequence of [8, Theorem 3.2].

Theorem 8.5. *Every non-periodic soluble N_{ab} -group is abelian.*

Theorem 8.6. [4, Theorem 1.7] *Let G be an infinite periodic group, and suppose that G is non-abelian and soluble. Then G is an N_{ab} -group if and only if $G = \langle x \rangle \rtimes G'$, where x is an element of prime power order p^n , $x^p \in C_G(G')$ and G' is an abelian group with no elements of order p .*

Finally, the structure of locally finite N_{ab} -groups is also completely described. Indeed, these groups are either finite or soluble [7, Theorem 3.4].

9. GROUPS IN WHICH ALL NON-CYCLIC SUBGROUPS ARE SELF-NORMALIZING

We denote this class by N_c . It is easy to see that the class N_c is subgroup and quotient closed. In this section we give a full classification of locally graded N_c -groups.

Lemma 9.1. *A non-cyclic abelian group lies in the class N_c if and only if it is isomorphic either to $C_p \times C_p$ or to \mathbb{Z}_{p^∞} for some prime p .*

Proof. Since $N_c \subseteq C_c$, the result is an obvious consequence of [5, Theorem 2.2]. □

Lemma 9.2. *The only finite non-abelian p -group lying in the class N_c is the quaternion group Q_8 .*

Proof. Let G be a finite non-abelian p -group in the class N_c . By [5, Theorem 2.4], G is either a 2-group of maximal class or an extraspecial p -group of order p^3 for some odd prime p .

Let $n \geq 3$. The dihedral group of order 2^n ,

$$D_{2^n} = \langle a, x \mid a^{2^{n-1}} = 1 = x^2, a^x = a^{-1} \rangle,$$

is not an N_c -group, since the non-cyclic subgroup $\langle a^2, x \rangle$ has index 2, and therefore it is normal. Analogously, the semidihedral group of order 2^n ,

$$SD_{2^n} = \langle a, x \mid a^{2^{n-1}} = 1 = x^2, a^x = a^{2^{n-2}-1} \rangle,$$

is not an N_c -group, because of the non-cyclic normal subgroup $\langle a^{2^{n-2}}, x \rangle$. If $n > 3$, the quaternion group of order 2^n ,

$$Q_{2^n} = \langle a, x \mid a^{2^{n-2}} = x^2, a^{2^{n-1}} = 1, a^x = a^{-1} \rangle,$$

is not an N_c -group, since it has $Q_{2^{n-1}}$ as a non-cyclic normal subgroup. On the other hand, Q_8 is obviously an N_c -group since each of its proper subgroups is cyclic.

Let now p be an odd prime. Clearly, the extraspecial group of order p^3 and exponent p is not an N_c -group, since the subgroup of order p^2 is not cyclic. Finally, the extraspecial group of order p^3 and exponent p^2 ,

$$M_3(p) = \langle a, b \mid a^{p^2} = 1 = b^p, ab = ba^{p+1} \rangle,$$

has a non-cyclic subgroup of order p^2 , namely $\langle a^p, b \rangle$, so it is not an N_c -group. \square

Proposition 9.3. *A nilpotent group lies in the class N_c if and only if it is either abelian or isomorphic to the group Q_8 .*

Proof. The result is an immediate consequence of [8, Proposition 2.6] and Lemma 9.2. \square

Lemma 9.4. *Every finite group in the class N_c is supersoluble.*

Proof. Let G be a finite N_c -group. Since $N_c \subset N_{ab}$, it immediately follows from [8, Theorem 2.14] that G is either soluble or simple. Suppose that G is simple and non-abelian. Then from [5, Theorem 3.7] and [8, Theorem 2.17] it follows that $G \cong \text{PSL}_2(4) \cong \text{Alt}(5)$. But $\text{Alt}(5)$ is not an N_c -group, since its 2-Sylow subgroup P is isomorphic to the group $C_2 \times C_2$, but $N_{\text{Alt}(5)}(P)$ has order 12. Thus G has to be soluble. Since every proper normal subgroup of an N_c -group is cyclic, it follows that G' is cyclic. Therefore G is supersoluble. \square

When checking whether or not a subgroup is self-normalizing, it is possible to move to each of its conjugates.

Lemma 9.5. *A subgroup H of a group G is self-normalizing if and only if H^g is self-normalizing, for all $g \in G$.*

Proof. It is an immediate consequence of the fact that $N_G(H^g) = N_G(H)^g$. \square

Proposition 9.6. *Every Frobenius group with cyclic kernel of odd order and complement of prime order is an N_c -group.*

Proof. Let G be a Frobenius group with cyclic kernel C of odd order and complement D of prime order. Consider any non-cyclic proper subgroup H of G . Then there exists an element $h \in H \setminus C$. Hence $h \in D^c$ for a suitable $c \in C$. It follows that $D^c < H$, and so $D < H^{c^{-1}}$. Therefore, by Lemma 9.5 we may assume that $D < H$. Write $K = H \cap C$. If $K = \{1\}$, then H is isomorphic to a subgroup of D , which is impossible since D is cyclic. On the other side, from $C \leq H$ it would follow $H = G$, a contradiction. Therefore we have $\{1\} < K < C$. Furthermore, $DK = H$. Thus $H = D \rtimes K$ is a Frobenius group with kernel K and complement D . In particular,

$$H = K \dot{\cup} \bigcup_{c \in K} (D^c \setminus \{1\}).$$

Let now $g \in N_G(H)$. Since $H^g = H$, we get $K^g = K$. Moreover there exists $k \in K$ such that $D^g = D^k$. It follows that $gk^{-1} \in N_G(D) = D < H$. As $k \in H$, we obtain $g \in H$. Therefore H is self-normalizing, as required. \square

Theorem 9.7. *A finite group G lies in the class N_c if and only if one of the following holds:*

- (i) G is cyclic;
- (ii) $G \cong C_p \times C_p$ for some prime p ;
- (iii) $G \cong Q_8$;
- (iv) G is a Frobenius group with cyclic kernel of odd order and complement of prime order;
- (v) $G \cong D \rtimes C$, where D is a cyclic q -group (q being the smallest prime dividing the order of G), C is a cyclic q' -group, $\{1\} < Z(G) < D$, $|D/Z(G)| = q$ and $G/Z(G)$ is a Frobenius group.

Proof. Let G be a finite N_c . If G is nilpotent, then from Proposition 9.3 and Lemmas 9.1 it follows that G is either cyclic, or isomorphic to the group $C_p \times C_p$ for some prime number p , or to the quaternion group Q_8 . Let G be not nilpotent. By Lemma 9.4 G is supersoluble. Let p denote the largest prime number dividing the order of G , and P denote a Sylow p -subgroup of G . Then P is a normal subgroup of G , so P has to be cyclic. Thus from [7, Theorem 2.1] it follows that one of the following holds:

- (1) $G \cong D \rtimes C$, where C is cyclic, D is cyclic and every non-trivial element of D acts fixed point freely on C (so G is a Frobenius group);
- (2) $G \cong D \rtimes C$, where D is a cyclic q -group, C is a cyclic q' -group (here q denotes the smallest prime dividing the order of G), $\{1\} < Z(G) < D$ and $G/Z(G)$ is a Frobenius group.

Observe that those in (2.1.2) of [7, Theorem 2.1] are not N_c -groups, since $Q_8 \rtimes C$ has a non-cyclic subgroup of index 2 isomorphic to $C_4 \rtimes C$.

Let G be an N_c -group in (1). Clearly C has odd order, otherwise it has an unique involution, which is fixed by all elements of D , a contradiction. Suppose now that the order of D is not a prime. Then D has a non-trivial proper subgroup H . Take $d \in D \setminus H$. Thus $d \in N_G(HC)$, so either $d \in HC$ or HC is cyclic. In the former case $d = hc$ for some $h \in H$ and $c \in C \setminus \{1\}$, a contradiction since $c = h^{-1}d \in D$. In the latter case every non-trivial element of H fixes all elements of C , again a contradiction. Therefore D has prime order, and G has the structure described in (iv).

Finally, let G be an N_c -group in (2). Then, using (iv), it follows that G has the structure described in (v).

Conversely, all the groups listed in (i), (ii) and (iii) are N_c -groups by Lemma 9.2. Those in (iv) are N_c -groups by Proposition 9.6. Finally, let G be a group having the structure described in (v). We will prove that G is an N_c -group. Write $|D| = q^n$, $|C| = m$ and $|Z(G)| = q^{n-1}$. Then $n > 1$, m is an (odd) integer not divisible by q , and $|G| = q^n m$. Since G is supersoluble, it has a subgroup of order $p^\alpha d$ for all integers $0 \leq \alpha \leq n$ and all positive integers t dividing m . Let t be any positive integer dividing m , and let π denote the set of all primes dividing m . Then a subgroup of G of order t is a π -subgroup, hence it is contained in C , which is the only π -Hall subgroup of G . Therefore the only subgroup of order t of G is the one contained in C . If H is any subgroup of G of order $q^\alpha t$, with $\alpha < n$ and t dividing m , then the q -Sylow subgroup of H is contained in $Z(G)$, and H has a subgroup K of

order t which is contained in C . It follows that $H = Z(G) \times K$ is cyclic. Therefore a non-cyclic proper subgroup H of G must have order $q^n t$ for some positive integer t dividing m . Then a q -Sylow subgroup of H is D^c for some $c \in C$, and H has a subgroup K of order t which is contained in C . It follows that $H = D^c \times K$. By Lemma 9.5 we can assume that $H = D \times K$. Let now $g \in N_G(H)$. Thus $D^g < H$, so D^g is a q -Sylow subgroup of H . Hence $D^g = D^h$ for a suitable $h \in H$. It follows that $gh^{-1} \in N_G(D)$. Write $gh^{-1} = ab$, where $a \in D$, $b \in C$. Then $D^b = D$. For all $d \in D$ we have $[d, b] \in [D, C] \leq C$ since C is normal in G . On the other hand, $[d, b] = d^{-1}d^b \in D$. Hence $[d, b] \in D \cap C$, so $[d, b] = 1$. Thus $[D, b] = 1$, and $b \in Z(G) \cap C$. It follows that $b = 1$, and $gh^{-1} \in D < H$. Hence $g \in H$. Therefore H is self-normalizing, and G is an N_c -group. \square

The structure of locally graded N_c -groups is completely determined, since they are finite or abelian.

Lemma 9.8. *Every infinite soluble N_c -group is abelian.*

Proof. By [7, Theorem 3.6], an infinite soluble N_c -group is either abelian or dihedral. On the other hand, the infinite dihedral group

$$D_\infty = \langle a, x \mid x^2 = 1, a^x = a^{-1} \rangle$$

has a normal subgroup $H = \langle a^2, x \rangle$ which is dihedral, therefore D_∞ is not an N_c -group. \square

Lemma 9.9. *Every infinite locally finite N_c -group is isomorphic to \mathbb{Z}_{p^∞} for some prime p .*

Proof. The result immediately follows from [7, Theorem 3.7]. \square

Theorem 9.10. *Every infinite locally graded N_c -group is abelian.*

Proof. Let G be an infinite locally graded N_c -group. By Lemma 9.9, we can assume that G is not locally finite. Thus G has an infinite finitely generated subgroup H . Let a and b be arbitrary elements of G , and put $L = \langle H, a, b \rangle$. Since G is locally graded, there exists a proper normal subgroup K of L such that L/K is finite. Hence K is infinite cyclic, and L is soluble by Lemma 9.4. It follows from Lemma 9.8 that L is abelian, so a and b commute. Therefore G is abelian. \square

10. GROUPS IN WHICH ALL NON-NILPOTENT SUBGROUPS ARE SELF-NORMALIZING

This class of groups, that we denote by N_{nil} , has been studied in [9]. Since all nilpotent groups, and therefore all finite p -groups are in the class N_{nil} , one cannot expect to classify all finite N_{nil} -groups. By definition, the class N_{nil} contains all minimal non-nilpotent groups, which are non-nilpotent groups whose proper subgroups are nilpotent. Finite minimal non-nilpotent groups are soluble, and their structure is well known (see [14, 9.1.9]). Infinite minimal non-nilpotent groups are either finitely generated or locally finite p -groups (see [2]).

It is easy to see that every N_{nil} -group is either soluble or perfect. The structure of all soluble N_{nil} -groups can be described.

Theorem 10.1. [9, Theorem 3.4] *A non-periodic soluble group lies in the class N_{nil} if and only if it is nilpotent.*

When a cyclic group $\langle x \rangle$ acts on a group H by means of an automorphism x , we write $L \leq_x H$ to say that the subgroup L of H is invariant with respect to $\langle x \rangle$. We consider the induced map

$$\rho_x: H \rightarrow H, \quad \rho_x(h) = [x, h] = h^{-x}h,$$

and the following property of ρ_x :

$$(\star) \quad \forall K \leq_x H, (\exists n \geq 1: \rho_x^n(K) = 1 \vee \langle \rho_x(K) \rangle = K).$$

Theorem 10.2. [9, Theorem 3.5] *Let G be a periodic soluble group, and assume that G is not locally nilpotent. Then G is an N_{nil} -group if and only if G splits as $G = \langle x \rangle \rtimes H$, where $\langle x \rangle$ is a p -group for some prime p , H is a nilpotent p' -group, x^p acts trivially on H and ρ_x has property (\star) .*

Theorem 10.3. [9, Theorem 3.7] *A locally nilpotent soluble group lies in the class N_{nil} if and only if it is either nilpotent or minimal non-nilpotent.*

It follows that every locally nilpotent soluble N_{nil} -group is either nilpotent or a p -group for some prime p ([9, Corollary 3.8]).

Theorem 10.4. [9, Theorem 4.8] *A finite perfect group G belongs to the class N_{nil} if and only if it is either isomorphic to $\text{SL}_2(5)$ or to $\text{PSL}_2(2^n)$ where $2^n - 1$ is a prime.*

If G is a perfect N_{nil} -group, and F denotes its Fitting subgroup, then G/F is a non-abelian simple group ([9, Proposition 4.11]). It is still an open question whether or not there exist infinite perfect N_{nil} -groups which are not simple.

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