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A SURVEY ON GROUPS WITH SOME RESTRICTIONS ON NORMALIZERS OR CENTRALIZERS

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Abstract. We consider conditions on normalizers or centralizers in a group and we collect results showing how such conditions influence the structure of the group.

1. Introduction

The analysis of groups which satisfy some restriction related to normality is a common topic and an active area of research in group theory. Classical examples are the determination by Dedekind [4] and Baer [1] of the groups with all subgroups normal (now known as *Dedekind groups*), the characterization by Neumann [19] of the groups G with $|G : N_G(H)| < \infty$ for every subgroup H as the central-by-finite groups, or the characterization in the same paper of the groups with $|H^G : H| < \infty$ for every subgroup H as the groups with finite derived subgroup. Thus, it is clear that the behaviour of normalizer subgroups of a group has often a strong impact on the structure of the group itself. Here we analyze conditions on normalizers or centralizers both in finite and in infinite groups.

2. Finite groups

In papers [29] and [30], a new condition has been considered in connection to normality, in the realm of nilpotent groups. If G is nilpotent and H is a proper subgroup of G, it is known that $|N_G(H) : H| > 1$. The normalizer $N_G(H)$ will be as large as the whole G if H is normal in G, but what happens if one impose a bound to the index $|N_G(H) : H|$ for every non-normal subgroup H when G is non-Dedekind? For finite nilpotent groups, this problem reduces to finite p-groups for p a

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prime (see for instance, [29, Lemma 2.12]). Thus, the following question arises: *If k is a fixed positive integer, what can be said about the finite p-groups G which satisfy the condition*

$$|N_G(H) : H| \leq p^k \quad \text{for every } H \ntrianglelefteq G, \tag{2.1}$$

and which are not Dedekind groups?

In [29], Q. Zhang and Gao answered a question posed by Berkovich [2, Problem 116 (i)], classifying all finite p-groups satisfying condition (2.1) for $k = 1$.

Theorem 2.1. [29, Theorem 3.5] *Let G be a finite non-abelian p-group, $p > 2$ and $|G| \geq p^4$. Then $|N_G(H) : H| \leq p$ for every $H \ntrianglelefteq G$ if and only if $G \cong \langle a, b \mid a^{p^2} = b^{p^2} = 1,\ a^b = a^{1+p}\rangle$.*

Theorem 2.2. [29, Theorem 3.7] *Let G be a non-Dedekind group of order 2^n. Then $|N_G(H) : H| \leq p$ for every $H \ntrianglelefteq G$ if and only if G is one of the following mutually non-isomorphic groups:*

(i) 2*-group of maximal class*
(ii) $\langle a, b \mid a^{2^{n-2}} = b^4 = 1,\ a^b = a^{-1}\rangle$
(iii) $\langle a, b \mid a^{2^{n-2}} = b^4 = 1,\ a^b = a^{-1+2^{n-3}}\rangle$.

Berkovich, in [3], investigated a more specific class of finite p-groups in which all non-normal cyclic subgroups of minimal order have index p in their normalizers, and proposed the following problem: *Classify finite p-groups G satisfying the condition*

$$|N_G(\langle g\rangle) : \langle g\rangle| \leq p^2 \quad \text{for every } \langle g\rangle \ntrianglelefteq G.$$

In line with this, X. Zhang and Guo, in [30], given a positive integer k, considered finite p-groups G satisfying the following condition

$$|N_G(\langle g\rangle) : \langle g\rangle| \leq p^k \quad \text{for every } \langle g\rangle \ntrianglelefteq G, \tag{2.2}$$

and they completely classified the non-Dedekind p-groups, $p > 2$, satisfying condition (2.2) for $k = 2$ (Theorem 4.6, Theorem 4.7).

Notice that the order of non-abelian p-groups satisfying (2.1), for $k = 1$, as classified in Theorem 2.1, is at most p^4 for odd p. This is not particular to the case $k = 1$. In fact, as shown in the following theorem, it is possible to bound the order of a non-abelian p-group (p odd), satisfying the seemingly weaker condition (2.2), for arbitrary k.

Theorem 2.3. [30, Theorem 3.4] *Let G be a finite non-abelian p-group, $p > 2$. If G satisfies condition (2.2) then $|G| \leq p^{(2k+1)(k+1)}$.*

Some related problems raised about centralizers of elements. By similarity with condition (2.2), for a fixed positive integer k, one may ask what can be said about a finite p-group with the restriction

$$|C_G(g) : \langle g\rangle| \leq p^k \quad \text{for every } \langle g\rangle \ntrianglelefteq G. \tag{2.3}$$

The question was addressed in [8], where finite non-Dedekind p-groups satisfying any of the conditions (2.1), (2.2) or (2.3) have been studied.

Clearly, (2.1) implies (2.2) which in turn implies (2.3). Actually, as quoted below, it is sufficient to deal with the two conditions (2.1) and (2.3).

Proposition 2.4. [8, Proposition 2.2] *Let G be a finite non-Dedekind p-group. Then conditions* (2.1) *and* (2.2) *are equivalent.*

Theorem 2.5. [8, Theorem A] *Let G be a finite non-abelian p-group, where $p > 2$. If G satisfies either condition* (2.1) *or* (2.3)*, then $|G| \leq p^{2k+2}$.*

This result is an improvement of the aforementioned bound in Theorem 2.3 for condition (2.1) and $p > 2$, from a quadratic to a linear function in the exponent of p. Furthermore, as shown in the following example, the bound in Theorem 2.5 is the best possible.

Example 2.6. [8, Example 3.4] *Let p be an arbitrary prime, and let k be a positive integer. Consider the group G given by the following presentation:*

$$G = \langle a, b \mid a^{p^{k+1}} = b^{p^{k+1}} = 1,\ a^b = a^{1+p^k} \rangle.$$

Then $Z(G) = \langle a^p, b^p \rangle$ and $o(g) = p^{k+1}$ for every $g \in G \smallsetminus Z(G)$. By using these two facts, one can readily check that G satisfies both condition (2.1) *and condition* (2.3).

The bound in Theorem 2.5 is no longer valid for $p = 2$. Nevertheless, the next theorem states that the finite 2-groups satisfying any of the conditions (2.1) or (2.3) are either of bounded order, or belong to two infinite families.

Theorem 2.7. [8, Theorem B] *Let G be a finite non-Dedekind 2-group, satisfying either the condition* (2.1) *or the condition* (2.3)*. Then there exists a polynomial function $f(k)$ of degree four such that, if $|G| > 2^{f(k)}$, then G belongs to one of the families $\mathcal{F}_1$ or $\mathcal{F}_2$ described below.*

As usual, if G is a finite p-group then $\Omega_1(G)$ denotes the subgroup generated by the elements of G of order p.

Then, those two mentioned families consist of 2-groups of the form $G = \langle b, A \rangle$, where A is normal abelian, and $b^2 \in \Omega_1(A)$.

In the family $\mathcal{F}_1$, A is taken of exponent 2^n and $a^b = a^s$ for every $a \in A$, where either $s = -1$ and $n \geq 1$, or $s = -1 + 2^{n-1}$ and $n \geq 3$. These groups can be constructed with the help of the theory of cyclic extensions (see Section III.7 of [28]), and any element in $\Omega_1(A)$ is a valid choice for b^2.

On the other hand, in the family $\mathcal{F}_2$, $A = \langle a_1 \rangle \times A^*$, being $o(a_1) = 2^n$ and A^* non-trivial of order 2^m. The action of b on A is given by $a_1^b = a_1^s z$ and $(a^*)^b = (a^*)^s$ for every $a^* \in A^*$, where $z \in \Omega_1(A^*)$, $z \neq 1$, and either $s = -1$ or $s = -1 + 2^{n-1}$. It is assumed that $n \geq 2$ if $s = -1$, and that $n \geq 3$ and $n \geq m$ if $s = -1 + 2^{n-1}$. Again by the theory of cyclic extensions, for given A and s, any choice of $z \in \Omega_1(A^*) \smallsetminus \{1\}$ and $b^2 \in \Omega_1(A)$ will define a group in $\mathcal{F}_2$. Since $\langle a_1 \rangle \ntrianglelefteq G$, the family $\mathcal{F}_2$ consists entirely of non-Dedekind groups.

In any case, the values of k in (2.1) and (2.3) are independent of n ([8, Theorem 4.2, Theorem 4.4]), which allows to obtain 2-groups of arbitrarily large order while the indices in (2.1) and (2.3) remain bounded.

Observe that the infinite families obtained in Theorem 2.2 all belong to the family $\mathcal{F}_1$, by choosing $A \cong C_{2^{n-1}}$ in the case of 2-groups of maximal class, and $A \cong C_{2^{n-2}} \times C_2$ for the other cases. On the other hand, the groups of the family $\mathcal{F}_2$ are not present in the case $k = 1$.

Another natural restriction on centralizers has been introduced in [7] and concerns centralizers of non-central elements, in a finite group. If G is a group and $g \in G$, the inclusion $\langle g \rangle \leq C_G(g)$ holds. If $g \in Z(G)$ then $C_G(g)$ is as large as the whole group G, but otherwise one can require that the index $|C_G(g) : \langle g \rangle|$ should be small. Thus one may ask what can be said about a finite group G if $|C_G(g) : \langle g \rangle|$ is bounded as g runs over $G \smallsetminus Z(G)$.

Theorem 2.8. [7, Theorems A, C] *Let G be a finite non-abelian group such that $|C_G(g) : \langle g \rangle| \leq m$ for every $g \in G \smallsetminus Z(G)$. Then there exists a function f_1 depending only on m such that:*

1. *If G is a finite p-group and $m = p^k$ then $f_1(m) = p^{2k+2}$ and $|G| \leq f_1(m)$, unless $G \cong Q_8$.*
2. *If G is nilpotent then $|G| \leq f_1(m)$.*
3. *If G is soluble then $|G| \leq D\, f_1(m)$, where D is a product of at most two prime divisors of the order of G.*
4. *In general $|G| \leq E\, f_1(m)$, where E is a product of at most four prime divisors of the order of G. Thus the order of G can be bounded in terms of m and the largest prime divisor of $|G|$.*

Moreover, the bound given for a finite p-group is best possible.

Example 2.9. [7, Example 2.5] *Let p be an arbitrary prime, and let G be the group given by the following presentation:*

$$G = \langle a, b \mid a^{p^{k+1}} = b^{p^{k+1}} = 1,\ a^b = a^{1+p^k} \rangle.$$

Then $|G| = p^{2k+2}$, $Z(G) = \langle a^p, b^p \rangle$ and $o(g) = p^{k+1}$ for every $g \in G \smallsetminus Z(G)$. By using these facts, one can readily check that $|C_G(g) : \langle g \rangle| \leq p^k$ for every $g \in G \smallsetminus Z(G)$.

3. Infinite groups

On the other hand, some natural conditions on centralizers or normalizers have been considered also in the realm of infinite groups (see [5], [6], [9], [10], [11], [21], [24]). Following the notation introduced in [9], a given group G is said to be an *FCI-group* (FCI for 'finite centralizer index') provided that

$$|C_G(x) : \langle x \rangle| < \infty \quad \text{for every } \langle x \rangle \ntrianglelefteq G. \tag{3.1}$$

One can then ask for the existence of a uniform bound for the indices in (3.1), thus getting the condition that, for some positive integer m,

$$|C_G(x) : \langle x \rangle| \leq m \quad \text{for every } \langle x \rangle \ntrianglelefteq G, \tag{3.2}$$

in which case G is said to be a *BCI-group* (BCI for 'bounded centralizer index').

Interesting examples of BCI-groups are the generalized dihedral groups [9, Example 2.1], and the Tarski monster groups (i.e. infinite simple p-groups, for p a prime, all of whose proper non-trivial subgroups are of order p). It is easy to see that free groups satisfy condition (3.1) for every non-trivial element, and it is also known that the same happens with hyperbolic torsion-free groups [14]. Apart from the abelian case, such groups are examples of FCI-groups which are not BCI-groups [10, Lemma 4.2].

Equivalently, one could say that a group G is an FCI-group if either $\langle x\rangle \trianglelefteq G$ or $|N_G(\langle x\rangle) : \langle x\rangle|$ is finite for all $x \in G$. Thus a stronger property than FCI arises when the condition on normalizers is applied to arbitrary subgroups, not just the cyclic ones [11]. A group G is said to be an *FNI-group* (FNI for 'finite normalizer index') if

$$|N_G(H) : H| < \infty \quad \text{for every } H \ntrianglelefteq G. \tag{3.3}$$

Also, one can ask for the existence of a uniform bound for the indices in (3.3), thus getting the condition that, for some positive integer m,

$$|N_G(H) : H| \leq m \quad \text{for every } H \ntrianglelefteq G, \tag{3.4}$$

in which case G is said to be a *BNI-group* (BNI for 'bounded normalizer index').

Of course, there are obvious connections between these conditions and they hold in Dedekind groups, as well as in finite groups, so they can be considered as finiteness conditions. Clearly an FNI-group is an FCI-group, but the converse is false, since non-cyclic free groups are not FNI-groups.

3.1. **General properties and facts.** All the conditions (3.1), (3.2), (3.3), (3.4) are clearly hereditary for subgroups. Also, (3.3) and (3.4) can be easily checked to be hereditary for quotients. However, the rest of the conditions do not usually transfer to quotients. For example, consider the generalized dihedral group $G = \mathrm{Dih}(A)$, where A is torsion-free abelian of infinite 0-rank. Then G satisfies conditions (3.1), (3.2), but if N is the subgroup consisting of all fourth powers of elements of A, then $G/N \cong \mathrm{Dih}(A/N)$ does not satisfy any of them (see [9, Section 2] for more details).

Recall that the *rank* of an abelian p-group is the dimension, as a vector space over $\mathbb{F}_p$, of the subgroup formed by the elements of order at most p. This is equivalent to requiring that the group is a direct sum of finitely many cyclic and quasicyclic groups [20, 4.3.13]. If A is an abelian group, A_p denotes the unique Sylow p-subgroup of A. Then the *p-rank* of A is defined as the rank of A_p and the *0-rank* of A is the cardinality of a maximal independent subset of elements of infinite order from A. More generally, a group G is said to have *finite (Prüfer) rank* $r = r(G)$ if every finitely generated subgroup of G can be generated by r elements and r is the least such integer.

However, things change making the quotient with respect to a finite normal subgroup.

Proposition 3.1. [9, Proposition 2.2] *Let G be a group and let N be a finite normal subgroup of G. If G is an FCI-group then so is G/N. Similarly for BCI-groups.*

One can readily check that the centre of any periodic FCI-group is finite. Then the central quotient of a periodic FCI-group is again an FCI-group. Similarly for BCI-groups.

Notice that, for periodic groups, condition (3.1) is the same as the simpler

$$|C_G(x)| < \infty \quad \text{for every } \langle x \rangle \ntrianglelefteq G,$$

and there are well-known results that give information about the structure of a group simply from the knowledge that the centralizer of one element is finite.

Theorem 3.2. [26, page 263] *Let G be a periodic group having an involution with finite centralizer. Then G is soluble-by-finite, and so it is locally finite.*

Theorem 3.3. [16, page 286] *Let G be a periodic group having an involution with finite centralizer. Then G is nilpotent-by-finite.*

This property does not extend to non-involutions, but the following theorem holds.

Theorem 3.4. [17, Corollary 5.4.1]*, see also* [25] *for a more recent account. Let G be a locally finite group admitting an automorphism of prime order p whose centralizer has finite order m. Then G contains a nilpotent subgroup whose index is finite and bounded by a function of m and p and whose nilpotency class is bounded by a function of p.*

Then, the investigation of infinite locally finite FCI-groups [9] naturally started. Recall that in any group G the set of all elements admitting only finitely many conjugates is a characteristic subgroup, called the *FC-centre* of G and usually denoted by $FC(G)$. The next result shows the key role played by the FC-centre in the theory of FCI-groups.
A *power automorphism* of a group G is an automorphism sending every element $x \in G$ to a power of x.

Proposition 3.5. [9, Proposition 2.3, Proposition 2.5] *Let G be an infinite FCI-group.*

1. *If G is periodic then $FC(G)$ consists of all elements $x \in G$ such that $\langle x \rangle \trianglelefteq G$, and every element of G acts on $FC(G)$ by conjugation as a power automorphism. In particular, $FC(G)$ is a Dedekind group.*
2. *If G is locally finite then $G/FC(G)$ is a finite cyclic group.*

In [24], Shalev considered groups G satisfying the condition

$$|C_G(x)| < \infty \text{ or } |G : C_G(x)| < \infty \quad \text{for every } x \in G,$$

or the equivalent condition

$$|C_G(x)| < \infty \quad \text{for every } x \notin FC(G), \tag{3.5}$$

and he proved the following result.

Proposition 3.6. [24, Proposition 2.5] *Let G be a locally finite group satisfying condition (3.5). Then $G/FC(G)$ is finite.*

More recently, in [6], De Falco, de Giovanni, Musella and Trabelsi have considered a class which generalizes the one studied by Shalev. In fact, G is said to be an *AFC- group* if

$$|C_G(x) : \langle x\rangle| < \infty \quad \text{for every } x \notin FC(G). \tag{3.6}$$

Using this terminology, Proposition 3.6 can be reformulated as follows.

Theorem 3.7. [5, Theorem 3] *Let G be a locally finite AFC-group.*
Then $G/FC(G)$ is finite.

In the periodic case, by (1) of Proposition 3.5, condition (3.1) is equivalent to (3.6) which is in turn equivalent to the property (3.5). Then, the finiteness of the FC-central quotient in (2) of Proposition 3.5, follows by Proposition 3.6. However, contrary to Proposition 3.6, the proof of (2) of Proposition 3.5, as well as the one of Theorem 3.7, does not depend on the classification of finite simple groups.
Theorem 3.7 cannot be extended to non-periodic AFC-groups; in fact, the following example shows that there exists a metabelian AFC-group such that the factor group $G/FC(G)$ is infinite.

Example 3.8. [5] *Let P be the set of all primes. For each odd prime p let $\langle a_p\rangle$ be a group of order p, and let A be the direct product of the collection $(\langle a_p\rangle)_{p\in P}$. Consider an automorphism x of A inducing an automorphism of order $p-1$ on every $\langle a_p\rangle$, and let G be the semidirect product of A by $\langle x\rangle$. Clearly, x has infinite order, and A is the set of all elements of finite order of G. Then $C_A(x^k) = \langle a_p \mid p-1 \text{ divides } k\rangle$, for each positive integer k. In particular, the index $|C_G(x^k) : \langle x^k\rangle|$ is finite for each positive integer k, and it follows that every infinite cyclic subgroup of G has finite index in its centralizer. Moreover, for any $\in A$, the subgroup $\langle a\rangle$ is normal in G, and so $A = FC(G)$. Therefore G is an AFC-group, but $G/FC(G)$ is infinite cyclic.*

Nevertheless, one of the main results proved in [6] gives a strong information on the factor group of an AFC- group with respect to its FC-centre, within the universe of locally (soluble-by-finite) groups.

Theorem 3.9. [6, Theorem 3.7] *Let G be a locally (soluble-by-finite) AFC-group.*
Then the factor group $G/FC(G)$ is a soluble-by-finite group of finite rank.

3.2. **Characterization theorems.** Despite the complexity of the class of FCI-groups, in many cases it is possible to give a precise description of these groups. For instance, in the class of locally finite groups, it turns out that infinite FCI-groups are certain cyclic extensions of Dedekind groups.
We write $\pi(G)$ for the set of prime divisors of the orders of the elements of a group G. If G is a periodic and nilpotent group, G_p denotes the unique Sylow p-subgroup of G, and if φ is an automorphism of G, then φ_p stands for the restriction of φ to G_p.
Thus, a first characterization theorem comes to describe the structure of infinite locally finite FCI-groups.

Theorem 3.10. [9, Theorem 2.7] *A group G is an infinite locally finite FCI-group if and only if either*

(*i*) *G is an infinite periodic Dedekind group, or*

(*ii*) *$G = \langle g, D\rangle$, where D is an infinite periodic Dedekind group with D_2 of finite rank, and g acts on D as a power automorphism φ of order $m > 1$ such that $|G : D| = m$ and*

$$\prod_{p\in\pi_0\cup\pi_1} |D_p| < \infty, \tag{3.7}$$

where

$$\pi_0 = \{p \in \pi(D) \mid o(\varphi_p) < m\},$$

and

$$\pi_1 = \{p \in \pi(D) \mid o(\varphi_p) = m,\ p > 2 \text{ and } p \not\equiv 1 (\mathrm{mod}\ m)\}.$$

Furthermore, if (ii) holds, then for every $\langle x\rangle \ntrianglelefteq G$ we have

$$|C_G(x)| \leq mM \prod_{p\in(\pi_0\cup\pi_1)\setminus\{2\}} |D_p|,$$

where $M = |D_2|$ if D_2 is finite, and $M = 2^{r(D_2)}$ if D_2 is infinite.

Actually, in the locally finite case, the previous classification applies to FNI-groups, as well as to BCI-groups, and to BNI-groups. In fact, the following theorem holds.

Theorem 3.11. [11, Theorem 2.4] *Let G be a locally finite group.*
The following conditions are equivalent:

(1) *G is an FNI-group.*
(2) *G is an FCI-group.*
(3) *G is a BNI-group.*
(4) *G is a BCI-group.*

Concerning the periodic case, a restriction to locally graded groups is needed.
Recall that a group is *locally graded* if every non-trivial finitely generated subgroup has a non-trivial finite image. The class of locally graded groups is rather wide, since it includes locally (soluble-by-finite) groups, as well as residually finite groups, and so in particular free groups.
The restriction to locally graded groups is motivated to avoid Tarski monster groups and leads to the theorem below.

Theorem 3.12. [9, Theorem 3.2] *Every locally graded periodic BCI-group is locally finite.*

Thus, the following corollary of Theorem 3.11 holds.

Corollary 3.13. [11, Corollary 2.5] *Let G be a periodic locally graded group.*
The following conditions are equivalent:

(1) *G is a BNI-group.*
(2) *G is a BCI-group.*

It remains unclear whether Theorem 3.12 holds for all FCI-groups. This is related to the following question: *Given a periodic residually finite group G in which the centralizer of each non-trivial element is finite, is G finite?* Observe that the well-known examples of finitely generated infinite periodic groups constructed by Golod [12], Grigorchuk [13], and Gupta-Sidki [15] are residually finite but not FCI-groups ([18, Theorem 1.1], [23, Theorem 1] and [27, Theorem 2], respectively).

Now, let A be an abelian p-group, as shown in [22, Lemma 4.1.2], every power automorphism φ of A can be written in the form $a \longmapsto a^t$ for some $t \in \mathbb{Z}_p^\times$, where $\mathbb{Z}_p^\times$ stands for the group of units of the ring of p-adic integers. In the sequel, we write $\exp \varphi = t$, if $\varphi(a) = a^t$ for every $a \in A$.
As in the locally finite case, infinite FCI-groups have been also characterized in the locally nilpotent case. If an infinite locally nilpotent FCI-group is periodic, then Theorem 3.10 is available, whereas for non-periodic locally nilpotent FCI-groups the following theorem applies.

Theorem 3.14. [10, Theorem 3.7] *Let D be a periodic Dedekind group such that $\pi(D)$ is finite and D_p is of finite rank for every $p \in \pi(D)$. Let φ be a power automorphism of D, and write $t_p = \exp \varphi_p$ whenever D_p is abelian. Assume that the following conditions hold:*

- (*i*) *If D_2 is non-abelian, φ_2 is the identity automorphism.*
- (*ii*) *If $p > 2$ then $t_p \equiv 1 \pmod p$, and if D_p is infinite also $t_p \neq 1$.*
- (*iii*) *If $p = 2$ and D_2 is infinite, then $t_2 \neq 1, -1$.*

Then the semidirect product $G = \langle g \rangle \ltimes D$, where g is of infinite order and acts on D via φ, is a locally nilpotent FCI-group. Conversely, every non-periodic locally nilpotent FCI-group is either abelian or isomorphic to a group as above, D being the torsion subgroup of G.

A restriction to BCI-groups leads to abelian ones.

Theorem 3.15. [10, Corollary 4.4] *Let G be a non-periodic locally nilpotent group.*
If G is a BCI-group then G is abelian.

Hence, there exist non-periodic locally nilpotent FCI-groups which are not BCI-groups, while in the realm of locally finite groups, FCI- and BCI-conditions are equivalent, as pointed out before. Nevertheless, this is not the case for FCI- and FNI-conditions.

Theorem 3.16. [11, see Theorem 3.4] *Let G be a non-periodic locally nilpotent group.*
The following conditions are equivalent:

(1) *G is an FNI-group.*
(2) *G is an FCI-group.*

Then, non-periodic BCI- and BNI-groups have been studied in general.

Theorem 3.17. [10, see Theorem 4.3] *Let G be a non-periodic group.*
The following conditions are equivalent:

(1) *G is a BCI-group.*
(2) *Either G is abelian or $G = \langle g, A\rangle$, where A is a non-periodic abelian group of finite 2-rank and g is an element of order at most 4 such that $g^2 \in A$ and $a^g = a^{-1}$ for all $a \in A$.*

Theorem 3.18. [11, see Theorem 3.5] *Let G be a non-periodic group.*
The following conditions are equivalent:

(1) *G is a BNI-group.*
(2) *Either G is abelian or $G = \langle g, A\rangle$, where A is a non-periodic abelian group of finite 0-rank and finite 2-rank, and g is an element of order at most 4 such that $g^2 \in A$ and $a^g = a^{-1}$ for all $a \in A$.*

Every locally finite BCI-group is a BNI-group. According with Theorems 3.17 and 3.18, this is no longer true for non-periodic BCI-groups, where the extra condition of having finite 0-rank is needed to be a BNI-group.

A generalization of Theorem 3.10 and Theorem 3.14 has been obtained in [21] by Robinson, who classified the FCI-groups and FNI-groups belonging to a large class of infinite groups.
Let $\mathfrak{X}$ denote the smallest class of groups containing all finite groups and all abelian groups which is locally closed and closed with respect to forming ascending series with factors in the class. For example, $\mathfrak{X}$ contains all locally soluble-by-finite groups.
Following [21], a subgroup H is said to be *d-embedded* in a group G if $\langle x\rangle \trianglelefteq G$ for all $x \in H$. Thus H is a Dedekind group and elements of G induce power automorphisms in H.
Denoting by $A[n]$ the subgroup of elements in an abelian group A with order dividing n, there are five types of groups which appear in the classification of FCI-groups and FNI-groups.

(i) Dedekind groups;
(ii) non-abelian groups G with a finite d-embedded subgroup F such that G/F is infinite cyclic or infinite dihedral;
(iii) $G = \langle x, A\rangle$ where A is a non-periodic abelian group, $a^x = a^{-1}$ for all $a \in A$, $x^2 \in A[2]$ and $A[2]$ is finite;
(iv) $G = \langle x, A\rangle$ where A is an infinite periodic abelian subgroup which is d-embedded in G and $C_A(y)$ is finite for all $y \in G \smallsetminus A$;
(v) $G = H\bar{G}$, a central product, where H is a finite non-abelian Dedekind 2-group which is d-embedded in G, and $\bar{G} = \langle x, B\rangle$ is a group of type (iv), with $H \cap \bar{G} = B_2 \leq H[2] \cap \langle x\rangle$ and $|B_2| = 1$ or 2.

In fact,

Theorem 3.19. [21, Theorem 2] *Groups of types (i), (ii), (iv) and (v) are FNI-groups. A group of type (iii) is an FCI-group and if it has finite 0-rank, then it is an FNI-group.*

Theorem 3.20. [21, Theorem 3] *Let G be an infinite group belonging to the class $\mathfrak{X}$. If G is an FCI-group, then it is of types (i) to (v). If G is an FNI-group which is of type (iii), then it has finite 0-rank.*

As in the finite case, we finish with a restriction on centralizers of non-central elements

$$|C_G(x) : \langle x \rangle| < \infty \quad \text{for every } x \in G \smallsetminus Z(G). \tag{3.8}$$

It is easy to see that every periodic group containing an infinite abelian subgroup, and in particular every infinite locally finite group [20, 14.3.7], is abelian whenever condition (3.8) is satisfied. This can be extended to infinite locally nilpotent groups as follows.

Theorem 3.21. [10, Corollary 3.6] *Let G be an infinite locally nilpotent group. If $|C_G(x) : \langle x \rangle|$ is finite for every $x \in G \smallsetminus Z(G)$, then G is abelian.*

This is no longer true if G is only non-periodic. Indeed, the infinite dihedral group is an example of a group satisfying condition (3.8) which is not abelian. Nevertheless, this is not the case if the requirement of finiteness is replaced with a bound.

Proposition 3.22. [10, Proposition 4.5] *Let G be a non-periodic group, and assume that for some n we have $|C_G(x) : \langle x \rangle| \leq n$ for every $x \in G \smallsetminus Z(G)$. Then G is abelian.*

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