GROUPS WITH NUMERICAL RESTRICTIONS
ON MINIMAL GENERATING SETS

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ABSTRACT. We study an inverse problem of small doubling type. We investigate the structure of a finitely generated group $G$ such that for any set $S$ of generators of $G$ of minimal order we have $S^2 \leq 3|S| - \beta$, where $\beta \in \{1, 2, 3\}$.

1. Introduction

Let $G$ denote an arbitrary group. If $S$ is a subset of $G$, then we write

$$S^2 = \{xy \mid x, y \in S\}.$$ 

If $G$ is an additive group, then we put

$$2S = \{x + y \mid x, y \in S\}.$$ 

A well-known problem in additive number theory is to find the precise structure of $S$ in the case when $S$ is a finite subset of $G$ and

$$|S^2| \leq \alpha|S| + \beta$$

with $\alpha$ (the doubling coefficient) and $|\beta|$ small. Problems of this kind are called inverse problems of small doubling type. In the additive group of integers, these problems were detailed investigated by

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G. A. Freeman in [6], [7], [8] and [9]. It is very easy to prove that if $S$ is a finite subset of integers, $|S| = k$, then $|2S| \geq 2|S| - 1$, and $|2S| = 2|S| - 1$ if and only if there exist integers $a, q$ such that $S = \{a, a + q, a + 2q, \ldots, a + (k - 1)q\}$, i.e. $S$ is an arithmetic progression of length $k$. In his famous theorem, Freiman proved that if $S$ is a finite set of integers with $k \geq 3$ elements and $|2S| \leq 3k - 4$, then there exist integers $a, q$ such that $q > 0$ and $S \subseteq \{a, a + q, a + 2q, \ldots, a + (2k - 4)q\}$. He obtained similar results if $|2S| \leq 3|S| - 3$, or $|2S| \leq 3|S| - 2$. For other authors results of this type please see [27], [5], [22], [32], [33] and [34].

In arbitrary abelian groups, inverse problems have been studied by many other authors (see, for example, [1], [20], [18], [27], [30] and [19]). This study was initiated by M. Kneser [26].

More recently, small doubling problems in non-abelian groups have also been studied, see for example [2], [36], [4]. We also refer to recent surveys [19], [31], [3] and books [28] and [35].

In a series of papers with G. A. Freiman, M. Herzog, Y. V. Stanchescu, A. Plagne and D. J. S. Robinson (see [10, 11, 12, 13, 14, 15, 16] and [24]) the last two authors of the current paper studied small doubling problems in an orderable group.

J. H. B Kemperman showed that if $S$ is a finite subset of any torsion-free group, then $|S^2| \geq 2|S| - 1$ (see [25]), while G. A. Freiman and B. M. Schein proved that if $|S| = k$, then $|S^2| = 2|S| - 1$ if and only if $S = \{a, aq, \ldots, aq^{k-1}\}$, i.e. $S$ is a geometric progression and either $aq = qa$ or $aqa^{-1} = q^{-1}$ (see [17]). Therefore it is quite natural to ask what is the structure of $S$ in the case when $S$ is a finite subset of a group $G$, $|S| = k \geq 3$ and $|S^2| \leq 3|S| - \beta$, where $\beta = 1, 2, 3, 4$. It could be also interesting to know the structure of $\langle S \rangle$ if $S$ is a finite subset of a group and $|S^2| \leq 3|S| - \beta$ where $\beta = 1, 2, 3, 4$. There is an old conjecture by G. Freiman stating that if $S$ is a finite subset of a torsion-free group, $1 \in S$ and $|S^2| \leq 3k - 4$, then $\langle S \rangle$ is abelian (see [21, p. 250]).

If $G$ is a finitely generated group, we denote by $d(G)$ the minimal order of a finite set of generators of $G$. In the current paper, we investigate the structure of a finitely generated group $G$ in which $|S^2| \leq 3|S| - \beta$, $\beta = 1, 2, 3$ for any generating subset $S$ of $G$ of minimal order.

The present paper is organized as follows.

In section 2, we studied the case $|S^2| \leq 3k - 3$ and we proved the following Theorem.

**Theorem 2.4.** Let $G$ be a finitely generated group with $d(G) = n$. Suppose that $|S^2| \leq 3|S| - 3$, for any generating subset $S$ of $G$ such that $|S| = n$. Then $G$ is a group of one of the following types:

(i) $G$ is the quaternion group of order 8,
(ii) $G = \langle x_1 \rangle \times \langle x_2 \rangle \times \langle x_3 \rangle \times \langle x_4 \rangle \times \langle x_5 \rangle$, where $x_2^2 = x_3^2 = x_4^2 = x_5^2 = 1$,
(iii) $G = \langle x_1 \rangle \times \langle x_2 \rangle \times \langle x_3 \rangle \times \langle x_4 \rangle$, where $x_3^2 = x_4^2 = 1$,
(iv) $G$ is abelian and $n \leq 3$.

Conversely, if $G$ satisfies one of (i) – (iii), or (iv) with $n > 1$, then $|S^2| \leq 3|S| - 3$, for any generating subset $S$ of $G$ with $|S| = d(G)$.

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In section 3, we studied the case $|S^2| \leq 3|S|-2$. Obviously, if $|S| = 2$, then $|S^2| \leq 3|S|-2$, therefore we assume $d(G) \geq 3$. We proved the following Theorem.

**Theorem 3.5.** Let $G$ be a finitely generated group with $d(G) = n \geq 3$. Suppose that $|S^2| \leq 3|S|-2$, for any generating subset $S$ of $G$ such that $|S| = n$. Then $G$ is a group of one of the following types:

(i) $G = \langle x_1, x_2, x_3 \mid x_1^2 = x_2^2 = x_3^2 = 1, x_1x_2^{-1}x_3x_1^{-1} = x_2^3, i \neq j, 1 \leq i, j \leq 3 \rangle \simeq \mathbb{Z}_2 \times \langle d \rangle, |d| = 2$,

(ii) $G = \langle x_1 \times x_2 \times x_3 \times x_4 \times x_5 \times x_6 \rangle$, where $x_1^2 = x_2^2 = x_3^2 = x_4^2 = x_5^2 = x_6^2 = 1$,

(iii) $G = \langle x_1 \times x_2 \times x_3 \times x_4 \times x_5 \rangle$, where $x_3^2 = x_4^2 = x_5^2 = 1$,

(iv) $G$ is abelian and $n \leq 4$.

Conversely, if $G$ satisfies one of (i) – (iv), then $|S^2| \leq 3|S|-2$, for any generating subset $S$ of $G$ with $|S| = d(G)$.

Finally, in section 4, we studied the case $|S^2| \leq 3|S|-1$. We proved the following result.

**Theorem 4.6.** Let $G$ be a finitely generated group with $d(G) = n \geq 3$. Suppose that $|S^2| \leq 3|S|-1$, for any generating subset $S$ of $G$ such that $|S| = n$. Then $G$ is a group of one of the following types:

(i) $G = \langle x_1, x_2 \rangle \langle x_3 \rangle$, $x_1^2 = 1, x_1x_2 = x_2x_1, x_3^{-1}x_1x_3 = x_1^{-1}, x_3^{-1}x_2x_3 = x_2^{-1}$,

(ii) $G = \langle x_1, x_2 \rangle \langle x_3 \rangle$, $x_1^2 = x_2^2 = 1, x_1x_3 = x_3x_1, x_2x_3 = x_3x_2, x_2^{-1}x_1x_2 = x_1^{-1}, x_1^2x_2x_3 = 1$,

(iii) $G = \langle x_1, x_2 \rangle \langle x_3 \rangle$, $x_1^2 = x_2^2 = 1, x_1x_2^{-1}x_1x_2^{-1} = x_1^{-1}, x_1^2x_2x_3 = x_1^{-1}, x_3^{-1}x_2x_3 = x_2^{-1}$,

(iv) $G \simeq D_4 \times C_2$,

(v) $G \simeq Q_8 \times C_2$,

(vi) $G = \langle x_1 \times x_2 \times x_3 \times x_4 \times x_5 \times x_6 \rangle$, where $x_2^2 = x_3^2 = x_4^2 = x_5^2 = x_6^2 = 1$,

(vii) $G = \langle x_1 \times x_2 \times x_3 \times x_4 \times x_5 \rangle$, where $x_4^2 = x_5^2 = 1$,

(viii) $G$ is abelian and $n \leq 4$.

Conversely, if $G$ satisfies one of (i) – (viii), then $|S^2| \leq 3|S|-1$, for any generating subset $S$ of $G$ with $|S| = d(G)$.

We refer to [29] for notation, in particular we will denote $C_n$ the cyclic group of order $n$, $Q_8$ the quaternion group of order 8, and $D_n$ the dihedral group of order $2n$.

2. Minimal generating subsets $S$ with $|S^2| \leq 3|S|-3$

We start this section with the following useful Proposition.

**Proposition 2.1.** Let $G$ be a finitely generated group with $d(G) = n \geq 3$.

If $G$ is not abelian, then $G$ includes a subset $Y$ such that $G = \langle Y \rangle$, $|Y| = n$, and $Y^2$ contains at least $\frac{1}{2}(n^2 + n)$ elements.

**Proof.** Let $S$ be a finite subset of $G$ such that $G = \langle S \rangle$ and $|S| = n$. Write $S = \{x_1, x_2, \ldots, x_n\}$. Since $G$ is not abelian, there exist $i, j \in \{1, \ldots, n\}$, $i \neq j$ such that $x_ix_j \neq x_jx_i$. Without loss of generality
we can suppose \( x_1 x_2 \neq x_2 x_1 \). Suppose now that \( x_1 x_j = x_j x_1 \) for some \( j > 2 \), and let \( j \) be minimum with this property. Then \( x_1 (x_2 x_j) = (x_1 x_2) x_j \neq (x_2 x_1) x_j = x_2 (x_1 x_j) = x_2 (x j x_1) = (x_2 x_2) x_1 \). Put \( S_1 = \{ x_1, \ldots, x_{j-1}, x_2 x_j, x_{j+1}, \ldots, x_n \} \), then \( G = \langle S_1 \rangle \) and \( |S_1| = n \). Using the above arguments, after finitely many steps we obtain a subset \( Y = \{ y_1, y_2, \ldots, y_n \} \) of order \( n \) such that \( y_1 y_j \neq y_j y_1 \) for all \( j > 1 \), and such that \( G = \langle Y \rangle \). Suppose that \( y_j y_k = y_m y_l \) where \( j \neq k, m \neq t, j \neq m \). Then \( k \neq t \).

If \( j \neq t \) then \( y_j \in Y \setminus \{ y_j \} \). If \( j = t \) then \( y_k = y_j^{-1} y_m y_j \), so that \( y_k \in Y \setminus \{ y_k \} \). In both cases we obtain \( d(G) \leq n - 1 \). This contradiction shows that \( y_j y_k \neq y_m y_l \) whenever \( \{ j, k, m, t \} \geq 3 \). Consider now the elements:

\[
y_1 y_2, \ldots, y_1 y_n, y_2 y_3, \ldots, y_2 y_n, \ldots, y_{n-1} y_n.
\]

By the previous arguments these elements are pairwise different. Also the elements

\[
y_2 y_1, \ldots, y_n y_1
\]

are pairwise different. By the previous arguments \( y_j y_1 \neq y_s y_k \) for \( j \geq 2, 1 \leq s < k \leq n \). Finally the element \( y_1^2 \) is different from the previous ones. It follows that the subset \( Y^2 \) has at least

\[
(n - 1) + (n - 2) + \cdots + 2 + 1 + (n - 1) + 1 = \frac{1}{2} (n^2 - n) + n = \frac{1}{2} (n^2 + n)
\]
elements.

\[\square\]

**Corollary 2.2.** Let \( G \) be a finitely generated group with \( d(G) = n \geq 3 \).

Suppose that \( |S^2| \leq 3|S| - 3 \) for each generating subset \( S \) of \( G \) such that \( |S| = n \).

Then \( G \) is abelian.

**Proof.** Suppose \( G \) non-abelian. By Proposition 2.1, \( G \) has a finite subset \( Y \) such that \( G = \langle Y \rangle \), \( |Y| = n \) and \( |Y^2| \geq \frac{1}{2} (n^2 + n) \). Since \( |Y^2| \leq 3|Y| - 3 \), we obtain \( \frac{1}{2} (n^2 + n) \leq 3n - 3 \). It follows that \( n \leq 3 \). Thus \( Y = \{ y_1, y_2, y_3 \} \). From the proof of Proposition 2.1 we can see that \( Y^2 \) contains the elements: \( y_1^2, y_1 y_2, y_1 y_3, y_2 y_1, y_3 y_1, y_2 y_3 \), that are pairwise different. Since \( |Y^2| = 6 \), \( Y^2 = \{ y_1^2, y_1 y_2, y_1 y_3, y_2 y_1, y_3 y_1, y_2 y_3 \} \).

On the other hand \( y_3 y_2 \in Y^2 \) and by the previous remarks we have only the possibility \( y_2 y_3 = y_3 y_2 \). Put \( M = \{ y_1, y_1 y_2, y_3 \} \). Now it is easy to check that the set \( \{ y_1^2, y_1 y_3, y_3 y_1, y_1 y_1 y_2, y_1 y_2 y_1, y_1 y_2 y_3, y_3 y_1 y_2 \} \) has order 7: in particular \( y_1 y_2 y_3 \neq y_3 (y_1 y_2) \) since \( y_2 y_3 = y_3 y_2 \). It follows that \( M^2 \) contains at least 7 elements and we obtain the final contradiction.

\[\square\]

**Corollary 2.3.** Let \( G \) be a finitely generated group with \( d(G) = 2 \).

Suppose that \( |S^2| \leq 3|S| - 3 \) for each generating subset \( S \) of \( G \) such that \( |S| = 2 \).

Then \( G \) is abelian or \( G \) is the quaternion group of order 8.

Conversely if either \( G \) is abelian or \( G \simeq Q_8 \), then \( |S^2| \leq 3|S| - 3 \) for each generating set \( S \) of \( G \) such that \( |S| = 2 \).

**Proof.** Suppose that \( G \) is non-abelian. Then \( G = \langle x, y \rangle \) where \( xy \neq yx \). By our condition \( |S^2| \leq 3|S| - 3 = 3 \). We have \( S^2 = \{ x^2, y^2, xy, yx \} \). Then \( x^2 = y^2 \). Suppose that \( x^2 = y^2 = c \neq 1 \). Then
c ∈ ζ(G). We have also \( G = \langle x, xy \rangle \). Then \( x(xy) \neq (xy)x \), hence \( x^2 = (xy)^2 \) and also \( y^2 = (xy)^2 \).

Furthermore, from \( G = \langle x, cy \rangle \) and \( c \in ζ(G) \) we get \( y^2 = x^2 = (cy)^2 = c^2y^2 \) from which \( c^2 = 1 \). Using \( x^2 = (xy)^2 \), we obtain that \( x = yxy \). It follows that \( xy^{-1} = yx \) and hence \( y^{-1} = x^{-1}yx \). Thus the subgroup \( \langle y \rangle \) is normal in \( G \). By the same reason also \( \langle x \rangle \) is normal in \( G \). Finally \( x^4 = (x^2)^2 = c^2 = 1 \) and similarly \( y^4 = 1 \). It follows that \( G \) is the quaternion group of order 8. Suppose now \( x^2 = y^2 = 1 \). Since \( G = \langle x, xy \rangle \), we obtain \( (xy)^2 = x^2 = 1 \). It follows that \( y^{-1}xy = yxy = x^{-1}x \), so \( xy = yx \) and we obtain a contradiction.

Conversely, if \( G \) is abelian and \( S = \{x, y\} \), then \( S^2 = \{x^2, xy, y^2\} \), and \( |S^2| ≤ 3 = 3|S| - 3 \). If \( G \cong Q_8 \), and \( S = \{x, y\}, G = \langle S \rangle \), then \( x^2 = y^2 \) and \( S^2 = \{xy, yx, x^2\} \), as required.

Now we can prove Theorem 2.4.

**Theorem 2.4.** Let \( G \) be a finitely generated group with \( d(G) = n \). Suppose that \( |S^2| ≤ 3|S| - 3 \) for any generating subset \( S \) of \( G \) such that \( |S| = n \). Then \( G \) is a group of one of the following types:

(i) \( G \) is the quaternion group of order 8,

(ii) \( G = \langle x_1 \rangle \times \langle x_2 \rangle \times \langle x_3 \rangle \times \langle x_4 \rangle \), where \( x_2^2 = x_3^2 = x_4^2 = 1 \),

(iii) \( G = \langle x_1 \rangle \times \langle x_2 \rangle \times \langle x_3 \rangle \times \langle x_4 \rangle \), where \( x_2^2 = x_3^2 = 1 \),

(iv) \( G \) is abelian and \( n ≤ 3 \).

Conversely, if \( G \) satisfies one of (i) – (iii), or (iv) with \( n > 1 \), then \( |S^2| ≤ 3|S| - 3 \), for any generating subset \( S \) of \( G \) with \( |S| = d(G) \).

**Proof.** If \( n = 2 \), then Corollary 2.3 shows that either \( G \) is the quaternion group of order 8 or \( G \) is abelian. If \( n = 3 \), then Corollary 2.2 shows that \( G \) is abelian. Hence either (i) or (iv) holds.

Now suppose \( n ≥ 4 \). Then \( G \) is abelian by Corollary 2.2 and \( G \) is a finitely generated abelian group with \( d(G) = n \). Choose an arbitrary generating subset \( S \) of \( G \) such that \( |S| = n \). Let \( S = \{g_1, \ldots, g_n\} \).

Clearly \( S^2 = A \cup B \) where \( A = \{g_jg_m \mid 1 ≤ j < m ≤ n\} \), \( B = \{g_j^2 \mid 1 ≤ j ≤ n\} \). Suppose that \( g_jg_m = g_sg_k \) where \( (j, m) \neq (s, k), j < m, s < k \). If \( \{j, m\} \cap \{s, k\} = \emptyset \), then \( g_m = g_j^{-1}g_s g_k \) and we obtain a contradiction with the minimality of \( S \). Suppose that \( s ∈ \{j, m\} \). If \( s = j \), then \( g_m = g_k \), which is impossible. If \( s = m \), then \( g_j = g_s g_k g_s^{-1} \), and we obtain a contradiction with the minimality of \( S \). Using similar arguments we obtain a contradiction if \( k ∈ \{j, m\} \). It follows that the elements of the subset \( A \) are pairwise different, and arguing analogously, that \( A \) and \( B \) are disjoint. Then \( |S^2| = \frac{1}{2}n(n - 1) + d \) where \( d = |B| \). We note that \( d ≤ n \). Thus \( \frac{1}{2}n(n - 1) + 1 ≤ 3n - 3 \), so that \( n ≤ 5 \). If \( n = 4 \), then \( \frac{1}{2}n(n - 1) + d = 6 + d ≤ 3n - 3 = 9 \). It follows that \( d ≤ 3 \), so that we can suppose \( g^2_3 = g^2_4 \). Since \( G \) is a finitely generated abelian group with \( d(G) = 4 \), we have \( G = \langle x_1 \rangle \times \langle x_2 \rangle \times \langle x_3 \rangle \times \langle x_4 \rangle \), for some elements \( x_1, x_2, x_3, x_4 \). Clearly \( \{x_1, x_2, x_3, x_4\} \) is a minimal generating subset for \( G \). Then, arguing as before, we can suppose \( x^2_3 = x^2_4 \). That is possible only in the case when \( x^2_3 = x^2_1 = 1 \). If \( n = 5 \), then \( \frac{1}{2}n(n - 1) + d = 10 + d ≤ 3n - 3 = 12 \). It follows that \( d ≤ 2 \), so that we can suppose \( g^2_3 = g^2_5 = g^2_4 \). Since \( G \) is a finitely generated abelian group with
\(d(G) = 5\), we have \(G = \langle x_1 \rangle \times \langle x_2 \rangle \times \langle x_3 \rangle \times \langle x_4 \rangle \times \langle x_5 \rangle\), for some elements \(x_1, x_2, x_3, x_4, x_5\). We can suppose \(x_2^2 = x_3^2 = x_4^2 = x_5^2\) and, as in the previous case, we obtain that \(x_2^2 = x_3^2 = x_4^2 = x_5^2 = 1\).

Conversely, suppose that \(G\) satisfies (i), or (ii), or (iii), or (iv) with \(n > 1\). It is easy to prove that \(|S|^2 \leq 3|S| - 3\) if \(|S| = n = d(G)\) and \(G = \langle S \rangle\). In fact, if \(G\) is abelian and \(n = 3\), then \(|S|^2 \leq \frac{1}{2}n(n-1) + 3 = 6 = 3|S| - 3\). (If (iii) holds with \(n = 4\), then \(|B| \leq 3\), where \(B = \{g^2 \mid g \in S\}\), and \(|S|^2 \leq \frac{1}{2}n(n-1) + 3 = 8 = 3|S| - 3\). If (ii) holds with \(n = 5\), then \(|B| \leq 2\), and \(|S|^2 \leq \frac{1}{2}n(n-1) + 2 = 12 = 3|S| - 3\), and finally if (i) holds, then the result follows from Corollary 2.3. □

From Theorem 2.4 it follows the following easy Corollary.

**Corollary 2.5.** Let \(G\) be a finitely generated group with \(d(G) = n\).

Suppose that \(|S|^2 \leq 3|S| - 3\) for each generating subset \(S\) of \(G\) such that \(|S| = n\).

If \(G\) is torsion-free, then \(G\) is abelian and \(n \leq 3\).

### 3. Minimal generating subsets \(S\) with \(|S|^2 \leq 3|S| - 2\)

In this section we will consider finitely generated groups \(G\) such that \(|S|^2 \leq 3|S| - 2\) for each generating subset \(S\) of \(G\) with the property \(|S| = d(G)\).

Suppose that \(G\) is non-abelian and \(d(G) = 2\). If \(S = \{g_1, g_2\}\) is a generating subset of \(G\), then clearly \(g_1g_2 \neq g_2g_1\), so that \(3 \leq |S|^2 \leq 4 = 3|S| - 2\).

Therefore in the sequel we will assume that \(d(G) \geq 3\).

We start with an easy more general Lemma.

**Lemma 3.1.** Let \(G\) be a finitely generated group with \(d(G) = n \geq 3\). Suppose that \(|S|^2 \leq 3|S| - 1\) for each generating subset \(S\) of \(G\) such that \(|S| = n\). If \(G\) is non-abelian, then \(n \in \{3, 4\}\).

**Proof.** Since \(G\) is non-abelian, Proposition 2.1 shows that \(G\) has a finite subset \(Y\) such that \(G = \langle Y \rangle\), \(|Y| = n\) and \(|Y|^2 \geq \frac{1}{2}(n^2 + n)\). Thus we have \(\frac{1}{2}(n^2 + n) \leq 3n - 1\). It follows that \(n^2 - 5n + 2 \leq 0\). This is possible only if \(n \in \{3, 4\}\). □

Now we show that in the case \(n = d(G) = 4\) and \(|S|^2 \leq 3|S| - 1\) for each generating subset \(S\) of \(G\) such that \(|S| = n\), again we obtain that \(G\) is abelian.

**Lemma 3.2.** Let \(G\) be a finitely generated group with \(d(G) = 4\).

Suppose that \(|S|^2 \leq 11\) for each generating subset \(S\) of \(G\) such that \(|S| = 4\).

Then \(G\) is abelian.

**Proof.** Suppose the contrary, so let \(G\) be non-abelian. Then, arguing as in the proof of Proposition 2.1, we obtain that \(G\) contains a subset \(S = \{x_1, x_2, x_3, x_4\}\) such that \(G = \langle S \rangle\) and the set

\[L = \{x_1^2, x_1x_2, x_1x_3, x_1x_4, x_2x_3, x_2x_4, x_3x_4, x_4x_1, x_3x_1, x_2x_1\}\]

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has order 10. Now consider the set $T = \{y_1, y_2, y_3, y_4\}$, where $y_2 = x_1x_2$, $y_j = x_j$ whenever $j \neq 2$. It is easy to see that the set

$$V = \{y_1^2, y_1y_2, y_1y_3, y_1y_4, y_2y_3, y_2y_4, y_3y_4, y_4y_1, y_3y_1, y_2y_1\}$$

has order 10. If $x_2x_3 = x_3x_2$, then $y_2y_3 \neq y_3y_2$ and $T^2 = V \cup \{y_3y_2\}$. Therefore $x_2x_4 \neq x_4x_2$ otherwise $y_2y_4 \neq y_4y_2$ and $|T^2| > 11$. Arguing analogously with $x_3$ we get that it is not possible to have simultaneously $x_3x_2 = x_2x_3$ and $x_3x_4 = x_4x_3$ and, arguing with $x_4$, to have simultaneously $x_4x_2 = x_2x_4$ and $x_4x_3 = x_3x_4$. From this it follows that $x_\alpha x_\beta \neq x_\beta x_\alpha, x_\alpha x_\gamma \neq x_\gamma x_\alpha$ for suitable $\alpha, \beta, \gamma$ such that $\{2, 3, 4\} = \{\alpha, \beta, \gamma\}$. But this implies $|S^2| > 11$, the final contradiction. \hfill \Box

Now we study the case $n = d(G) = 3$ and $|S^2| \leq 3|S| - 2$ for each generating subset $S$ of $G$ such that $|S| = n$. Our first remark is the following Lemma.

**Lemma 3.3.** Let $G$ be a finitely generated group with $d(G) = 3$.

Suppose that $|X^2| \leq 7$ for each generating subset $X$ of $G$ such that $|X| = 3$. If $G$ is non-abelian and $S = \{x_1, x_2, x_3\}$ is a subset of $G$ such that $G = \langle S \rangle$ and $|\{x_1^2, x_1x_2, x_1x_3, x_2x_3, x_2x_1, x_3x_1\}| = 6$, then

$$x_1^2 = x_2^2 = x_3^2.$$

**Proof.** Write $L = \{x_1^2, x_1x_2, x_1x_3, x_2x_3, x_2x_1, x_3x_1\}$ and assume by contradiction that $x_2^2 \neq x_1^2$. Then $S^2 = L \cup \{x_2^2\}$ has order 7. This implies, arguing as usual, that $x_2x_3 = x_3x_2$.

Consider the set $T = \{y_1, y_2, y_3\}$ where $y_1 = x_1, y_2 = x_2, y_3 = x_1x_3$. Obviously $G = \langle T \rangle$ and it is easy to see that $V = \{y_1^2, y_2^2, y_1y_2, y_1y_3, y_2y_3, y_3y_1, y_2y_2\}$ has order 7. But $x_2x_3 = x_3x_2$ implies $y_2y_3 \neq y_3y_2$, then the subset $V \cup \{y_3y_2\}$ of $T^2$ has order 8, a contradiction. A similar argument holds if $x_3^2 \neq x_1^2$, so $x_1^2 = x_2^2 = x_3^2$, as required. \hfill \Box

Now we can prove the following Proposition that gives a description of $G$ if $d(G) = 3$ and $|S^2| \leq 7$.

**Proposition 3.4.** Let $G$ be a finitely generated group with $d(G) = 3$. Suppose that $|S^2| \leq 7$ for each generating subset $S$ of $G$ such that $|S| = 3$. If $G$ is non-abelian, then

$$G = \langle x_1, x_2, x_3 \mid x_1^2 = x_2^2 = x_3^2 = c \in \xi(G), c^2 = 1, x_jx_kx_j^{-1} = x_k^2, 1 \leq j, k \leq 3, j \neq k \rangle \simeq Q_8 \times \langle d \rangle, |d| = 2.$$

Conversely, if $G = Q_8 \times \langle d \rangle$ with $|d| = 2$, then $d(G) = 3$ and $|S^2| \leq 7$ for every generating subset $S$ of $G$ with $|S| = 3$.

**Proof.** Arguing as in the proof of Proposition 2.1, we find a subset $S = \{x_1, x_2, x_3\}$ of $G$ such that $G = \langle S \rangle$ and $|\{x_1^2, x_1x_2, x_1x_3, x_2x_3, x_2x_1, x_3x_1\}| = 6$. Then by Lemma 3.3 we have

$$x_1^2 = x_2^2 = x_3^2.$$

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Put $y_1 = x_1, y_2 = x_1x_2, y_3 = x_3, S_1 = \{y_1, y_2, y_3\}$. Then $G = \langle S_1 \rangle$ and it is easy to prove that $\{y_1^2, y_1y_2, y_1y_3, y_2y_3, y_3y_1\} = 6$. Therefore, again by Lemma 3.3, we have

$$y_1^2 = y_2^2 = y_3^2.$$ 

From $y_1^2 = y_2^2$ it follows $x_1^2 = x_1x_2x_1x_2$, so $x_1 = x_2x_1x_2$ and $x_1^{-1}x_2x_1 = x_2^{-1}$. It follows that $[x_2, x_1] = x_2^{-2}$. On the other hand $x_1^2 = x_2^2$, so we have $x_2x_2 = x_1x_2x_1x_2$, thus $x_2 = x_1x_2x_1$ and $x_2^{-1}x_1x_2 = x_1^{-1}$. It follows that $[x_1, x_2] = x_2^{-2}$. We have now $x_2^{-2} = [x_2, x_1] = [x_1, x_2]^{-1} = x_1^2 = x_2^2$, hence $x_4^2 = 1$. Then also $x_4^4 = (x_1^2)^2 = (x_2^2)^2 = 1$ and $x_4^3 = 1$. The equality $x_1^2 = x_2^2 = x_3^2 = c$ shows that $c \in Z(G)$ and $|c| = 2$. Arguing as before we obtain that $[x_1, x_3] = x_1^{-2} = x_3^2$ and $[x_2, x_3] = x_2^{-2} = x_2^3 = x_3^2$. Obviously $\langle x_1, x_2 \rangle \cong Q_8$. From $x_1x_2x_3x_1x_2x_3 = (x_1x_2)^2x_2^3 = 1$ we obtain the result.

Conversely, let $G = Q_8 \times \langle d \rangle$, $|d| = 2$. Then $x^2 = y^2$ for every $x, y \in G$ with $x^2 \neq 1$ and $y^2 \neq 1$. Moreover $s \in Z(G)$ for every $s$ in $G$ of order 2. Now let $S = \{x_1, x_2, x_3\}$. If there exists $i \in \{1, 2, 3\}$, say $i = 1$ such that $x_1^2 = 1$ then $x_1 \in Z(G)$,

$$S^2 = \{x_1^2, x_1x_2, x_1x_3, x_2x_3, x_2^2, x_3x_2, x_3^2\},$$

and $|S^2| \leq 7$. If $x_1^2 \neq 1$ for every $i \in \{1, 2, 3\}$, then $x_1^2 = x_2^2 = x_3^2$,

$$S^2 = \{x_1^2, x_1x_2, x_1x_3, x_2x_3, x_2x_1, x_3x_1, x_3x_2\}$$

and again $|S^2| \leq 7$. The result is proved.

Now we can prove Theorem 3.5.

**Theorem 3.5.** Let $G$ be a finitely generated group with $d(G) = n \geq 3$. Suppose that $|S^2| \leq 3|S| - 2$ for any generating subset $S$ of $G$ such that $|S| = n$. Then $G$ is a group of one of the following types:

(i) $G = \langle x_1, x_2, x_3 \mid x_1^2 = x_2^2 = x_3^2 = c \in Z(G), c^2 = 1, x_jx_i^{-1} = x_i^3, i \neq j, 1 \leq i, j \leq 3 \rangle \cong Q_8 \times \langle d \rangle, |d| = 2$,

(ii) $G = \langle x_1 \rangle \times \langle x_2 \rangle \times \langle x_3 \rangle \times \langle x_4 \rangle \times \langle x_5 \rangle \times \langle x_6 \rangle$, where $x_1^2 = x_2^2 = x_3^2 = x_4^2 = x_5^2 = x_6^2 = 1$,

(iii) $G = \langle x_1 \rangle \times \langle x_2 \rangle \times \langle x_3 \rangle \times \langle x_4 \rangle \times \langle x_5 \rangle$, where $x_1^2 = x_4^2 = x_5^2 = 1$,

(iv) $G$ is abelian and $n \leq 4$.

Conversely, if $G$ satisfies one of (i) – (iv), then $|S^2| \leq 3|S| - 2$ for any generating subset $S$ of $G$ with $|S| = d(G)$.

**Proof.** Suppose first that $G$ is non-abelian. Then Lemma 3.1 shows that $n \in \{3, 4\}$. More precisely Lemma 3.2 shows that $d(G) = 3$. Then Proposition 3.4 implies that $G$ is a group of type (i).

Assume now that $G$ is abelian. Choose an arbitrary generating subset $S$ of $G$ such that $|S| = n$. Let $S = \{g_1, \ldots, g_n\}$. Clearly $S^2 = A \cup B$ where $A = \{g_jg_m \mid 1 \leq j < m \leq n\}$, $B = \{g_j^2 \mid 1 \leq j \leq n\}$. As in a proof of Theorem 2.4 we can show that all elements of the subset $A$ are pairwise different and

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that $A$ and $B$ are disjoint. It follows that $|S^2| = \frac{1}{2}n(n-1)+d$ where $d = |B|$. We note that $d \leq n$. Thus $\frac{1}{2}n(n-1)+1 \leq 3n-2$, so that $n \leq 6$.

If $n = 5$, then $\frac{1}{2}n(n-1) + d = 10 + d$. From $|S^2| \leq 3|S| - 2$ we obtain that $10 + d \leq 13$, thus $d \leq 3$. Therefore we can suppose $g_3^2 = g_4^2 = g_5^2$, and $G$ is of type (iii).

If $n = 6$, then $\frac{1}{2}n(n-1) + d = 15 + d$. From $|S^2| \leq 3|S| - 2$ we obtain that $15 + d \leq 16$, which implies that $d \leq 1$, so that $g_1^2 = g_2^2 = g_3^2 = g_4^2 = g_5^2 = g_6^2$ and $G$ is a group of type (ii).

Conversely, suppose that $G$ satisfies (i), then $d(G) = 3$ and by Proposition 3.4 $|S^2| \leq 7$ for every generating subset of $G$ of order 3. Now suppose that (ii), or (iii), or (iv) holds, then $G$ is abelian and for every subset $S$ of $G$ of order $n = d(G)$ we have $|S^2| = \frac{1}{2}n(n-1) + d$ where $d = |B|$, $B = \{g_i^2 \mid g_i \in S\}$.

Now if (ii) holds, then $n = 6$ and $d = 1$, and we have $|S^2| = 15 + 1 = 3|S| - 2$, as required; if (iii) holds, then $n = 5$ and $d = 3$, and we have $|S^2| \leq 10 + 3 = 3|S| - 2$, as required, and finally if (iv) holds, then $n \in \{3, 4\}$, $d \leq n$ and in any case $|S^2| \leq 3|S| - 2$ as required. \hfill \Box

From Theorem 3.5 it follows the following easy Corollary.

**Corollary 3.6.** Let $G$ be a finitely generated group with $d(G) = n$.

Suppose that $S^2 \leq 3|S| - 2$ for each generating subset $S$ of $G$ such that $|S| = n$.

If $G$ is torsion-free, then $G$ is abelian and $n \leq 4$.

4. Minimal generating subsets $S$ with $|S^2| \leq 3|S| - 1$

In this section we consider finitely generated groups $G$ such that $|S^2| \leq 3|S| - 1$ for each generating subset $S$ of $G$ with the property $|S| = d(G)$.

First we assume $G$ non-abelian. Hence by Lemmas 3.1 and 3.2 we have $d(G) = 3$.

Let $S = \{x_1, x_2, x_3\}$, with $G = \langle S \rangle$ non-abelian. From $|S^2| \leq 3|S| - 1 = 8$, it follows that either two of the elements $x_1, x_2, x_3$ have the same square or two of them commute. We study first the situation $|\{x_1^2, x_2^2, x_3^2\}| = 3$. In this case we can suppose, without loss of generality, $x_2x_3 = x_3x_2$. We start with the following Lemma.

**Lemma 4.1.** Let $G = \langle x_1, x_2, x_3 \rangle$, with $d(G) = 3$, $x_2x_3 = x_3x_2$, $|\{x_1^2, x_2^2, x_3^2\}| = 3$. If $|T|^2 \leq 8$ for each generating subset $T$ of $G$ of order 3, then $x_2 \in Z(G)$ or $x_3 \in Z(G)$ or $x_2x_3 \in Z(G)$ or the following holds

$(i)$ $G = \langle x_2, x_3 \rangle \langle x_1 \rangle$, $x_4^4 = 1$, $x_2x_3 = x_3x_2$, $x_2^{-1}x_2x_1 = x_2^{-1}$, $x_3^{-1}x_3x_1 = x_3^{-1}$.

**Proof.** Suppose that $x_2 \notin Z(G)$, $x_3 \notin Z(G)$, $x_2x_3 \notin Z(G)$. Thus $x_1x_2 \neq x_2x_1$. Consider the subset $T = \{x_1, x_2, x_1x_2x_3\}$. Obviously $G = \langle T \rangle$, thus $|T|^2 \leq 8$. By the hypothesis $x_1x_2 \neq x_2x_1$ and $x_2$ does not commute with $x_1x_2x_3$. If $x_1$ commutes with $x_1x_2x_3$, then $x_2x_3 \in Z(G)$, which is not the case. Hence from $|T|^2 \leq 8$ we get that either $(x_1x_2x_3)^2 = x_1^2$ or $(x_1x_2x_3)^2 = x_2^2$. Arguing similarly on the subset $V = \{x_1, x_3, x_1x_2x_3\}$ we obtain that either $(x_1x_2x_3)^2 = x_1^2$ or $(x_1x_2x_3)^2 = x_3^2$. Then

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Lemma 4.2. Let \( G = \langle x_1, x_2, x_3 \rangle \), with \( d(G) = 3 \), \( |\{x_1^2, x_2^2, x_3^2\}| = 3 \). Suppose that \( G \) is non-abelian and that \( |T|^2 \leq 8 \) for each generating subset \( T \) of \( G \) of order 3. If \( x_3 \in Z(G) \), then one of the following holds:

\( (j) \) \( G = \langle a, b \rangle \times \langle c \rangle \), \( c^2 = 1 \), \( a^4 = 1 \), \( a^b = a^{-1} \),

\( (jj) \) \( G = \langle a, b \rangle \langle c \rangle \), \( a^4 = b^4 = c^4 = 1 \), \( ac = ca \), \( bc = cb \), \( a^b = a^{-1} \), \( a^2b^2c^2 = 1 \).

Proof. Consider the generating subset of pairwise non-commuting elements \( V = \{x_1x_3, x_2x_3, x_1x_2x_3\} \). Then either \( (x_1x_2x_3)^2 = (x_1x_3)^2 \) or \( (x_1x_2x_3)^2 = (x_2x_3)^2 \), therefore either \( x_1x_2^2x_3^2 = x_2^2x_3^2 \) or \( (x_1x_2)^2x_3^2 = x_2^2x_3^2 \), hence either \( (x_1x_2)^2 = x_1^2 \) or \( (x_1x_2)^2 = x_2^2 \). Without loss of generality we can suppose \( (x_1x_2)^2 = x_1^2 \), hence

\[ x_2^x = x_2^{-1}. \]

Now consider the generating subset of pairwise non-commuting elements \( W = \{x_1^{-1}x_3, x_2x_3, x_1x_2x_3\} \). Then \( (x_1^{-1}x_3)^2 = (x_2x_3)^2 \) or \( (x_1^{-1}x_3)^2 = (x_1^{-1}x_3)^2 \) or \( (x_1x_2x_3)^2 = (x_2x_3)^2 \). The last equality implies the contradiction \( x_1^2 = x_2^2 \). From the first equality we get \( x_1^{-2} = x_2^2 \), and from \( x_2^{x_1} = x_2^{-1} \) we get \( x_2^4 = 1 \).
and then $x_1^4 = 1$. Finally from $(x_1x_2x_3)^2 = (x_1^{-1}x_3)^2$ we obtain $x_1^2 = x_1^{-2}$. In any case

$$x_1^4 = 1.$$ 

Now consider the generating subset of pairwise non-commuting elements $\{x_1, x_2, x_1x_2x_3\}$. Then either $(x_1x_2x_3)^2 = x_1^2$ or $(x_1x_2x_3)^2 = x_2^2$. Since $(x_1x_2)^2 = x_1^2$, the first equality implies $x_3^2 = 1$ and (j) holds.

So assume $(x_1x_2x_3)^2 = x_2^2$, then

$$x_2^2x_3^2 = x_2^2.$$ 

Arguing analogously on the generating subset of pairwise non-commuting elements $\{x_1, x_2^{-1}, x_1x_2x_3\}$, we obtain that either $x_3^2 = 1$ and (j) holds, or $(x_1x_2x_3)^2 = x_2^{-2}$, or $x_1^2 = x_2^{-2}$. In the second case, from $x_1^2x_3^2 = x_2^2$ we get that $x_2^2 = 1$ and also that $x_3^2 = 1$ and $x_1^2x_2^2x_3^2 = 1$, therefore (jj) holds. Finally in the last case we have $x_1^{-2} = x_2^2 = x_2^{-2}$, which is a contradiction. \[\square\]

Notice that if (j) holds, then $G = \langle a, c \rangle \langle b \rangle$, with $a^b = a^{-1}, c^b = c = c^{-1}$, therefore (i) of Lemma 4.1 holds.

We will show later that if (jj) holds, then $|S^2| \leq 8$ for every generating subset $S$ of $G$ of order 3.

Now we assume that there exists a generating subset $S$ of $G$, with $S = \{x_1, x_2, x_3\}, |\{x_1^2, x_2^2, x_3^2\}| = 3, x_2x_3$ in $Z(G)$. Notice that in this case if $(x_2x_3)^2 \neq x_1^2, x_3^2$, then the subset $\{x_1, x_2, x_2x_3\}$ satisfies the hypothesis of Lemma 4.2 and (j) or (jj) of Lemma 4.2 holds. Similarly if $(x_2x_3)^2 \neq x_1^2, x_3^2$, then the subset $\{x_1, x_3, x_2x_3\}$ satisfies the hypothesis of Lemma 4.2. Hence we can suppose $(x_2x_3)^2 = x_1^2$. In this case we can prove:

**Lemma 4.3.** Let $G = \langle x_1, x_2, x_3 \rangle$, with $d(G) = 3, |\{x_1^2, x_2^2, x_3^2\}| = 3$. Suppose that $G$ is non-abelian and that $|T^2| \leq 8$ for each generating subset $T$ of $G$ of order 3. If $x_2x_3 \in Z(G)$ and $(x_2x_3)^2 = x_1^2$, then (i) of Lemma 4.1 holds.

**Proof.** Consider the generating subset of pairwise non-commuting elements $V = \{x_1, x_2, x_1x_3\}$. Then either $(x_1x_3)^2 = x_1^2$ or $(x_1x_3)^2 = x_2^2$. If $(x_1x_3)^2 = x_2^2$, consider the generating subset of pairwise non-commuting elements $W = \{x_1x_3, x_3, x_1^{-1}(x_2x_3)\}$. Then either $x_3^2 = (x_1^{-1}(x_2x_3))^2 = 1$, or $x_2^2 = (x_1x_3)^2 = (x_1^{-1}(x_2x_3))^2 = 1$. But if $x_3^2 = 1$ then from $x_2^2x_3^2 = (x_2x_3)^2 = x_1^2$ we get the contradiction $x_2^2 = x_1^2$, while if $x_2^2 = 1$ then from $x_2^2x_3^2 = (x_2x_3)^2 = x_1^2$ we obtain the contradiction $x_2^2 = x_1^2$. Therefore

$$x_1x_3)^2 = x_1^2, \text{ i.e. } (x_3)^{x_1} = x_3^{-1}.$$ 

Arguing analogously on the generating subset of pairwise non-commuting elements $V_1 = \{x_1, x_3, x_1x_2\}$ we obtain that either $(x_1x_2)^2 = x_1^2$ or $(x_1x_2)^2 = x_2^2$ and that the relation $(x_1x_2)^2 = x_3^2$ is not possible considering the subset $W_1 = \{x_1x_2, x_2, x_1^{-1}(x_2x_3)\}$. Therefore

$$(x_1x_2)^2 = x_1^2, \text{ i.e. } (x_2)^{x_1} = x_2^{-1}.$$ 

Finally

$$x_1^4 = 1.$$ 

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In fact, considering the subset $V_2 = \{x_1^{-1}, x_2, x_1 x_3\}$ we get $x_1^2 = (x_1 x_3)^2 = x_1^{-2}$ and $x_1^1 = 1$, or $x_1^{-2} = x_2^2$ and from $x_2 x_1 = x_2^{-1}$ it follows that $x_2^2 = (x_2^2 x_1) = x_2^{-2}$ thus $x_1^1 = x_2^4 = 1$. Therefore (i) of Lemma 4.1 holds.

Now we assume that $|\{x^2 \mid x \in S\}| \leq 2$ for each generating subset $S$ of order 3. First suppose that $|\{x^2 \mid x \in S\}| = 1$ for each generating subset $S$ of order 3. In this case $G$ is abelian, as the following Lemma shows.

**Lemma 4.4.** Let $G = \langle x_1, x_2, x_3 \rangle, d(G) = 3$, and suppose that $|\{x^2 \mid x \in S\}| = 1$ for each generating subset $S$ of order 3. Then $G$ is an elementary abelian 3-generated 2-group.

**Proof.** We have $x_1^2 = x_2^2 = x_3^2 = (x_1 x_2)^2 = (x_1 x_3)^2$, hence $x_1 x_2 = x_1^{-1}$, $x_2 x_3 = x_2^{-1}$, $x_3 x_1 = x_3^{-1}$. Considering the subset $\{x_1 x_2 x_3, x_2 x_3, x_3 x_2 x_3\}$ we have also that $(x_1 x_2 x_3)^2 = (x_2 x_3)^2$, thus $x_1 x_2 x_3 = x_1^{-1}$. But we have also that $x_1 x_2 x_3 = x_1$, therefore $x_1^2 = 1$. Thus $x_2 = x_3 = x_2^4 = 1$, then $G$ is abelian and an elementary abelian 2-group, as required.

Now suppose that there exists a generating subset $S$ of $G$ of order 3, with $|\{x^2 \mid x \in S\}| = 2$ and that $|\{x^2 \mid x \in T\}| \leq 2$ for each generating subset $T$ of order 3 of $G$.

We can suppose $G = \langle x_1, x_2, x_3 \rangle$ with $x_1^2 = x_2^2 \neq x_3^2$. The structure of $G$ follows from the following Proposition.

**Proposition 4.5.** Let $G = \langle x_1, x_2, x_3 \rangle, d(G) = 3$, $x_1^2 = x_2^2 \neq x_3^2$ and suppose that $|\{x^2 \mid x \in T\}| \leq 2$ for each generating subset $T$ of $G$ of order 3. Then either $G$ is abelian or one of the following holds:

- (a) $G = \langle x_1, x_2 \rangle \times \langle x_3 \rangle, \langle x_1, x_2 \rangle \simeq Q_8, x_3^2 = 1, x_1 x_2 = x_1^{-1}, x_2 x_3 = x_2 x_3^{-1}$;
- (b) $G = \langle a, b \rangle \times \langle c \rangle, a^4 = b^2 = c^2 = 1, a b = b a^{-1}, G \simeq D_4 \times C_2$;
- (c) $G = \langle a, b \rangle \times \langle c \rangle, a^4 = b^4 = c^2 = 1, a^2 = b^2, a b = a^{-1}, G \simeq Q_8 \times C_2$.

Conversely, if (a) or (b) or (c) holds, then $|\{x^2 \mid x \in T\}| \leq 2$ for each generating subset $T$ of $G$ of order 3.

**Proof.** Consider the generating subset $\{x_2, x_1 x_3, x_3\}$, then either $(x_1 x_3)^2 = x_3^2$ or $(x_1 x_3)^2 = x_2^2 = x_1^2$. Arguing similarly on $\{x_1, x_2 x_3, x_3\}$, then either $(x_2 x_3)^2 = x_2^2$ or $(x_2 x_3)^2 = x_2^2 = x_2^2$. Furthermore, considering the generating subsets $\{x_1, x_3, x_1 x_2 x_3\}$ and $\{x_1, x_3, x_1 x_2\}$ we obtain that either $(x_1 x_2 x_3)^2 = x_3^2$ or $(x_1 x_2 x_3)^2 = x_1^2$ and either $(x_1 x_2)^2 = x_3^2$ or $(x_1 x_2)^2 = x_3^2$.

First we show that

$$x_1^4 = x_2^2 = 1.$$

In fact, from $x_1^2 = x_2^2$ it follows that $x_1^2 \in C_G(x_2), x_2^2 \in C_G(x_1)$. Moreover, considering the generating subset $\{x_1^2, x_2, x_3\}$, we get either $(x_1 x_3)^2 = x_2^2$ and the contradiction $x_1^2 = x_3^2$ or $(x_1 x_3)^2 = x_1^2 = x_2^2$, thus $x_1^4 = C_G(x_3)$ and $x_3^2 = x_2^2 \in Z(G)$. If $x_2^2 \neq x_3^2$, then, considering the subset $\{x_1^{-1}, x_2, x_3\}$, we obtain $x_1^{-2} = x_2^2 = x_3^2$ and $x_1^4 = 1 = x_2^4$, as required. If $x_1^2 = x_2^2$, then $x_3^2 \in Z(G)$. Thus the relation
\((x_1x_3)^2 = x_1^2\) implies \(x_3^2 = x_3^{-1}\) and \(x_3^2 = (x_3^2)x_1 = x_3^{-2}\) implies \(x_3^4 = 1\) and then \(x_1^4 = 1\), while the relation \((x_1x_3)^2 = x_2^3\) implies \(x_1^{x_1} = x_1^{-1}\) and \(x_1^{x_1} = (x_1^2)x_3 = x_2^{-2}\) and again \(x_1^4 = 1\), as required.

Now our proof splits into four different cases.

Case I) \((x_1x_3)^2 = x_2^3\) and \((x_2x_3)^2 = x_3^2\). Then

\[
x_3^{x_3} = x_3^{-1}, x_2^{x_3} = x_2^{-1}.
\]

In this case \((x_1x_2x_3)^2 = x_1x_2x_3x_1x_2x_3 = x_1x_2x_1^{-1}x_2^{-1}x_3^2 = x_1x_2x_1^3x_2^2x_3^2 = x_1x_2x_1^3x_1x_2x_3^2 = (x_1x_2)^2x_3^2.
\]

If \((x_1x_2x_3)^2 = x_3^2\), then \((x_1x_2)^2 = 1 = x_1^2x_2^2\) and \(x_1x_2 = x_2x_1\). Furthermore, considering the generating subset \(\{x_1, x_3, x_1x_2\}\), we obtain that either \(x_1^2 = 1 = x_2^2\) and in this case \(G\) is abelian, or \(x_3^2 = 1\) and in this case \(G = \langle x_1x_2 \rangle \times \langle x_1, x_3 \rangle\) and \((\beta)\) holds.

If \((x_1x_2x_3)^2 = x_1^2\) we have \(x_1^2 = (x_1x_2)^2x_3^2\). Now, if \((x_1x_2)^2 = x_3^2\), then \(G \cong Q_8\). Furthermore the relation \((x_1x_2)^2x_3^2 = x_1^2\) implies \(x_3^2 = 1\), thus \(G\) has the structure in \((\alpha)\). If \((x_1x_2)^2 = x_3^2\), then \(x_3^2 = x_3^4\), moreover, considering the generating subset \(\{x_1, x_3^3, x_1x_2\}\) we get \(x_3^4 = 1\). Therefore \(x_3^2 = 1 = x_2^2\), then \(x_3^{x_3^2} = x_1, x_3^{x_3^2} = x_2\), and \(G = \langle x_1x_2, x_3^{-1} \rangle \times \langle x_2 \rangle\), with \(x_1x_2, x_3^{-1} \cong Q_8\)

\((x_1x_2)^2 = (x_1x_2)^{-1}, (x_1x_3)^2 = x_1x_3^3 = (x_1x_3)^{-1}\) and \((\alpha)\) holds.

Case II) Now suppose \((x_1x_3)^2 = x_2^2\) and \((x_3x_2)^2 = x_3^2\). Then

\[
x_2^{x_2} = x_2^{-1}, x_3^{x_2} = x_3^{-1}.
\]

First suppose \((x_1x_2x_3)^2 = x_3^2\). Then from \(x_3^2 = (x_1x_2x_3)^2 = (x_1x_2)^2x_3^2\) we obtain

\[
(x_1x_2)^2 = 1.
\]

Then \((x_1x_2)^2 = x_3^2\) implies \(x_1x_2 = x_2x_1\).

Furthermore, considering the generating subset \(\{x_1, x_3, x_1x_2\}\), we obtain that either \(x_1^2 = 1\) or \(x_3^2 = 1\).

If \(x_1^2 = 1\), then \(x_2^2 = x_1^2 = (x_1x_2)^2 = 1\), and, considering the generation subset \(\{x_3^{-1}, x_1, x_1x_2x_3\}\), we get \(x_3^{-2} = x_3^2\), thus \(x_3^4 = 1\). Therefore

\[
G = \langle x_3, x_1 \rangle \times \langle x_1x_2 \rangle
\]

and \((\beta)\) holds.

If \(x_3^2 = 1\), then \(x_3x_1 = x_1x_3\) and \(x_3x_2 = x_2x_3\), therefore

\[
G = \langle x_1, x_2 \rangle \times \langle x_3 \rangle
\]

and \(G\) is abelian.

Now suppose \((x_1x_2x_3)^2 = x_2^2\). Then \(x_2^2 = (x_1x_2)^2x_3^2\).

If \((x_1x_2)^2 = x_1^2\), then we obtain \(x_3^2 = 1\). Hence

\[
G = \langle x_1, x_2 \rangle \times \langle x_3 \rangle.
\]
Furthermore \( \langle x_1, x_2 \rangle \simeq Q_8 \), then (\( \gamma \)) holds.

If \( (x_1x_2)^2 = x_3^2 \), then \( x_1^2 = (x_1x_2)^2x_3^2 \) we obtain \( x_1^2 = x_3^4 \). Moreover, considering the subset \( \{ x_3^{-1}, x_1, x_1x_2 \} \), we get either \( x_3^{-2} = x_3^2 \) and \( x_3^4 = 1 \) or \( x_3^{-2} = x_1^2 \in C_G(x_1) \) and again \( x_3^4 = 1 \) since \( x_3^{-1} = x_3^1 \). Thus \( x_3^1 = 1 \) and \( x_1^2 = x_2^2 = 1 \), and from \( (x_1x_2)^2 = x_3^2 \) it follows that \( x_2^{-1} = x_2x_3 \). Therefore \( G = \langle x_3, x_1 \rangle \times \langle x_1x_2x_3 \rangle \), with \( (x_1x_2x_3)^2 = 1 \) and (\( \beta \)) holds.

Case III) Now suppose \( (x_1x_3)^2 = x_1^2 \), \( (x_2x_3)^2 = x_2^2 \).

Then
\[
x_3^1 = x_3^{-1}, x_2^3 = x_2^{-1}.
\]
In this case, arguing on the subset \( \{ x_3^{-1}, x_1, x_2x_3 \} \) we get
\[
x_3^4 = 1.
\]
We have \( (x_1x_2)^3 = x_1x_2x_3x_1x_2x_3 = x_1x_2x_1x_2^{-1} = (x_1x_2)^2x_2^2 \).

If \( (x_1x_2)^3 = x_3^2 \), then \( (x_1x_2)^2x_3^2 = x_3^2 \). Arguing as before, if \( (x_1x_2)^2 = x_3^2 \), then \( x_1^2 = x_2^1 = 1 \), \( x_3, x_1x_2 \simeq Q_8 \), \( x_3^1 = x_3^{-1}, (x_1x_2)^2x_1 = x_2x_1 \) (\( x_1x_2 \))^{-1} and (\( \alpha \)) holds. And the same happens if \( (x_1x_2)^2 = x_2^2 \) since in this case \( x_3^1 = 1 \) and \( G = \langle x_1, x_2 \rangle \times \langle x_2x_3 \rangle \), where \( \langle x_1, x_2 \rangle \simeq Q_8 \), \( (x_2x_3)^2 = 1 \), \( x_1^2x_3 = x_1^{-1}, x_2^x_3 = x_2^{-1} \).

If \( (x_1x_2)^3 = x_1^2 \), then \( (x_1x_2)^2x_2^2 = x_2^2 \) implies \( (x_1x_2)^2 = 1 \) and \( x_1x_2 = x_2x_1 \). Arguing as in previous cases, from \( (x_1x_2)^2 = 1 \) we obtain that either \( x_2 = 1 \) or \( x_1 = 1 \). In the first case \( x_3^1 = 1 \), implies that \( G = \langle x_1x_2, x_2 \rangle \times \langle x_3 \rangle \), with \( (x_1x_2)^2 = x_1^2 = x_2^2, (x_1x_2)^2x_2^3 = x_1x_2x_3 = (x_1x_2)^{-1} \), \( (x_1x_3)^3 = x_1x_2x_3^2 \), and (\( \alpha \)) holds.

Finally, if \( x_1^2 = x_2^2 = 1 \), then \( x_2 \in Z(G) \) and \( G = \langle x_2 \rangle \times \langle x_3, x_1 \rangle \simeq C_2 \times D_4 \).

Case IV) Finally suppose \( (x_1x_3)^2 = x_3^2 \), \( (x_2x_3)^2 = x_2^2 \). In this case we can argue as in case III) changing the role of \( x_1 \) and \( x_2 \).

Conversely, assume that (\( \alpha \)) holds, then, for every \( g \in G \), \( g = sx_3^\delta \) with \( \delta \in \{ 0, 1 \} \), \( s \in \langle x_1, x_2 \rangle \).

If \( \delta = 0 \), then \( g^2 \in \{ 1, x_1^2 = x_2^2 \} \). If \( \delta = 1 \), then \( g^2 = sx_3x_3 = ss^{-1}x_3^2 = 1 \) if \( x_3 \neq x_3s \), while \( g^2 = s^2x_3^2 = s^2 \in \{ 1, x_1 \} \) if \( x_3 = x_3s \). Thus \( |\{ g^2 \ | g \in G \}| = 2 \), and we have the result. If either (\( \beta \)) or (\( \gamma \)) holds, then, for every \( g \in G \) we have \( g = x_3^\delta \) with \( \delta \in \{ 0, 1 \} \) and \( x \in (a, b) \simeq Q_8 \) or \( \simeq D_4 \), then \( g^2 = x^2 \in \{ 1, a^2 \} \), and again \( |\{ g^2 \ | g \in G \}| = 2 \).

Now we can prove the main result of this section.

**Theorem 4.6.** Let \( G \) be a finitely generated group with \( d(G) = n \geq 3 \). Suppose that \( |S^2| \leq 3|S| - 1 \) for any generating subset \( S \) of \( G \) such that \( |S| = n \). Then \( G \) is a group of one of the following types:

(i) \( G = \langle x_1, x_2 \rangle \langle x_3 \rangle, x_3^4 = 1, x_1x_2 = x_2x_1, x_1x_3 = x_1^{-1}, x_3^{-1}x_2x_3 = x_2^{-1} \),

(ii) \( G = \langle x_1, x_2 \rangle \langle x_3 \rangle, x_1^4 = x_2^4 = x_3^4 = 1, x_1x_3 = x_3x_1, x_2x_3 = x_3x_2, x_1^{-1}x_1x_2 = x_1^{-1}, x_1^2x_2^2 = 1 \),

(iii) \( G = \langle x_1, x_2 \rangle \langle x_3 \rangle, x_1^4 = x_2^4 = x_3^4 = 1, x_1^4 = x_2^4 = x_1^{-1}, x_3^{-1}x_1x_3 = x_1^{-1}, x_3^{-1}x_1x_3 = x_1^{-1}, x_3^{-1}x_2x_3 = x_2^{-1} \),

(iv) \( G \simeq D_4 \times C_2 \),

(v) \( G \simeq Q_8 \times C_2 \).

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\((vi)\) \(G = \langle x_1 \rangle \times \langle x_2 \rangle \times \langle x_3 \rangle \times \langle x_4 \rangle \times \langle x_5 \rangle \times \langle x_6 \rangle\), where \(x_2^2 = x_3^2 = x_4^2 = x_5^2 = x_6^2 = 1\),

\((vii)\) \(G = \langle x_1 \rangle \times \langle x_2 \rangle \times \langle x_3 \rangle \times \langle x_4 \rangle \times \langle x_5 \rangle\), where \(x_4^2 = x_5^2 = 1\),

\((viii)\) \(G\) is abelian and \(n \leq 4\).

Conversely, if \(G\) satisfies one of \((i)-(viii)\), then \(|S^2| \leq 3|S| - 1\) for any generating subset \(S\) of \(G\) with \(|S| = d(G)\).

**Proof.** First assume \(G\) non-abelian then Lemma 3.1 shows that \(n \in \{3, 4\}\) and Lemma 3.2 shows that \(d(G) = 3\). Then Lemmas 4.1, 4.2, 4.3, 4.4 and Proposition 4.5 imply that \(G\) is a group of one of the types \((i)-(v)\).

Assume now that \(G\) is abelian. Choose an arbitrary generating subset \(S\) of \(G\) such that \(|S| = n\). Let \(S = \{g_1, \ldots, g_n\}\). Clearly \(S^2 = A \cup B\) where \(A = \{g_jg_m \mid 1 \leq j < m \leq n\}\), \(B = \{g_j^2 \mid 1 \leq j \leq n\}\).

As in a proof of Theorem 2.4 we can show that all elements of the subset \(A\) are pairwise different and that \(A\) and \(B\) are disjoint. It follows that \(|S^2| = \frac{1}{2}n(n - 1) + d\) where \(d = |B|\). We note that \(d \leq n\). Thus \(\frac{1}{2}n(n - 1) + 1 \leq 3n - 1\), so that \(n \leq 6\).

If \(n = 5\), then \(\frac{1}{2}n(n - 1) + d = 10 + d\). From \(|S^2| \leq 3|S| - 1\) we obtain that \(10 + d \leq 14\), thus \(d \leq 4\). Therefore we can suppose \(g_2^2 = g_5^2\), and \(G\) is of type \((vii)\).

If \(n = 6\), then \(\frac{1}{2}n(n - 1) + d = 15 + d\). From \(|S^2| \leq 3|S| - 1\) we obtain that \(15 + d \leq 17\), which implies that \(d \leq 2\), so that \(g_2^2 = g_3^2 = g_4^2 = g_5^2 = g_6^2\), and \(G\) is a group of type \((viii)\).

Conversely assume that \((i)\) holds. Write \(H = \langle x_1, x_2 \rangle \langle x_3^2 \rangle\). Then \(H\) is abelian, moreover \(d_{x_3} = d^{-1}\) for every \(d \in H\). Therefore for every \(g \in G \setminus H\) we have \(g = dx_3\), with \(d \in H\) and \(g^2 = dx_3dx_3 = dd^{-1}x_3^2 = x_3^2\). Hence if \(s, t, v \in G\), either two of them commute or two of them have the same square, thus \(|\{s, t, v\}| \leq 8\), as required.

If \((ii)\) holds, consider the central subgroup \(W = \langle x_1^2, x_2^2, x_3^2 \rangle\). Then \(d^2 = 1\) for any \(d \in W\) and \(G = W < x_1, x_2, x_3 >\). We have only to consider three elements \(s, t, v\) in \(GW\) which are pairwise non-commuting. But in this case it is not difficult to notice that the set \(\{s^2, t^2, v^2\}\) has always order 2, since \(x_1^2x_2^2x_3^2 = 1\).

A similar argument proves the result if \((ii)\) holds, while Proposition 4.5 shows that the result is true if one of \((iii)-(v)\) holds. Now suppose that one of \((vi)-(viii)\) holds, then \(G\) is abelian, and for every subset \(S\) of \(G\) of order \(n = d(G)\) we have \(|S^2| = \frac{1}{2}n(n - 1) + d\) where \(d = |B|\), \(B = \{g^2 \mid g \in S\}\).

Now if \((vi)\) holds, then \(n = 6\) and \(d = 2\), and we have \(|S^2| = 15 + 2 = 3|S| - 1\), as required; if \((vii)\) holds, then \(n = 5\) and \(d = 4\), and we have \(|S^2| \leq 10 + 4 = 3|S| - 1\), as required; finally if \((viii)\) holds, then \(n \in \{3, 4\}\), \(d \leq 4\) and in any case \(|S^2| \leq 3|S| - 1\) as required.

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