GROUPS WITH NUMERICAL RESTRICTIONS
ON MINIMAL GENERATING SETS

LEONID A. KURDACHENKO, PATRIZIA LONGOBARDI AND MERCEDE MAJ

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ABSTRACT. We study an inverse problem of small doubling type. We investigate the structure of a finitely generated group $G$ such that for any set $S$ of generators of $G$ of minimal order we have $S^2 \leq 3|S| - \beta$, where $\beta \in \{1, 2, 3\}$.

1. Introduction

Let $G$ denote an arbitrary group. If $S$ is a subset of $G$, then we write

$$S^2 = \{xy \mid x, y \in S\}.$$ 

If $G$ is an additive group, then we put

$$2S = \{x + y \mid x, y \in S\}.$$ 

A well-known problem in additive number theory is to find the precise structure of $S$ in the case when $S$ is a finite subset of $G$ and

$$|S^2| \leq \alpha|S| + \beta$$

with $\alpha$ (the doubling coefficient) and $|\beta|$ small. Problems of this kind are called inverse problems of small doubling type. In the additive group of integers, these problems were detailed investigated by G. A. Freeman in [6], [7], [8] and [9]. It is very easy to prove that if $S$ is a finite subset of integers,
\[ |S| = k, \text{ then } |2S| \geq 2|S| - 1, \text{ and } |2S| = 2|S| - 1 \text{ if and only if there exist integers } a, q \text{ such that } S = \{a, a + q, a + 2q, \ldots, a + (k - 1)q\}, \text{ i.e. } S \text{ is an arithmetic progression of length } k. \]

In his famous theorem, Freiman proved that if \( S \) is a finite set of integers with \( k \geq 3 \) elements and \( |2S| \leq 3k - 4 \), then there exist integers \( a, q \) such that \( q > 0 \) and \( S \subseteq \{a, a + q, a + 2q, \ldots, a + (2k - 4)q\} \). He obtained similar results if \( |2S| \leq 3|S| - 3 \), or \( |2S| \leq 3|S| - 2 \). For other authors results of this type please see [27], [5], [22], [32], [33] and [34].

In arbitrary abelian groups, inverse problems have been studied by many other authors (see, for example, [1], [20], [18], [27], [30] and [19]). This study was initiated by M. Kneser [26].

More recently, small doubling problems in non-abelian groups have also been studied, see for example [2], [36], [4]. We also refer to recent surveys [19], [31], [3] and books [28] and [35].

In a series of papers with G. A. Freiman, M. Herzog, Y. V. Stanchescu, A. Plagne and D. J. S. Robinson (see [10, 11, 12, 13, 14, 15, 16] and [24]) the last two authors of the current paper studied small doubling problems in an orderable group.

J. H. B Kemperman showed that if \( S \) is a finite subset of any torsion-free group, then \( |S^2| \geq 2|S| - 1 \) (see [25]), while G. A. Freiman and B. M. Schein proved that if \( |S| = k \), then \( |S^2| = 2|S| - 1 \) if and only if \( S = \{a, aq, \ldots, aq^{k - 1}\} \), i.e. \( S \) is a geometric progression and either \( aq = qa \) or \( aqa^{-1} = q^{-1} \) (see [17]). Therefore it is quite natural to ask what is the structure of \( S \) in the case when \( S \) is a finite subset of a group \( G \), \( |S| = k \geq 3 \) and \( |S^2| \leq 3|S| - \beta \), where \( \beta = 1, 2, 3, 4 \). It could be also interesting to know the structure of \( \langle S \rangle \) if \( S \) is a finite subset of a group and \( |S^2| \leq 3|S| - \beta \) where \( \beta = 1, 2, 3, 4 \).

There is an old conjecture by G. Freiman stating that if \( S \) is a finite subset of a torsion-free group, \( 1 \in S \) and \( |S^2| \leq 3k - 4 \), then \( \langle S \rangle \) is abelian (see [21, p. 250]).

If \( G \) is a finitely generated group, we denote by \( d(G) \) the minimal order of a finite set of generators of \( G \). In the current paper, we investigate the structure of a finitely generated group \( G \) in which \( |S^2| \leq 3|S| - \beta \), \( \beta = 1, 2, 3 \) for any generating subset \( S \) of \( G \) of minimal order.

The present paper is organized as follows.

In section 2, we studied the case \( |S^2| \leq 3k - 3 \) and we proved the following Theorem.

**Theorem 2.4.** Let \( G \) be a finitely generated group with \( d(G) = n \). Suppose that \( |S^2| \leq 3|S| - 3 \), for any generating subset \( S \) of \( G \) such that \( |S| = n \). Then \( G \) is a group of one of the following types:

(i) \( G \) is the quaternion group of order 8,

(ii) \( G = \langle x_1 \rangle \times \langle x_2 \rangle \times \langle x_3 \rangle \times \langle x_4 \rangle \times \langle x_5 \rangle \), where \( x_2^2 = x_3^2 = x_4^2 = x_5^2 = 1 \),

(iii) \( G = \langle x_1 \rangle \times \langle x_2 \rangle \times \langle x_3 \rangle \times \langle x_4 \rangle \), where \( x_3^2 = x_4^2 = 1 \),

(iv) \( G \) is abelian and \( n \leq 3 \).

Conversely, if \( G \) satisfies one of (i) – (iii), or (iv) with \( n > 1 \), then \( |S^2| \leq 3|S| - 3 \), for any generating subset \( S \) of \( G \) with \( |S| = d(G) \).

In section 3, we studied the case \( |S^2| \leq 3|S| - 2 \). Obviously, if \( |S| = 2 \), then \( |S^2| \leq 3|S| - 2 \), therefore we assume \( d(G) \geq 3 \). We proved the following Theorem.

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Theorem 3.5. Let $G$ be a finitely generated group with $d(G) = n \geq 3$. Suppose that $|S^2| \leq 3|S| - 2$, for any generating subset $S$ of $G$ such that $|S| = n$. Then $G$ is a group of one of the following types:

(i) $G = \langle x_1, x_2, x_3 \mid x_1^2 = x_2^3 = x_3^3 = c \in Z(G), c^2 = 1, x_jx_ix_j^{-1} = x_i^3, i \neq j, 1 \leq i, j \leq 3 \rangle \simeq Q_8 \times \langle d \rangle, |d| = 2$,

(ii) $G = \langle x_1 \rangle \times \langle x_2 \rangle \times \langle x_3 \rangle \times \langle x_4 \rangle \times \langle x_5 \rangle \times \langle x_6 \rangle$, where $x_1^2 = x_2^2 = x_3^2 = x_4^2 = x_5^2 = x_6^2 = 1$,

(iii) $G = \langle x_1 \rangle \times \langle x_2 \rangle \times \langle x_3 \rangle \times \langle x_4 \rangle \times \langle x_5 \rangle$, where $x_2^3 = x_3^2 = x_4^3 = x_5^2 = 1$,

(iv) $G$ is abelian and $n \leq 4$.

Conversely, if $G$ satisfies one of (i) – (iv), then $|S^2| \leq 3|S| - 2$, for any generating subset $S$ of $G$ with $|S| = d(G)$.

Finally, in section 4, we proved the case $|S^2| \leq 3|S| - 1$. We proved the following result.

Theorem 4.6. Let $G$ be a finitely generated group with $d(G) = n \geq 3$. Suppose that $|S^2| \leq 3|S| - 1$, for any generating subset $S$ of $G$ such that $|S| = n$. Then $G$ is a group of one of the following types:

(i) $G = \langle x_1, x_2 \rangle \langle x_3 \rangle, x_3^3 = 1, x_1x_2 = x_2x_1, x_3^{-1}x_1x_3 = x_1^{-1}, x_3^{-1}x_2x_3 = x_2^{-1}$,

(ii) $G = \langle x_1, x_2 \rangle \langle x_3 \rangle, x_3^4 = x_2 = x_4^2 = x_5^2 = 1, x_1x_3 = x_3x_1, x_2x_3 = x_3x_2, x_2^{-1}x_1x_2 = x_1^{-1}, x_1^2x_2^2x_3^2 = 1$,

(iii) $G = \langle x_1, x_2 \rangle \times \langle x_3 \rangle, x_3^4 = x_2^3 = 1, x_1^2 = x_2^2, x_2^{-1}x_1x_2 = x_1^{-1}, x_3^{-1}x_1x_3 = x_1^{-1}, x_3^{-1}x_2x_3 = x_2^{-1}$,

(iv) $G \simeq D_4 \times C_2$,

(v) $G \simeq Q_8 \times C_2$,

(vi) $G = \langle x_1 \rangle \times \langle x_2 \rangle \times \langle x_3 \rangle \times \langle x_4 \rangle \times \langle x_5 \rangle \times \langle x_6 \rangle$, where $x_2^3 = x_3^2 = x_4^2 = x_5^2 = x_6^2 = 1$,

(vii) $G = \langle x_1 \rangle \times \langle x_2 \rangle \times \langle x_3 \rangle \times \langle x_4 \rangle \times \langle x_5 \rangle$, where $x_2^3 = x_3^2 = 1$,

(viii) $G$ is abelian and $n \leq 4$.

Conversely, if $G$ satisfies one of (i) – (viii), then $|S^2| \leq 3|S| - 1$, for any generating subset $S$ of $G$ with $|S| = d(G)$.

We refer to [29] for notation, in particular we will denote $C_n$ the cyclic group of order $n$, $Q_8$ the quaternion group of order 8, and $D_n$ the dihedral group of order $2n$.

2. Minimal generating subsets $S$ with $|S^2| \leq 3|S| - 3$

We start this section with the following useful Proposition.

Proposition 2.1. Let $G$ be a finitely generated group with $d(G) = n \geq 3$.

If $G$ is not abelian, then $G$ includes a subset $Y$ such that $G = \langle Y \rangle$, $|Y| = n$, and $Y^2$ contains at least $\frac{1}{2}(n^2 + n)$ elements.

Proof. Let $S$ be a finite subset of $G$ such that $G = \langle S \rangle$ and $|S| = n$. Write $S = \{x_1, x_2, \ldots, x_n\}$. Since $G$ is not abelian, there exist $i, j \in \{1, \ldots, n\}, i \neq j$ such that $x_ix_j \neq x_jx_i$. Without loss of generality we can suppose $x_1x_2 \neq x_2x_1$. Suppose now that $x_1x_j = x_jx_1$ for some $j > 2$, and let $j$ be minimum with this property. Then $x_1(x_2x_j) = x_1x_2x_j \neq x_2x_1x_j = x_2(x_1x_j) = x_2(x_jx_1) = (x_2x_j) x_1$. Put $S_1 = \{x_1, \ldots, x_{j-1}, x_2x_j, x_{j+1}, \ldots, x_n\}$, then $G = \langle S_1 \rangle$ and $|S_1| = n$. Using the above arguments, after
finitely many steps we obtain a subset $Y = \{y_1, y_2, \ldots, y_n\}$ of order $n$ such that $y_1 y_j \neq y_j y_1$ for all $j > 1$, and such that $G = \langle Y \rangle$. Suppose that $y_j y_k = y_m y_t$ where $j \neq k$, $m \neq t$, $j \neq m$. Then $k \neq t$. If $j \neq t$ then $y_j \in (Y \setminus \{y_k\})$. If $j = t$ then $y_k = y_j^{-1} y_m y_j$, so that $y_k \in (Y \setminus \{y_k\})$. In both cases we obtain $d(G) \leq n - 1$. This contradiction shows that $y_j y_k \neq y_m y_t$ whenever $|\{j, k, m, t\}| \geq 3$. Consider now the elements:

$$y_1 y_2, \ldots, y_1 y_n; y_2 y_3, \ldots, y_2 y_n, \ldots, y_{n-1} y_n.$$

By the previous arguments these elements are pairwise different. Also the elements

$$y_2 y_1, \ldots, y_n y_1$$

are pairwise different. By the previous arguments $y_j y_1 \neq y_s y_k$ for $j \geq 2$, $1 \leq s < k \leq n$. Finally the element $y_2^2$ is different from the previous ones. It follows that the subset $Y^2$ has at least

$$(n - 1) + (n - 2) + \cdots + 2 + 1 + (n - 1) + 1 = \frac{1}{2} (n^2 - n) + n = \frac{1}{2} (n^2 + n)$$

elements.

\[\square\]

**Corollary 2.2.** Let $G$ be a finitely generated group with $d(G) = n \geq 3$.

Suppose that $|S^2| \leq 3|S| - 3$ for each generating subset $S$ of $G$ such that $|S| = n$.

Then $G$ is abelian.

**Proof.** Suppose $G$ non-abelian. By Proposition 2.1, $G$ has a finite subset $Y$ such that $G = \langle Y \rangle$, $|Y| = n$ and $|Y^2| \geq \frac{1}{2} (n^2 + n)$. Since $|Y^2| \leq 3|Y| - 3$, we obtain $\frac{1}{2} (n^2 + n) \leq 3n - 3$. It follows that $n \leq 3$. Thus $Y = \{y_1, y_2, y_3\}$. From the proof of Proposition 2.1 we can see that $Y^2$ contains the elements: $y_1^2, y_1 y_2, y_1 y_3, y_2 y_1, y_2 y_3, y_3 y_1, y_3 y_2$, that are pairwise different. Since $|Y^2| = 6$, $Y^2 = \{y_1^2, y_1 y_2, y_1 y_3, y_2 y_1, y_3 y_1, y_3 y_2\}$.

On the other hand $y_3 y_2 \in Y^2$ and by the previous remarks we have only the possibility $y_2 y_3 = y_3 y_2$. Put $M = \{y_1, y_1 y_2, y_3\}$. Now it is easy to check that the set $\{y_1^2, y_1 y_3, y_3 y_1, y_1 y_1 y_2, y_1 y_2 y_1, y_1 y_2 y_3, y_3 y_1 y_2\}$ has order 7: in particular $(y_1 y_2) y_3 \neq y_3 (y_1 y_2)$ since $y_2 y_3 = y_3 y_2$. It follows that $M^2$ contains at least 7 elements and we obtain the final contradiction. \[\square\]

**Corollary 2.3.** Let $G$ be a finitely generated group with $d(G) = 2$.

Suppose that $|S^2| \leq 3|S| - 3$ for each generating subset $S$ of $G$ such that $|S| = 2$.

Then $G$ is abelian or $G$ is the quaternion group of order 8.

Conversely if either $G$ is abelian or $G \simeq Q_8$, then $|S^2| \leq 3|S| - 3$ for each generating set $S$ of $G$ such that $|S| = 2$.

**Proof.** Suppose that $G$ is non-abelian. Then $G = \langle x, y \rangle$ where $xy \neq yx$. By our condition $|S^2| \leq 3|S| - 3 = 3$. We have $S^2 = \{x^2, y^2, xy, yx\}$. Then $x^2 = y^2$. Suppose that $x^2 = y^2 = c \neq 1$. Then $c \in \zeta(G)$. We have also $G = \langle x, y \rangle$. Then $x(xy) \neq (xy)x$, hence $x^2 = (xy)^2$ and also $y^2 = (xy)^2$.

Furthermore, from $G = \langle x, cy \rangle$ and $c \in \zeta(G)$ we get $y^2 = x^2 = (cy)^2 = c^2 y^2$ from which $c^2 = 1$. Using $x^2 = (xy)^2$, we obtain that $x = xy$. It follows that $xy^{-1} =yx$ and hence $y^{-1} = x^{-1}yx$. Thus the subgroup $\langle y \rangle$ is normal in $G$. By the same reason also $\langle x \rangle$ is normal in $G$. Finally $x^4 = (x^2)^2 = c^2 = 1$.
and similarly $y^4 = 1$. It follows that $G$ is the quaternion group of order 8. Suppose now $x^2 = y^2 = 1$. Since $G = \langle x, xy \rangle$, we obtain $(xy)^2 = x^2 = 1$. It follows that $y^{-1}xy = yxy = x^{-1} = x$, so $xy = yx$ and we obtain a contradiction.

Conversely, if $G$ is abelian and $S = \{x, y\}$, then $S^2 = \{x^2, xy, y^2\}$, and $|S^2| \leq 3 = 3|S| - 3$. If $G \cong Q_8$, and $S = \{x, y\}$, $G = \langle S \rangle$, then $x^2 = y^2$ and $S^2 = \{xy, yx, x^2\}$, as required. □

Now we can prove Theorem 2.4.

**Theorem 2.4.** Let $G$ be a finitely generated group with $d(G) = n$. Suppose that $|S^2| \leq 3|S| - 3$ for any generating subset $S$ of $G$ such that $|S| = n$. Then $G$ is a group of one of the following types:

(i) $G$ is the quaternion group of order 8,

(ii) $G = \langle x_1 \rangle \times \langle x_2 \rangle \times \langle x_3 \rangle \times \langle x_4 \rangle \times \langle x_5 \rangle$, where $x_2^2 = x_3^2 = x_4^2 = x_5^2 = 1$,

(iii) $G = \langle x_1 \rangle \times \langle x_2 \rangle \times \langle x_3 \rangle \times \langle x_4 \rangle$, where $x_2^3 = x_4^2 = 1$,

(iv) $G$ is abelian and $n \leq 3$.

Conversely, if $G$ satisfies one of (i) – (iii), or (iv) with $n > 1$, then $|S^2| \leq 3|S| - 3$, for any generating subset $S$ of $G$ with $|S| = d(G)$.

**Proof.** If $n = 2$, then Corollary 2.3 shows that either $G$ is the quaternion group of order 8 or $G$ is abelian. If $n = 3$, then Corollary 2.2 shows that $G$ is abelian. Hence either (i) or (iv) holds.

Now suppose $n \geq 4$. Then $G$ is abelian by Corollary 2.2 and $G$ is a finitely generated abelian group with $d(G) = n$. Choose an arbitrary generating subset $S$ of $G$ such that $|S| = n$. Let $S = \{g_1, \ldots, g_n\}$. Clearly $S^2 = A \cup B$ where $A = \{g_jg_m \mid 1 \leq j < m \leq n\}$, $B = \{g_j^2 \mid 1 \leq j \leq n\}$. Suppose that $g_jg_m = g_sg_k$ where $(j, m) \neq (s, k), j < m, s < k$. If $\{j, m\} \cap \{s, k\} = \emptyset$, then $g_m = g_j^{-1}g_sg_k$, and we obtain a contradiction with the minimality of $S$. Suppose that $s \in \{j, m\}$. If $s = j$, then $g_m = g_k$, which is impossible. If $s = m$, then $g_j = g_sg_kg_j^{-1}$, and we obtain a contradiction with the minimality of $S$. Using similar arguments we obtain a contradiction if $k \in \{j, m\}$. It follows that the elements of the subset $A$ are pairwise different, and arguing analogously, that $A$ and $B$ are disjoint. Then $|S^2| = \frac{1}{2}n(n - 1) + d$ where $d = |B|$. We note that $d \leq n$. Thus $\frac{1}{2}n(n - 1) + 1 \leq 3n - 3$, so that $n \leq 5$. If $n = 4$, then $\frac{1}{2}n(n - 1) + d = 6 + d \leq 3n - 3 = 9$. It follows that $d \leq 3$, so that we can suppose $g_3^2 = g_4^2$. Since $G$ is a finitely generated abelian group with $d(G) = 4$, we have $G = \langle x_1 \rangle \times \langle x_2 \rangle \times \langle x_3 \rangle \times \langle x_4 \rangle$, for some elements $x_1, x_2, x_3, x_4$. Clearly $\{x_1, x_2, x_3, x_4\}$ is a minimal generating subset for $G$. Then, arguing as before, we can suppose $x_3^2 = x_4^2$. That is possible only in the case when $x_3^2 = x_4^2 = 1$. If $n = 5$, then $\frac{1}{2}n(n - 1) + d = 10 + d \leq 3n - 3 = 12$. It follows that $d \leq 2$, so that we can suppose $g_3^2 = g_4^2 = g_2^2 = g_3^2$. Since $G$ is a finitely generated abelian group with $d(G) = 5$, we have $G = \langle x_1 \rangle \times \langle x_2 \rangle \times \langle x_3 \rangle \times \langle x_4 \rangle \times \langle x_5 \rangle$, for some elements $x_1, x_2, x_3, x_4, x_5$. We can suppose $x_2^2 = x_3^2 = x_4^2 = x_5^2$ and, as in the previous case, we obtain that $x_2^2 = x_3^2 = x_4^2 = x_5^2 = 1$.

Conversely, suppose that $G$ satisfies (i), or (ii), or (iii), or (iv) with $n > 1$. It is easy to prove that $|S^2| \leq 3|S| - 3$ if $|S| = n = d(G)$ and $G = \langle S \rangle$. In fact, if $G$ is abelian and $n = 3$, then $|S^2| \leq \frac{1}{2}n(n - 1) + 3 = 6 = 3|S| - 3$. If (iii) holds with $n = 4$, then $|B| \leq 3$, where $B = \{g^2 \mid g \in S\}$.

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and \( |S^2| \leq \frac{1}{2} n(n - 1) + 3 = 8 = 3|S| - 3 \). If \((ii)\) holds with \(n = 5\), then \( |B| \leq 2 \), and \( |S^2| \leq \frac{1}{2} n(n - 1) + 2 = 12 = 3|S| - 3 \), and finally if \((i)\) holds, then the result follows from Corollary 2.3. \(\square\)

From Theorem 2.4 it follows the following easy Corollary.

**Corollary 2.5.** Let \( G \) be a finitely generated group with \( d(G) = n \).

Suppose that \( |S^2| \leq 3|S| - 3 \) for each generating subset \( S \) of \( G \) such that \( |S| = n \).

If \( G \) is torsion-free, then \( G \) is abelian and \( n \leq 3 \).

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### 3. Minimal generating subsets \( S \) with \( |S^2| \leq 3|S| - 2 \)

In this section we will consider finitely generated groups \( G \) such that \( |S^2| \leq 3|S| - 2 \) for each generating subset \( S \) of \( G \) with the property \( |S| = d(G) \).

Suppose that \( G \) is non-abelian and \( d(G) = 2 \). If \( S = \{g_1, g_2\} \) is a generating subset of \( G \), then clearly \( g_1g_2 \neq g_2g_1 \), so that \( 3 \leq |S^2| \leq 4 = 3|S| - 2 \).

Therefore in the sequel we will assume that \( d(G) \geq 3 \).

We start with an easy more general Lemma.

**Lemma 3.1.** Let \( G \) be a finitely generated group with \( d(G) = n \geq 3 \). Suppose that \( |S^2| \leq 3|S| - 1 \) for each generating subset \( S \) of \( G \) such that \( |S| = n \). If \( G \) is non-abelian, then \( n \in \{3, 4\} \).

**Proof.** Since \( G \) is non-abelian, Proposition 2.1 shows that \( G \) has a finite subset \( Y \) such that \( G = \langle Y \rangle \), \( |Y| = n \) and \( |Y^2| \geq \frac{1}{2}(n^2 + n) \). Thus we have \( \frac{1}{2}(n^2 + n) \leq 3n - 1 \). It follows that \( n^2 - 5n + 2 \leq 0 \).

This is possible only if \( n \in \{3, 4\} \). \(\square\)

Now we show that in the case \( n = d(G) = 4 \) and \( |S^2| \leq 3|S| - 1 \) for each generating subset \( S \) of \( G \) such that \( |S| = n \), again we obtain that \( G \) is abelian.

**Lemma 3.2.** Let \( G \) be a finitely generated group with \( d(G) = 4 \).

Suppose that \( |S^2| \leq 11 \) for each generating subset \( S \) of \( G \) such that \( |S| = 4 \).

Then \( G \) is abelian.

**Proof.** Suppose the contrary, so let \( G \) be non-abelian. Then, arguing as in the proof of Proposition 2.1, we obtain that \( G \) contains a subset \( S = \{x_1, x_2, x_3, x_4\} \) such that \( G = \langle S \rangle \) and the set

\[
L = \{x_1^2, x_1x_2, x_1x_3, x_1x_4, x_2x_3, x_2x_4, x_3x_4, x_4x_1, x_3x_1, x_2x_1\}
\]

has order 10. Now consider the set \( T = \{y_1, y_2, y_3, y_4\} \), where \( y_2 = x_1x_2, y_j = x_j \) whenever \( j \neq 2 \). It is easy to see that the set

\[
V = \{y_1^2, y_1y_2, y_1y_3, y_1y_4, y_2y_3, y_2y_4, y_3y_4, y_4y_1, y_3y_1, y_2y_1\}
\]

has order 10. If \( x_2x_3 = x_3x_2 \), then \( y_2y_3 \neq y_3y_2 \) and \( T^2 = V \cup \{y_3y_2\} \). Therefore \( x_2x_4 \neq x_4x_2 \) otherwise \( y_2y_4 \neq y_4y_2 \) and \( |T^2| > 11 \). Arguing analogously with \( x_3 \) we get that it is not possible to have simultaneously \( x_3x_2 = x_2x_3 \) and \( x_3x_4 = x_4x_3 \) and, arguing with \( x_4 \), to have simultaneously

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Lemma 3.3. Let $G$ be a finitely generated group with $d(G) = 3$.

Suppose that $|X^2| \leq 7$ for each generating subset $X$ of $G$ such that $|X| = 3$. If $G$ is non-abelian and $S = \{x_1, x_2, x_3\}$ is a subset of $G$ such that $G = \langle S \rangle$ and $|\{x_1^2, x_1x_2, x_1x_3, x_2x_3, x_2x_1, x_3x_1\}| = 6$, then

$$x_1^2 = x_2^2 = x_3^2.$$ 

Proof. Write $L = \{x_1^2, x_1x_2, x_1x_3, x_2x_3, x_2x_1, x_3x_1\}$ and assume by contradiction that $x_2^2 \neq x_1^2$. Then $S^2 = L \cup \{x_2^2\}$ has order 7. This implies, arguing as usual, that $x_2x_3 = x_3x_2$.

Consider the set $T = \{y_1, y_2, y_3\}$ where $y_1 = x_1, y_2 = x_2, y_3 = x_1x_3$. Obviously $G = \langle T \rangle$ and it is easy to see that $V = \{y_1^2, y_2^2, y_1y_2, y_1y_3, y_2y_3, y_3y_1, y_2y_2\}$ has order 7. But $x_2x_3 = x_3x_2$ implies $y_2y_3 \neq y_3y_2$, then the subset $V \cup \{y_3y_2\}$ of $T^2$ has order 8, a contradiction. A similar argument holds if $x_3^2 \neq x_1^2$, so $x_1^2 = x_2^2 = x_3^2$, as required.

Now we can prove the following Proposition that gives a description of $G$ if $d(G) = 3$ and $|S^2| \leq 7$.

Proposition 3.4. Let $G$ be a finitely generated group with $d(G) = 3$. Suppose that $|S^2| \leq 7$ for each generating subset $S$ of $G$ such that $|S| = 3$. If $G$ is non-abelian, then

$$G = \langle x_1, x_2, x_3 \mid x_1^2 = x_2^2 = x_3^2 = c \in \zeta(G), c^2 = 1, x_jx_kx_j^{-1} = x_k^3, 1 \leq j, k \leq 3, j \neq k \rangle \cong Q_8 \times \langle d \rangle, |d| = 2.$$ 

Conversely, if $G = Q_8 \times \langle d \rangle$ with $|d| = 2$, then $d(G) = 3$ and $|S^2| \leq 7$ for every generating subset $S$ of $G$ with $|S| = 3$.

Proof. Arguing as in the proof of Proposition 2.1, we find a subset $S = \{x_1, x_2, x_3\}$ of $G$ such that $G = \langle S \rangle$ and $|\{x_1^2, x_1x_2, x_1x_3, x_2x_3, x_2x_1, x_3x_1\}| = 6$. Then by Lemma 3.3 we have

$$x_1^2 = x_2^2 = x_3^2.$$ 

Put $y_1 = x_1, y_2 = x_1x_2, y_3 = x_3, S_1 = \{y_1, y_2, y_3\}$. Then $G = \langle S_1 \rangle$ and it is easy to prove that $\{y_1^2, y_1y_2, y_1y_3, y_2y_3, y_2y_1, y_3y_1\} = 6$. Therefore, again by Lemma 3.3, we have

$$y_1^2 = y_2^2 = y_3^2.$$ 

From $y_1^2 = y_2^2$ it follows $x_1^2 = x_1x_2x_1x_2$, so $x_1 = x_2x_1x_2$ and $x_1^{-1}x_2x_1 = x_2^{-1}$. It follows that $[x_2, x_1] = x_2^{-2}$. On the other hand $x_1^2 = x_2^2$, so we have $x_2x_2 = x_1x_2x_1x_2$, thus $x_2 = x_1x_2x_1$ and $x_2^{-1}x_1x_2 = x_1^{-1}$. It follows that $[x_1, x_2] = x_1^{-2}$. We have now $x_2^{-2} = [x_2, x_1] = [x_1, x_2]^{-1} = x_1^{-2} = x_2^{-2}$, hence $x_2^{-2} = 1$. Then also $x_1^2 = (x_1^2)^2 = (x_2^2)^2 = 1$ and $x_3^2 = 1$. The equality $x_1^2 = x_2^2 = x_3^2 = c$ shows that $c \in Z(G)$ and $|c| = 2$. Arguing as before we obtain that $[x_1, x_3] = x_1^{-2} = x_3^2 = x_2^2$ and

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\[ [x_2, x_3] = x_2^{-2} = x_3^2 = x_2^2. \] Obviously \( \langle x_1, x_2 \rangle \simeq Q_8. \) From \( x_1 x_2 x_3 x_1 x_2 x_3 = (x_1 x_2)^2 x_3^2 = 1 \) we obtain the result.

Conversely, let \( G = Q_8 \times \langle d \rangle, |d| = 2. \) Then \( x^2 = y^2 \) for every \( x, y \in G \) with \( x^2 \neq 1 \) and \( y^2 \neq 1. \) Moreover \( s \in Z(G) \) for every \( s \) in \( G \) of order 2. Now let \( S = \{x_1, x_2, x_3\}. \) If there exists \( i \in \{1, 2, 3\}, \) say \( i = 1 \) such that \( x_1^2 = 1 \) then \( x_1 \in Z(G), \)
\[
S^2 = \{x_1^2, x_1 x_2, x_1 x_3, x_2 x_3, x_2^2, x_3 x_2, x_3^2\},
\]
and \( |S^2| \leq 7. \) If \( x_i^2 \neq 1 \) for every \( i \in \{1, 2, 3\}, \) then \( x_1^2 = x_2^2 = x_3^2, \)
\[
S^2 = \{x_1^2, x_1 x_2, x_1 x_3, x_2 x_3, x_2 x_1, x_3 x_1, x_3 x_2\}
\]
and again \( |S^2| \leq 7. \) The result is proved. \( \square \)

Now we can prove Theorem 3.5.

**Theorem 3.5.** Let \( G \) be a finitely generated group with \( d(G) = n \geq 3. \) Suppose that \( |S^2| \leq 3|S| - 2 \) for any generating subset \( S \) of \( G \) such that \( |S| = n. \) Then \( G \) is a group of one of the following types:

(i) \( G = \langle x_1, x_2, x_3 \mid x_1^2 = x_2^2 = x_3^2 = c \in Z(G), c^2 = 1, x_j x_i x_j^{-1} = x_i^3, i \neq j, 1 \leq i, j \leq 3 \rangle \simeq Q_8 \times \langle d \rangle, |d| = 2, \)
(ii) \( G = \langle x_1 \rangle \times \langle x_2 \rangle \times \langle x_3 \rangle \times \langle x_4 \rangle \times \langle x_5 \rangle \times \langle x_6 \rangle, \) where \( x_1^2 = x_2^2 = x_3^2 = x_4^2 = x_5^2 = x_6^2 = 1, \)
(iii) \( G = \langle x_1 \rangle \times \langle x_2 \rangle \times \langle x_3 \rangle \times \langle x_4 \rangle \times \langle x_5 \rangle, \) where \( x_3^2 = x_4^2 = x_5^2 = 1, \)
(iv) \( G \) is abelian and \( n \leq 4. \)

Conversely, if \( G \) satisfies one of (i) – (iv), then \( |S^2| \leq 3|S| - 2 \) for any generating subset \( S \) of \( G \) with \( |S| = d(G). \)

**Proof.** Suppose first that \( G \) is non-abelian. Then Lemma 3.1 shows that \( n \in \{3, 4\}. \) More precisely Lemma 3.2 shows that \( d(G) = 3. \) Then Proposition 3.4 implies that \( G \) is a group of type (i).

Assume now that \( G \) is abelian. Choose an arbitrary generating subset \( S \) of \( G \) such that \( |S| = n. \) Let \( S = \{g_1, \ldots, g_n\}. \) Clearly \( S^2 = A \cup B \) where \( A = \{g_j g_m \mid 1 \leq j < m \leq n\}, \) \( B = \{g_j^2 \mid 1 \leq j \leq n\}. \)

As in a proof of Theorem 2.4 we can show that all elements of the subset \( A \) are pairwise different and that \( A \) and \( B \) are disjoint. It follows that \( |S^2| = \frac{1}{2} n(n-1) + d \) where \( d = |B|. \) We note that \( d \leq n. \) Thus \( \frac{1}{2} n(n-1) + 1 \leq 3n - 2, \) so that \( n \leq 6. \)

If \( n = 5, \) then \( \frac{1}{2} n(n-1) + d = 10 + d. \) From \( |S^2| \leq 3|S| - 2 \) we obtain that \( 10 + d \leq 13, \) thus \( d \leq 3. \)
Therefore we can suppose \( g_5^2 = g_4^2 = g_3^2, \) and \( G \) is of type (iii).

If \( n = 6, \) then \( \frac{1}{2} n(n-1) + d = 15 + d. \) From \( |S^2| \leq 3|S| - 2 \) we obtain that \( 15 + d \leq 16, \) which implies that \( d \leq 1, \) so that \( g_5^2 = g_3^2 = g_2^2 = g_1^2 = g_0^2 \) and \( G \) is a group of type (ii).

Conversely, suppose that \( G \) satisfies (i), then \( d(G) = 3 \) and by Proposition 3.4 \( |S^2| \leq 7 \) for every generating subset of \( G \) of order 3. Now suppose that (ii), or (iii), or (iv) holds, then \( G \) is abelian and for every subset \( S \) of \( G \) of order \( n = d(G) \) we have \( |S^2| = \frac{1}{2} n(n-1) + d \) where \( d = |B|, B = \{g^2 \mid g \in S\}. \)

Now if (ii) holds, then \( n = 6 \) and \( d = 1, \) and we have \( |S^2| = 15 + 1 = 3|S| - 2, \) as required; if (iii)
holds, then \( n = 5 \) and \( d = 3 \), and we have \(|S^2| \leq 10 + 3 = 3|S| - 2\), as required, and finally if (iv) holds, then \( n \in \{3, 4\} \), \( d \leq n \) and in any case \(|S^2| \leq 3|S| - 2\) as required. \( \square \)

From Theorem 3.5 it follows the following easy Corollary.

**Corollary 3.6.** Let \( G \) be a finitely generated group with \( d(G) = n \).

Suppose that \(|S^2| \leq 3|S| - 2\) for each generating subset \( S \) of \( G \) such that \(|S| = n \).

If \( G \) is torsion-free, then \( G \) is abelian and \( n \leq 4 \).

4. Minimal generating subsets \( S \) with \(|S^2| \leq 3|S| - 1\)

In this section we consider finitely generated groups \( G \) such that \(|S^2| \leq 3|S| - 1\) for each generating subset \( S \) of \( G \) with the property \(|S| = d(G)\).

First we assume \( G \) non-abelian. Hence by Lemmas 3.1 and 3.2 we have \( d(G) = 3 \).

Let \( S = \{x_1, x_2, x_3\} \), with \( G = \langle S \rangle \) non-abelian. From \(|S^2| \leq 3|S| - 1 = 8\), it follows that either two of the elements \( x_1, x_2, x_3 \) have the same square or two of them commute. We study first the situation \(|\{x_1^2, x_2^2, x_3^2\}| = 3\). In this case we can suppose, without loss of generality, \( x_2x_3 = x_3x_2 \). We start with the following Lemma.

**Lemma 4.1.** Let \( G = \langle x_1, x_2, x_3 \rangle \), with \( d(G) = 3 \), \( x_2x_3 = x_3x_2 \), \(|\{x_1^2, x_2^2, x_3^2\}| = 3 \). If \(|T|^2 \leq 8\) for each generating subset \( T \) of \( G \) of order 3, then \( x_2 \in Z(G) \) or \( x_3 \in Z(G) \) or \( x_2x_3 \in Z(G) \) or the following holds

\[(i) \ G = \langle x_2, x_3 \rangle \langle x_1 \rangle, \quad x_1^4 = 1, x_2x_3 = x_3x_2, \ x_1^{-1}x_2x_1 = x_2^{-1}, \ x_1^{-1}x_3x_1 = x_3^{-1}.
\]

**Proof.** Suppose that \( x_2 \notin Z(G) \), \( x_3 \notin Z(G) \), \( x_2x_3 \notin Z(G) \). Thus \( x_1x_2 \neq x_2x_1 \). Consider the subset \( T = \{x_1, x_2, x_1x_2x_3\} \). Obviously \( G = \langle T \rangle \), thus \(|T|^2 \leq 8\). By the hypothesis \( x_1x_2 \neq x_2x_1 \) and \( x_2 \) does not commute with \( x_1x_2x_3 \). If \( x_1 \) commutes with \( x_1x_2x_3 \), then \( x_2x_3 \in Z(G) \), which is not the case. Hence from \(|T|^2 \leq 8\) we get that either \((x_1x_2x_3)^2 = x_1^2\) or \((x_1x_2x_3)^2 = x_2^2\). Arguing similarly on the subset \( V = \{x_1, x_3, x_1x_2x_3\} \) we obtain that either \((x_1x_2x_3)^2 = x_1^2\) or \((x_1x_2x_3)^2 = x_2^2\). Then \((x_1x_2x_3)^2 = x_1^2\) since \(x_2x_3 = x_3x_2\), hence

\[(x_2x_3)^{x_1} = (x_2x_3)^{-1}.
\]

Now consider the generating set \( W = \{x_1, x_1x_2, x_3\} \) consisting of pairwise non-commuting elements. From \(|W|^2 \leq 8\) we get that either \((x_1x_2)^2 = x_1^2\) or \((x_1x_2)^2 = x_2^2\). Arguing similarly on \(W_1 = \{x_1, x_1x_3, x_2\} \) we obtain that either \((x_1x_3)^2 = x_1^2\) or \((x_1x_3)^2 = x_2^2\).

First suppose that either \((x_1x_2)^2 = x_1^2\) or \((x_1x_3)^2 = x_2^2\), and without loss of generality, \((x_1x_2)^2 = x_1^2\). Then \(x_2^{-1} = x_2\) and from \((x_2x_3)^{x_1} = (x_2x_3)^{-1}\) we obtain that also \(x_3^{-1} = x_3\). Thus the equality \((x_1x_3)^2 = x_2^2\) is impossible, otherwise \(x_2^2 = x_1x_3x_1x_3 = x_1^2x_1^{-1}x_3x_1x_3 = x_1^2\). Therefore \((x_1x_3)^2 = x_1^2\) and considering the generating subset of pairwise non-commuting elements \(\{x_1^{-1}, x_2, x_1x_3\} \) we obtain that \((x_1x_3)^2 = x_1^{-2}\) and \(x_1^4 = 1\), or \(x_2^2 = x_1^{-2}\) and similarly, considering the subset \(\{x_1^{-1}, x_1x_2, x_3\} \) that \(x_1^4 = 1\), or \(x_2^2 = x_1^{-2}\), thus \(x_1^4 = 1\) since \(x_2^2 \neq x_2^{-2}\). Therefore (i) holds.

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Finally suppose \((x_1 x_2)^2 = x_3^2\) and \((x_1 x_3)^2 = x_2^2\). In this case the generating subset \(\{x_1, x_1 x_2, x_1 x_3\}\) has elements with different squares. Hence two of them commute and the unique possibility is \(x_1 x_2 x_1 x_3 = x_1 x_3 x_1 x_2\), then \((x_2^{-1} x_3)x_1 = x_1(x_2^{-1} x_3)\). But from \((x_1 x_3)^2 = x_2^2\) we have also that \(x_2^2\) commutes with \(x_1\). Hence \(x_2 x_3\) commutes with \(x_1\) and then it is in \(Z(G)\) a contradiction.

We will prove later that if \(G\) satisfies \((i)\), then \(|S^2| \leq 8\), for every generating subset \(S\) of \(G\) of order 3.

We continue our investigation assuming that there exists a generating subset \(S\) of \(G\), with \(S = \{x_1, x_2, x_3\}\), \(|\{x_1^2, x_2^2, x_3^2\}| = 3\), \(x_2^2 \neq x_3^2\), either \(x_2\) or \(x_3\) in \(Z(G)\). Assume for example that \(x_3 \in Z(G)\). Then \(x_1 x_2 \neq x_2 x_1\), since \(G\) is not abelian. In this case the structure of \(G\) is described in the following Lemma.

**Lemma 4.2.** Let \(G = \langle x_1, x_2, x_3 \rangle\), with \(d(G) = 3, |\langle x_1^2, x_2^2, x_3^2 \rangle| = 3\). Suppose that \(G\) is non-abelian and that \(|T^2| \leq 8\) for each generating subset \(T\) of \(G\) of order 3. If \(x_3 \in Z(G)\), then one of the following holds:

\[
\begin{align*}
(j) & \ G = \langle a, b \rangle \times \langle c \rangle, c^2 = 1, a^4 = 1, a^b = a^{-1}, \\
(jj) & \ G = \langle a, b \rangle \langle c \rangle, a^4 = b^4 = c^4 = 1, ac = ca, bc = cb, a^b = a^{-1}, a^2 b^2 c^2 = 1.
\end{align*}
\]

**Proof.** Consider the generating subset of pairwise non-commuting elements \(V = \{x_1 x_2, x_2 x_3, x_1 x_2 x_3\}\). Then either \((x_1 x_2 x_3)^2 = (x_1 x_3)^2\) or \((x_1 x_2 x_2 x_3)^2 = (x_2 x_3)^2\), therefore either \((x_1 x_2)^2 x_3^2 = x_1^2 x_3^2\) or \((x_1 x_2)^2 x_3^2 = x_2^2 x_3^2\), hence either \((x_1 x_2)^2 = x_1^2\) or \((x_1 x_2)^2 = x_2^2\). Without loss of generality we can suppose \((x_1 x_2)^2 = x_1^2\), hence
\[
x_1 x_2 = x_1^{-1}.
\]

Now consider the generating subset of pairwise non-commuting elements \(W = \{x_1^{-1} x_3, x_2 x_3, x_1 x_2 x_3\}\). Then \((x_1^{-1} x_3)^2 = (x_2 x_3)^2\) or \((x_1 x_2 x_3)^2 = (x_1^{-1} x_3)^2\) or \((x_1 x_2 x_3)^2 = (x_2 x_3)^2\). The last equality implies the contradiction \(x_1^2 = x_2^2\). From the first equality we get \(x_1^{-2} = x_2^2\), and from \(x_2 x_1 = x_2^{-1}\) we get \(x_2^2 = 1\) and then \(x_4^2 = 1\). Finally from \((x_1 x_2 x_3)^2 = (x_1^{-1} x_3)^2\) we obtain \(x_1^2 = x_1^{-2}\). In any case
\[
x_1^4 = 1.
\]

Now consider the generating subset of pairwise non-commuting elements \(\{x_1, x_2, x_1 x_2 x_3\}\). Then either \((x_1 x_2 x_3)^2 = x_1^2\) or \((x_1 x_2 x_3)^2 = x_2^2\). Since \((x_1 x_2)^2 = x_1^2\), the first equality implies \(x_3^2 = 1\) and \((j)\) holds. So assume \((x_1 x_2 x_3)^2 = x_2^2\), then
\[
x_1^2 x_3^2 = x_2^2.
\]

Arguing analogously on the generating subset of pairwise non-commuting elements \(\{x_1, x_2^{-1}, x_1 x_2 x_3\}\), we obtain that either \(x_3^2 = 1\) and \((j)\) holds, or \((x_1 x_2 x_3)^2 = x_2^{-2}\), or \(x_1^2 = x_2^{-2}\). In the second case, from \(x_1^2 x_3^2 = x_2^2\) we get that \(x_2^2 = 1\) and also that \(x_1^2 = 1\) and \(x_1^2 x_2^2 x_3^2 = 1\), therefore \((jj)\) holds. Finally in the last case we have \(x_1^{-2} = x_2^2 = x_2^{-2}\), which is a contradiction.

Notice that if \((j)\) holds, then \(G = \langle a, c \rangle \langle b \rangle, \) with \(a^b = a^{-1}, c^b = c = c^{-1}\), therefore \((i)\) of Lemma 4.1 holds.

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We will show later that if (jj) holds, then $|S^2| \leq 8$ for every generating subset $S$ of $G$ of order 3.

Now we assume that there exists a generating subset $S$ of $G$, with $S = \{x_1, x_2, x_3\}, |\{x_1^2, x_2^2, x_3^2\}| = 3, x_2x_3$ in $Z(G)$. Notice that in this case if $(x_2x_3)^2 \neq x_1^2, x_2^2$, then the subset $\{x_1, x_2, x_2x_3\}$ satisfies the hypothesis of Lemma 4.2 and (j) or (jj) of Lemma 4.2 holds. Similarly if $(x_2x_3)^2 \neq x_1^2, x_3^2$, then the subset $\{x_1, x_3, x_2x_3\}$ satisfies the hypothesis of Lemma 4.2. Hence we can suppose $(x_2x_3)^2 = x_1^2$. In this case we can prove:

**Lemma 4.3.** Let $G = \langle x_1, x_2, x_3 \rangle$, with $d(G) = 3, |\{x_1^2, x_2^2, x_3^2\}| = 3$. Suppose that $G$ is non-abelian and that $|T^2| \leq 8$ for each generating subset $T$ of $G$ of order 3. If $x_2x_3 \in Z(G)$ and $(x_2x_3)^2 = x_1^2$, then (i) of Lemma 4.1 holds.

**Proof.** Consider the generating subset of pairwise non-commuting elements $V = \{x_1, x_2, x_1x_3\}$. Then either $(x_1x_3)^2 = x_1^2$ or $(x_1x_3)^2 = x_2^2$. If $(x_1x_3)^2 = x_2^2$, consider the generating subset of pairwise non-commuting elements $W = \{x_1x_3, x_3, x_1^{-1}(x_2x_3)\}$. Then either $x_3^2 = (x_1^{-1}(x_2x_3))^2 = 1$, or $x_2^2 = (x_1x_3)^2 = (x_1^{-1}(x_2x_3))^2 = 1$. But if $x_3^2 = 1$ then from $x_2^2x_3^2 = (x_2x_3)^2 = x_1^2$ we get the contradiction $x_2^2 = x_1^2$, while if $x_2^2 = 1$ from $x_2^2x_3^2 = (x_2x_3)^2 = x_1^2$ we obtain the contradiction $x_3^2 = x_1^2$. Therefore $(x_1x_3)^2 = x_1^2$, i.e. $(x_3)^{x_1} = x_3^{-1}$.

Arguing analogously on the generating subset of pairwise non-commuting elements $V_1 = \{x_1, x_3, x_1x_2\}$ we obtain that either $(x_1x_2)^2 = x_1^2$ or $(x_1x_2)^2 = x_3^2$ and that the relation $(x_1x_2)^2 = x_3^2$ is not possible considering the subset $W_1 = \{x_1x_2, x_2, x_1^{-1}(x_2x_3)\}$. Therefore $(x_1x_2)^2 = x_1^2$, i.e. $(x_2)^{x_1} = x_2^{-1}$.

Finally $x_1^4 = 1$.

In fact, considering the subset $V_2 = \{x_1^{-1}, x_2, x_1x_3\}$ we get $x_1^2 = (x_1x_3)^2 = x_1^{-2}$ and $x_1^4 = 1$, or $x_2^2 = x_2^{-1}$ and from $x_1^{x_2} = x_2^{-1}$ it follows that $x_2^2 = (x_2^2)^{x_2} = x_2^{-2}$ thus $x_1^4 = x_2^4 = 1$. Therefore (i) of Lemma 4.1 holds.

Now we assume that $|\{x^2 \mid x \in S\}| \leq 2$ for each generating subset $S$ of order 3. First suppose that $|\{x^2 \mid x \in S\}| = 1$ for each generating subset $S$ of order 3. In this case $G$ is abelian, as the following Lemma shows.

**Lemma 4.4.** Let $G = \langle x_1, x_2, x_3 \rangle, d(G) = 3$, and suppose that $|\{x^2 \mid x \in S\}| = 1$ for each generating subset $S$ of order 3. Then $G$ is an elementary abelian 3-generated 2-group.

**Proof.** We have $x_1^2 = x_2^2 = x_3^2 = (x_1x_2)^2 = (x_1x_3)^2$, hence $x_1^{x_2} = x_1^{-1}, x_2^{x_1} = x_2^{-1}, x_3^{x_1} = x_3^{-1}$. Considering the subset $\{x_1x_2x_3, x_3, x_2x_3\}$ we have also that $(x_1x_2x_3)^2 = (x_2x_3)^2$, thus $x_1^{x_2}x_3 = x_3^{-1}$. But we have also that $x_1^{x_2}x_3 = x_1$, therefore $x_1^2 = 1$. Thus $x_2^2 = x_3^2 = x_1^2 = 1$, then $G$ is abelian and an elementary abelian 2-group, as required.

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Now suppose that there exists a generating subset $S$ of $G$ of order 3, with $|\{x^2 \mid x \in S\}| = 2$ and that $|\{x^2 \mid x \in T\}| \leq 2$ for each generating subset $T$ of order 3 of $G$.

We can suppose $G = \langle x_1, x_2, x_3 \rangle$ with $x_1^2 = x_2^3 \neq x_3^2$. The structure of $G$ follows from the following Proposition.

**Proposition 4.5.** Let $G = \langle x_1, x_2, x_3 \rangle$, $d(G) = 3$, $x_1^2 = x_2^3 \neq x_3^2$ and suppose that $|\{x^2 \mid x \in T\}| \leq 2$ for each generating subset $T$ of $G$ of order 3. Then either $G$ is abelian or one of the following holds:

1. $G = \langle x_1, x_2 \rangle \times \langle x_3 \rangle, \langle x_1, x_2 \rangle \simeq Q_8, x_3^2 = 1, x_1 x_3 = x_1^{-1}, x_2 x_3 = x_2^{-1}$;
2. $G = \langle a, b \rangle \times \langle c \rangle, a^4 = b^2 = c^2 = 1, a b = a^{-1}, G \simeq D_4 \times C_2$;
3. $G = \langle a, b \rangle \times \langle c \rangle, a^4 = b^4 = c^2 = 1, a^2 = b^2, a b = a^{-1}, G \simeq Q_8 \times C_2$.

Conversely, if (a) or (b) or (γ) holds, then $|\{x^2 \mid x \in T\}| \leq 2$ for each generating subset $T$ of $G$ of order 3.

**Proof.** Consider the generating subset $\{x_2, x_1x_3, x_3\}$, then either $(x_1x_3)^2 = x_3^2$ or $(x_1x_3)^2 = x_2^2 = x_1^2$.

Arguing similarly on $\{x_1, x_2x_3, x_3\}$, then either $(x_2x_3)^2 = x_2^2$ or $(x_2x_3)^2 = x_1^2 = x_2^2$.

Furthermore, considering the generating subsets $\{x_1, x_3, x_1x_2x_3\}$ and $\{x_1, x_3, x_1x_2\}$ we obtain that either $(x_1x_2x_3)^2 = x_3^2$ or $(x_1x_2x_3)^2 = x_1^2$ and either $(x_1x_2)^2 = x_2^2$ or $(x_1x_2)^2 = x_2^2$.

First we show that $x_1^4 = x_2^4 = 1$.

In fact, from $x_1^4 = x_2^4$ it follows that $x_1^4 \in C_G(x_2), x_2^4 \in C_G(x_1)$. Moreover, considering the generating subset $\{x_1x_3, x_2, x_3\}$, we get either $(x_1x_3)^2 = x_3^2$ and the contradiction $x_1^2 = x_3^2$, or $(x_1x_3)^2 = x_2^2 = x_2^2$, thus $x_1^4 \in C_G(x_2)$ and $x_2^4 \in C_G(x_1)$. If $x_1^4 \neq x_3^2$, then, considering the subset $\{x_1^4, x_2, x_3\}$, we obtain $x_1^{-2} = x_2^2 = x_1^2$ and $x_1^4 = 1 = x_2^4$, as required. If $x_1^4 = x_3^2$, then $x_3^2 \in Z(G)$. Thus the relation $(x_1x_3)^2 = x_2^2$ implies $x_1x_3 = x_3$, and $x_2^2 = (x_3^2)^2 = x_3^{-2}$ implies $x_3^2 = 1$ and then $x_1^4 = 1$, while the relation $(x_1x_3)^2 = x_3^2$ implies $x_1x_3 = x_1^{-1}$ and $x_2^2 = (x_1^2)^2 = x_1^{-2}$ and again $x_1^4 = 1$, as required.

Now our proof splits into four different cases.

Case I) $(x_1x_3)^2 = x_3^2$ and $(x_2x_3)^2 = x_2^2$. Then

$$x_1^{-1}, x_2^{-1}, x_3^{-1}.$$

In this case $(x_1x_2x_3)^2 = x_1x_2x_3x_1x_2x_3 = x_1x_2x_3^{-1}x_2^{-1}x_3 = x_1x_2^{-1}x_2^{-2}x_3 = x_1x_2^{-1}x^2_1x_2x_3^2 = x_1x_2^{-1}x_2x_3^2 = (x_1x_2)^2x_3^2.$

If $(x_1x_2x_3)^2 = x_3^2$, then $(x_1x_2)^2 = 1 = x_1x_2^2$ and $x_1x_2 = x_2x_1$. Furthermore, considering the generating subset $\{x_1, x_3, x_1x_2\}$ we obtain that either $x_1^{-1} = 1 = x_2^2$ and in this case $G$ is abelian, or $x_3^2 = 1$ and in this case $G = \langle x_1x_2 \rangle \times \langle x_1, x_3 \rangle$ and (β) holds.

If $(x_1x_2x_3)^2 = x_2^2$ we have $x_1^2 = (x_1x_2)^2x_3^2$. Now, if $(x_1x_2)^2 = x_1^2$, then $\langle x_1, x_2 \rangle \simeq Q_8$. Furthermore the relation $(x_1x_2)^2x_3^2 = x_1^2$ implies $x_3^2 = 1$, thus $G$ has the structure in (α). If $(x_1x_2)^2 = x_3^2$, then $x_1^2 = x_3$, moreover, considering the generating subset $\{x_1, x_3^{-1}, x_1x_2\}$ we get $x_3^4 = 1$. Therefore $x_1 = 1 = x_3^2$, then $x_1x_3 = x_1, x_3x_2 = x_2$, and $G = \langle x_1x_2, x_1x_3 \rangle \times \langle x_2 \rangle$, with $\langle x_1x_2, x_1x_3 \rangle \simeq Q_8, (x_1x_2)^2 = (x_1x_2)^{-1}, (x_1x_3)^2 = x_1^2x_3^2 = (x_1x_3)^{-1}$ and (α) holds.

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Case II) Now suppose \((x_1x_3)^2 = x_1^2\) and \((x_2x_3)^2 = x_2^2\). Then
\[
x_3^{x_1} = x_3^{-1}, x_3^{x_2} = x_3^{-1}.
\]

First suppose \((x_1x_2x_3)^2 = x_3^2\). Then from \(x_3^2 = (x_1x_2x_3)^2 = (x_1x_2)^2 x_3^2\) we obtain
\[
(x_1x_2)^2 = 1.
\]

Then \((x_1x_2)^2 = x_1^2 x_2^2\) implies \(x_1x_2 = x_2x_1\).

Furthermore, considering the generating subset \(\{x_1, x_3, x_1x_2\}\), we obtain that either \(x_1^2 = 1\) or \(x_3^2 = 1\).

If \(x_1^2 = 1\), then \(x_2^2 = x_1^2 = (x_1x_2)^2 = 1\), and, considering the generation subset \(\{x_3^{-1}, x_1, x_1x_2x_3\}\), we get \(x_3^{-2} = x_3^2 = x_3\), thus \(x_3^4 = 1\). Therefore
\[
G = \langle x_3, x_1 \rangle \times \langle x_1x_2 \rangle
\]
and \((\beta)\) holds.

If \(x_3^2 = 1\), then \(x_3x_1 = x_1x_3\) and \(x_3x_2 = x_2x_3\), therefore
\[
G = \langle x_1, x_2 \rangle \times \langle x_3 \rangle
\]
and \(G\) is abelian.

Now suppose \((x_1x_2x_3)^2 = x_3^2\). Then \(x_1^2 = (x_1x_2)^2 x_3^2\).

If \((x_1x_2)^2 = x_1^2\), then we obtain \(x_3^2 = 1\). Hence
\[
G = \langle x_1, x_2 \rangle \times \langle x_3 \rangle.
\]

Furthermore \(\langle x_1, x_2 \rangle \simeq Q_8\), then \((\gamma)\) holds.

If \((x_1x_2)^2 = x_3^2\), then from \(x_1^2 = (x_1x_2)^2 x_3^2\) we obtain \(x_1^2 = x_3^4\). Moreover, considering the subset \(\{x_3^{-1}, x_1, x_1x_2\}\), we get either \(x_3^{-2} = x_3^2\) and \(x_3^4 = 1\) or \(x_3^{-2} = x_1^2 \in C_G(x_1)\) and again \(x_3^4 = 1\) since \(x_3^{x_1} = x_3^{-1}\). Thus \(x_1^2 = 1\) and \(x_2^2 = x_3^2 = 1\), and from \((x_1x_2)^2 = x_3^2\) it follows that \(x_2^{x_1} = x_2x_3^2\). Therefore
\[
G = \langle x_3, x_1 \rangle \times \langle x_1x_2x_3 \rangle, \text{ with } (x_1x_2x_3)^2 = 1 \text{ and } (\beta) \text{ holds}.
\]

Case III) Now suppose \((x_1x_3)^2 = x_1^2\), \((x_2x_3)^2 = x_2^2\).

Then
\[
x_3^{x_1} = x_3^{-1}, x_3^{x_2} = x_2^{-1}.
\]

In this case, arguing on the subset \(\{x_3^{-1}, x_1, x_2x_3\}\) we get
\[
x_3^4 = 1.
\]

We have \((x_1x_2x_3)^2 = x_1x_2x_3x_1x_2x_3 = x_1x_2x_3x_1^{-1} = (x_1x_2)^2 x_2^2\).

If \((x_1x_2x_3)^2 = x_3^2\), then \((x_1x_2)^2 x_2^2 = x_3^2\). Arguing as before, if \((x_1x_2)^2 = x_3^2\), then \(x_1^2 = x_2^2 = 1\,\langle x_3, x_1, x_2 \rangle \simeq Q_8\), \(x_3^{x_1} = x_3^{-1}\), \((x_1x_2)^x_1 = x_2x_1 = (x_1x_2)^{-1}\) and \((\alpha)\) holds. And the same happens if \((x_1x_2)^2 = x_3^2 = x_2^2\) since in this case \(x_3^2 = 1\) and \(G = \langle x_1, x_2 \rangle \times \langle x_2x_3 \rangle\), where \(\langle x_1, x_2 \rangle \simeq Q_8\), \((x_2x_3)^2 = 1\), \(x_1^{x_2x_3} = x_1^{-1}\), \(x_2^{x_2x_3} = x_2^{-1}\).
Thus for any generating subset $G$ of $S$ let $x_1$ be such that $x_1$ is not abelian and $S$.

Proof. Theorem 4.6. Suppose that $|S^2| = 3|S| - 1$ for any generating subset $S$ of $G$ such that $|S| = n$. Then $G$ is a group of one of the following types:

(i) $G = \langle x_1, x_2 \rangle \langle x_3 \rangle, \quad x_3^2 = x_1 = x_1 x_2 = x_2 x_1, x_1^{-1} x_1 x_3 = x_1 x_3 x_1, x_1 x_3 x_1 x_3 = x_1 x_2 x_1 x_2 = x_2 x_1 x_2 x_1.$

(ii) $G = \langle x_1, x_2 \rangle \langle x_3 \rangle, \quad x_3 = 1 = x_1 x_3, x_1 x_3 x_1 = x_3 x_2, x_3 x_2 = x_2^{-1} x_1 x_2 = x_1^{-1} x_1 x_2 = x_2 x_1 x_2 x_1 = x_2 x_1 x_2 x_1.$

(iii) $G = \langle x_1, x_2 \rangle \langle x_3 \rangle, \quad x_3 = 1 = x_1 x_3, x_1 x_3 x_1 = x_3 x_2, x_3 x_2 = x_2^{-1} x_1 x_2 = x_1^{-1} x_1 x_2 = x_2 x_1 x_2 x_1 = x_2 x_1 x_2 x_1.$

(iv) $G \cong D_4 \times C_2.$

(v) $G \cong Q_8 \times C_2.$

(vi) $G = \langle x_1 \rangle \langle x_2 \rangle \langle x_3 \rangle \langle x_4 \rangle \langle x_5 \rangle \langle x_6 \rangle, \quad x_2 = x_3 = x_4 = x_5 = x_6 = 1.$

(vii) $G = \langle x_1 \rangle \langle x_2 \rangle \langle x_3 \rangle \langle x_4 \rangle \langle x_5 \rangle, \quad x_4 = x_5 = 1.$

(viii) $G$ is abelian and $n \leq 4.$

Conversely, if $G$ satisfies one of (i)-(viii), then $|S^2| = 3|S| - 1$ for any generating subset $S$ of $G$ with $|S| = d(G).$

Proof. First assume $G$ non-abelian then Lemma 3.1 shows that $n \in \{3, 4\}$ and Lemma 3.2 shows that $d(G) = 3.$ Then Lemmas 4.1, 4.2, 4.3, 4.4 and Proposition 4.5 imply that $G$ is a group of one of the types (i) – (v).

Assume now that $G$ is abelian. Choose an arbitrary generating subset $S$ of $G$ such that $|S| = n.$ Let $S = \{g_1, \ldots, g_n\}.$ Clearly $S^2 = A \cup B$ where $A = \{g_j g_m \mid 1 \leq j < m \leq n\}, B = \{g_j^2 \mid 1 \leq j \leq n\}.$ As in a proof of Theorem 2.4 we can show that all elements of the subset $A$ are pairwise different and that $A$ and $B$ are disjoint. It follows that $|S^2| = \frac{1}{2} n(n - 1) + d$ where $d = |B|.$ We note that $d \leq n.$ Thus $\frac{1}{2} n(n - 1) + 1 \leq 3n - 1,$ so that $n \leq 6.$

If $n = 5,$ then $\frac{1}{2} n(n - 1) + d = 10 + d.$ From $|S^2| = 3|S| - 1$ we obtain that $10 + d \leq 14,$ thus $d \leq 4.$ Therefore we can suppose $g_5^2 = g_5^2,$ and $G$ is of type (vii).
If \( n = 6 \), then \( \frac{1}{2} n(n - 1) + d = 15 + d \). From \( |S^2| \leq 3|S| - 1 \) we obtain that \( 15 + d \leq 17 \), which implies that \( d \leq 2 \), so that \( g_2^2 = g_3^2 = g_4^2 = g_5^2 = g_6^2 \), and \( G \) is a group of type \((viii)\).

Conversely assume that \((i)\) holds. Write \( H = \langle x_1, x_2 | x_3^2 \rangle \). Then \( H \) is abelian, moreover \( d^x = d^{-1} \) for every \( d \in H \). Therefore for every \( g \in G \setminus H \) we have \( g = dx_3 \), with \( d \in H \) and \( g^2 = dx_3dx_3 = dd^{-1}x_3^2 = x_3^2 \). Hence if \( s, t, v \in G \), either two of them commute or two of them have the same square, thus \( |\{s, t, v\}| \leq 8 \), as required.

If \((ii)\) holds, consider the central subgroup \( W = \langle x_1^2, x_2^2, x_3^2 \rangle \). Then \( d^2 = 1 \) for any \( d \in W \) and \( G = W \langle x_1, x_2, x_3 \rangle \). We have only to consider three elements \( s, t, v \) in \( G \) which are pairwise non-commuting. But in this case it is not difficult to notice that the set \( \{s^2, t^2, v^2 \} \) has always order 2, since \( x_1^2x_2^2x_3^2 = 1 \).

A similar argument proves the result if \((ii)\) holds, while Proposition 4.5 shows that the result is true if one of \((iii) - (v)\) holds. Now suppose that one of \((vi) - (viii)\) holds, then \( G \) is abelian, and for every subset \( S \) of \( G \) of order \( n = d(G) \) we have \( |S^2| = \frac{1}{2} n(n - 1) + d \) where \( d = |B|, B = \{g^2 \mid g \in S\} \).

Now if \((vi)\) holds, then \( n = 6 \) and \( d = 2 \), and we have \( |S^2| = 15 + 2 = 3|S| - 1 \), as required; if \((vii)\) holds, then \( n = 5 \) and \( d = 4 \), and we have \( |S^2| \leq 10 + 4 = 3|S| - 1 \), as required; finally if \((viii)\) holds, then \( n \in \{3, 4\}, d \leq 4 \) and in any case \( |S^2| \leq 3|S| - 1 \) as required. \( \square \)

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Leonid A. Kurdachenko
Department of Algebra and Geometry, School of Mathematics and Mechanics, University of Dnipro, Gagarin prospect 72, Dnipro 10, 49010 Ukraine
Email: lkurdachenko@i.ua

Patrizia Longobardi
Dipartimento di Matematica, Università di Salerno, via Giovanni Paolo II, 132, 84084 Fisciano (Salerno), Italy
Email: plongobardi@unisa.it

Mercede Maj
Dipartimento di Matematica, Università di Salerno, via Giovanni Paolo II, 132, 84084 Fisciano (Salerno), Italy
Email: mmaj@unisa.it

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