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ON NORMAL AUTOMORPHISMS OF n-PERIODIC PRODUCTS OF FINITE CYCLIC GROUPS

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Abstract. We prove that each normal automorphism of the n-periodic product of cyclic groups of odd order $r \geq 1003$ is inner, whenever r divides n.

1. Introduction

In the paper [1] for any odd number $n \geq 665$ a group multiplication was introduced, called the *periodic product of a given period n or the n-periodic product*. These operations possess many properties of the classical operations of free and direct products of groups, including exactness, associativity and hereditary properties for subgroups. The latter property means that for every family of groups $\{G_i\}_{i\in I}$ and any subgroups H_i of G_i the identical embeddings $H_i \to G_i$ are extendable to the embedding of the n-periodic product $\prod\limits_{i\in I}{}^{n}H_i$ into the n-periodic product $\prod\limits_{i\in I}{}^{n}G_i$, that is the subgroup of factors generate their n-periodic product in $\prod\limits_{i\in I}{}^{n}G_i$.

The operations of the n-periodic product of groups solve Maltsev's problem on the existence of an operation that is associative, exact and hereditary on subgroups in the class of all groups, different from the direct and the free product (see also [4], [17], [13]). The papers [8]–[7] are devoted to the study of some properties of n-periodic products of groups.

The present paper studies normal automorphisms of n-periodic products of finite cyclic groups. First, let us give some definitions.

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Suppose that G is an arbitrary group and $\mathscr{N} = \mathscr{N}(G)$ is the set of all normal subgroups of G. Consider the set

$$Aut_{\mathscr{N}}(G) \rightleftharpoons \{\varphi \in Aut(G) |\, \varphi(H) = H \text{ for all } H \in \mathscr{N}\}.$$

Each automorphism from $Aut_{\mathscr{N}}(G)$ is called a **normal** automorphism. It is easy to see that

$$Inn(G) \trianglelefteq Aut_{\mathscr{N}}(G) \leq Aut(G),$$

where $Inn(G)$ is the group of all inner automorphisms of G.

It is clear that if φ is a normal automorphism of G, then it induces an automorphism of the quotient group ${}^{\mathrm{G}}/_{N}$.

A.Lubotzky showed in [14] that every normal automorphism of the free product of infinite cyclic groups is inner, i.e. the equality $Inn(F) = Aut_{\mathscr{N}}(F)$ is true, where

$$F = \mathbb{Z} * \mathbb{Z} * \cdots * \mathbb{Z}.$$

The equality $Inn(G) = Aut_{\mathscr{N}}(G)$ was proved by several authors for various groups G (see [18] – [6]). For example, Minasyan and Osin showed in [15] that if G is a non-cyclic relatively hyperbolic group without non-trivial finite normal subgroups, then $Inn(G) = Aut_{\mathscr{N}}(G)$. In the paper [6] it is proved that all normal automorphisms of the free Burnside group $B(m,n)$ of rank $m > 1$ and odd period $n \geq 1003$ are inner.

Improving a result of [14], M.V.Neshchadim proved in [16] that every normal automorphism of a free product of nontrivial groups is inner. Note that the corresponding statement is false in general for n-periodic products of groups. This was shown in [9].

The main result of this paper is the following theorem.

Theorem 1.1. *Let $F = \prod_{i\in I}{}^{n}\langle a_i\rangle$ be an n-periodic product of cyclic groups $\langle a_i\rangle$ of odd order $r \geq 1003$, where r divides n. Then each normal automorphism ϕ of the group F is inner.*

2. Some quotient groups of F and auxiliary lemmas

In the proof of Theorem 1.1 we will consider some quotient groups of the group F. For simplicity we denote the generators a_1 and a_2 by a and b respectively. Suppose that $F(n,0)$ is a free group with generators $a_1 = a$, $a_2 = b, a_3$ .... For any natural $\beta > 0$ by $F(n,\beta)$ denote a group with the same generators and system of defining relations $\{a_i^r = 1,\, i \in I\}$, and $\{A^n = 1$, where $A \not\equiv a_i$ for each $i \in I$ and $A \in \bigcup_{i\leq\beta} \mathscr{E}_i\}$ (see [2, ch. VI, §2]).

For $\alpha = 0\,,\,1\,,\,2$ let $\Gamma_\alpha \rightleftharpoons F(n,\alpha)$.

Suppose that $\alpha > 2$ and the groups Γ_δ are already defined for ordinals $\delta < \alpha$. Let Ψ_α be the set of all elementary periods C of rank $\alpha - 1$, satisfying the relation

$$C \overset{\alpha-2}{=} A^{-d}Z^{-1}\,B^{-d}Z\,A^{d}Z^{-1}\,B^{d}Z \tag{2.1}$$

where A and B are minimized elementary periods of ranks γ and β respectively, $Z \in \mathscr{M}_{\alpha-2}, \gamma \leqslant \beta \leqslant \alpha-2, d = 191$ (see [3, §1]). Recall that the period D of rank $\delta \leq \alpha - 1$ is called *minimized*, if the word

D^q is a subword of some word from $\mathscr{M}_{\delta-1}$ (see [3, Definition 2.2]. We choose the subset $\overline{\Psi}_\alpha \subset \Psi_\alpha$ so that each element $C \in \Psi_\alpha$ is conjugate to exactly one word D in the group $\Gamma_{\alpha-2}$, satisfying $D \in \overline{\Psi}_\alpha$ or $D^{-1} \in \overline{\Psi}_\alpha$. This subset $\overline{\Psi}_\alpha$ can be chosen in different ways, and we will use this in what follows.

In [3] the set of words for each period $C \in \overline{\Psi}_\alpha$ and elementary period A of rank $\gamma \le \alpha - 2$, appearing in (2.1), containing exactly two words $C^{200}AC^{200}A^2 \cdots A^{r-1}C^{200}a$, $C^{300}AC^{300}A^2 \cdots A^{r-1}C^{300}b$, is denoted by Φ_α. Unlike in [3], we denote by Φ_α the set of words for each period $C \in \overline{\Psi}_\alpha$ and fixed elementary period A of rank $\gamma \le \alpha - 2$ from (2.1) containing exactly two words

$$C^{200}AC^{200}A^2 \cdots A^{r-1}C^{200}x_C, \tag{2.2}$$

$$C^{300}AC^{300}A^2 \cdots A^{r-1}C^{300}y_C, \tag{2.3}$$

where the elements x_C and y_C are chosen such that one of them is equal to a, and the other is equal to b. Obviously, for each C there are two possibilities (a, b) and (b, a) for the choice of the pair (x_C, y_C). If a concrete pair is not mentioned, then we will assume it is chosen arbitrarily. In the following we will point out concrete values of some pairs (x_C, y_C) (see the definition of $\mathcal{K}'$).

We consider the groups

$$\Gamma_\alpha \rightleftharpoons \left\langle a_1, a_2, \ldots \mid a_i^r = 1, i \in I,\ R^n = 1, F = 1, R \in \bigcup_{\beta \le \alpha} \mathscr{E}_\beta,\ R \not\equiv a_i, i \in I,\ F \in \bigcup_{\beta \le \alpha} \Phi_\beta \right\rangle$$

and

$$\Gamma \rightleftharpoons \left\langle a_1, a_2, \ldots \mid a_i^r = 1, i \in I,\ R^n = 1, F = 1, R \in \bigcup_{\beta > 0} \mathscr{E}_\beta,\ R \not\equiv a_i, i \in I,\ F \in \bigcup_{\beta \ge 3} \Phi_\beta \right\rangle.$$

By $\mathcal{K}$ we denote the class of all groups Γ, which are obtained by the methods described above for different choices of the subset $\overline{\Psi}_\alpha \subset \Psi_\alpha$ and the elements x_C, y_C.

The following lemmas can be proved exactly in the same way as the corresponding statements in the papers [3], [6], [5].

Lemma 2.1. *Each $\Gamma \in \mathcal{K}$ is an infinite group such that every two non-commuting elements generate the whole group Γ.*

Lemma 2.2. *(see Lemma 13 in [6]) If $X^\delta \overset{F}{=} T X^\varepsilon T^{-1}$, then the subgroup $\langle X^\varepsilon, X^\delta, T\rangle_F$ is cyclic.*

Lemma 2.3. *(see Lemma 3 in [5]) If E is an elementary period of rank γ, $Z_1, Z_2 \in \mathscr{M}_\lambda \cap \mathscr{A}_{\lambda+1}$ for some $\lambda \ge \gamma$, $\left[E^d,\ Z^{-1} E^d Z\right] \ne 1$ and the commutators $\left[E^d,\ Z^{-1} E^d Z\right]$ and $\left[E^d,\ Z'^{-1} E^d Z'\right]$ are conjugate in F, where $\alpha \ge \gamma$, then for some integers u and v either $Z'^{-1}E^{-d}Z' = E^u Z^{-1} E^{-d} Z E^v$ or $Z' E^{-d} Z'^{-1} = E^u Z^{-1} E^d Z E^v$ in F.*

Lemma 2.4. *(see Lemma 4 in [5]) For $1 \le |k| \le \frac{r-1}{2}$ each of the commutators*

$$\left[a^k,\ b^{-9} a^k b^9\right] \equiv a^{-k} b^{-9} a^{-k} b^9 a^k b^{-9} a^k b^9,$$

is a minimized elementary period of rank 2.

To obtain the quotient groups that we will use later in the proof of Theorem 1.1, we will add some extra conditions on the set $\overline{\Psi}_3$ of groups from the class $\mathcal{K}$.

Obviously, a is a minimized elementary period of rank 1 and $b^9 \in \mathscr{M}_1$. According to Lemma 2.4 and the definition of the set Ψ_3 we have $\left[a^d,\ b^{-9}\, a^d\, b^9\right] \in \Psi_3$. Denote by $\mathcal{K}'$ the set of all groups $\Gamma \in \mathcal{K}$, for which the following conditions hold:

1. the set $\overline{\Psi}_3$ in the definition of the group Γ is chosen such that $\left[a^d,\ b^{-9}\, a^d\, b^9\right] \in \overline{\Psi}_3$,

2. for the period $C \rightleftharpoons \left[a^d,\ b^{-9}\, a^d\, b^9\right] \in \overline{\Psi}_3$ the elements x_C and y_C appearing in (2.2), (2.3), respectively, are chosen as $x_C \rightleftharpoons b$, $y_C \rightleftharpoons a$,

3. for all the other periods $C \in \overline{\Psi}_\alpha$, $C \overset{\alpha-2}{\neq} \left[a^d,\ b^{-9}\, a^d\, b^9\right]$ the elements x_C and y_C appearing in (2.2) and (2.3), respectively, are chosen as $x_C \rightleftharpoons a$, $y_C \rightleftharpoons b$.

The following lemmas can be proved exactly in the same way as the corresponding statements in [6].

Lemma 2.5. *(see Lemma 7 in [6]) For every $\Gamma \in \mathcal{K}'$ the following relations hold*

$$\left[a^d,\ b^{-9}\, a^d\, b^9\right]^{200} a \left[a^d,\ b^{-9}\, a^d\, b^9\right]^{200} a^2 \cdots a^{(r-1)} \left[a,\ b^{-9}\, a\, b^9\right]^{200} b = 1, \tag{2.4}$$

$$\left[a^d,\ b^{-9}\, a^d\, b^9\right]^{300} a \left[a^d,\ b^{-9}\, a^d\, b^9\right]^{300} a^2 \cdots a^{(r-1)} \left[a,\ b^{-9}\, a\, b^9\right]^{300} a = 1 \tag{2.5}$$

and

$$C^{200} A C^{200} A^2 \cdots A^{n-1} C^{200} a = 1,\ \ C^{300} A C^{300} A^2 \cdots A^{r-1} C^{300} b = 1,$$

for each period $C \in \overline{\Psi}_\alpha$ and $C \overset{\alpha-2}{\neq} \left[a^d,\ b^{-9}\, a^d\, b^9\right]$.

Lemma 2.6. *(see Lemma 11 in [6]) Let $a, b \in \{a_i\}, i \in I$, $\phi : \prod_{i\in I}{}^n \langle a_i\rangle \to \prod_{i\in I}{}^n \langle a_i\rangle$ be a normal automorphism and let $\phi(Z) = b^9$. Then the commutator $[a^d,\ Z^{-1} a^d Z]$ is not a conjugate of $[a^d,\ b^{-9}\, a^d\, b^9]^{-1}$ in the group $\prod_{i\in I}{}^n \langle a_i\rangle$.*

Lemma 2.7. *(see Lemma 8 in [6]) If in the group $\Gamma \in \mathcal{K}'$ the relation*

$$\left[a^k,\ b^{-9}\, a^k\, b^9\right]^s a^t \left[a^k,\ b^{-9}\, a^k\, b^9\right]^s a^{2t} \cdots a^{(r-1)t} \left[a^k,\ b^{-9}\, a^k\, b^9\right]^s a^t = 1$$

holds, where $1 \le |\,k\,|,\ |\,t\,| \le \frac{r-1}{2}$, $k \equiv d \cdot t (\mathrm{mod}\, r)$ and $q + 2 \le s \le \frac{r-1}{2} - 2$, then $k = \pm d$ and $t = \pm 1$.

3. Properties of the normal automorphisms of F

Lemma 3.1. *Let $F = \prod_{i\in I}{}^n \langle a_i\rangle$ be a n-periodic product of cyclic groups $\langle a_i\rangle$ of odd order $r \ge 1003$, where r divides n. If ϕ is a normal automorphism of F, then $\phi(a_i) = u_i a_i^{s_i} u_i^{-1}$ for some $u_i \in F$, and some s_i satisfying $(s_i, r) = 1$, $i \in I$.*

Proof. The operations of an n-periodic product for odd $n \ge 665$ are exact. Therefore, the elements a_i, $i \in I$ have order r in F. Hence, their automorphic images $\phi(a_i)$, $i \in I$ also have order r.

Since ϕ is a normal automorphism, then we have the equalities

$$N_{a_i} = \phi(N_{a_i}) = N_{\phi(a_i)} = N_{u a_j u^{-1}} = N_{a_j},$$

where N_x stands for the normal closure of the element x. According to the obvious relation $a_i \in N_{a_i}$, we get $a_i \in N_{a_j}$.

The sum of the exponents of the letter a_i in any word from the normal closure N_{a_j} is equal to 0 modulo r. Indeed, any defining relation of the group F has the form either a_i^r or A^n, where $A \in F$ is an elementary period of some rank. Thus, the sum of the exponents of the letter a_i in each word from the normal closure N_{a_j} has the form $ur + vn$, which is a multiple of r.

Assuming that $j \neq i$, we obtain that a_i is equal to some word W from the normal closure N_{a_j}. Moreover, the sum of degrees of the letter a_i in W is equal to 0 modulo r. Thus, we get an obvious contradiction.

So, we can conclude that $j = i$. Consequently, we have proved that $\phi(a_i) = ua_i^s u^{-1}$ for some integer s. Applying the automorphism ϕ^{-1} to both sides of this equality we get ${a_i}^{ss_1} = a_i$ for some integer s_1. This implies that $ss_1 \equiv 1(mod\ r)$. The lemma is proved.

□

Lemma 3.2. *Let $a, b \in \{a_i\}, i \in I$, $\phi : F \to F$ be a normal automorphism and let $\phi(a) = a^t$, $\phi(b) = ub^t u^{-1}$. Fix an element Z such that $\phi(Z) = b^9$. Then the commutators $[a^d,\, Z^{-1}a^dZ]$ and $[a^d,\ b^{-9}\,a^d\,b^{\,9}]$ are conjugate in the group $\prod_{i\in I}{}^n\langle a_i\rangle$.*

Proof. We will prove the lemma by way of contradiction. Assume that the commutators

$$[a^d,\, Z^{-1}a^dZ] \text{ and } [a^d,\ b^{-9}\,a^d\,b^{\,9}]$$

are not conjugate in the group $F = \prod_{i\in I}{}^n\langle a_i\rangle$.

Since $\phi(Z) = b^9$, we obtain

$$\phi(\left[a^d,\, Z^{-1}a^dZ\right]) \overset{\prod_{i\in I}{}^n\langle a_i\rangle}{=} \left[a^d,\ b^{-9}\,a^d\,b^{\,9}\right].$$

Then, according to [2, ch. VI, §2, i. 4] and [2, ch. IV, §3, i. 12], one can assume that $Z \in \mathscr{M}_\alpha \cap \mathscr{A}_{\alpha+1}$ for some $\alpha \geq 1$. Choose a reduced form G_1 of the commutator $[a^d,\, Z^{-1}a^dZ]$ according to Lemma 3.2 of [3]. By definition of the reduced forms we have $G_1 \overset{0}{=} w[a^d,\, Z^{-1}a^dZ]w^{-1}$ for some $w \equiv a^j$ (see relation (3.6) in [3]). By Lemma 7.2 of [3] G_1 is an elementary period of some rank $\delta \geq 2$ for each $\Gamma \in \mathcal{K}$. Since by assumption the commutators $[a^d,\, Z^{-1}a^dZ]$ and $[a^d,\ b^{-9}\,a^d\,b^{\,9}]$ are not conjugate in the group $\prod_{i\in I}{}^n\langle a_i\rangle$, using Lemma 11 of [6], we obtain that the elements $[a^d,\, Z^{-1}a^dZ]$ and $[a^d,\ b^{-9}\,a^d\,b^{\,9}]^{\pm1}$ are not conjugate in the group Γ_1. Therefore, there exist groups from the class $\mathcal{K}' \subset \mathcal{K}$, such that $G_1 \in \overline{\Psi}_{\delta+1}$. Let Γ^+ be one of these groups.

By Lemma 2.5, the relations (2.4), (2.5) and

$$G_1^{200}aG_1^{200}a^2\cdots a^{(r-1)}G_1^{200}a = 1, \tag{3.1}$$

$$G_1^{300}aG_1^{300}a^2\cdots a^{(r-1)}G_1^{300}b = 1 \tag{3.2}$$

hold in the group Γ^+.

Since $G_1 \overset{0}{=} a^j[a^d,\ Z^{-1}a^dZ]a^{-j}$, we get $\phi(G_1) \overset{\prod_{i\in I}{}^n\langle a_i\rangle}{=} a^j[a^d,\ b^{-9}a^db^9]a^{-j}$.

From the definition of the group Γ^+, for some normal subgroup N of the group $\prod_{i\in I}{}^n\langle a_i\rangle$ we have $\Gamma^+ = \prod_{i\in I}{}^n\langle a_i\rangle \Big/ N$.

Applying ϕ to both sides of the relation (3.1), we obtain

$$(a^{jt}[a^k,b^{-9}a^kb^9]a^{-jt})^{200}a^t(a^{jt}[a^k,b^{-9}a^kb^9]a^{-jt})^{200}\cdots a^{(r-1)t}(a^{jt}[a^k,b^{-9}a^kb^9]a^{-jt})^{200}\,a^t \in N.$$

Therefore,

$$[a^k,b^{-9}a^kb^9]^{200}a^t[a^k,b^{-9}a^kb^9]^{200}\cdots a^{(r-1)t}[a^k,b^{-9}a^kb^9]^{200}\,a^t \in N,$$

that is

$$[a^k,b^{-9}a^kb^9]^{200}a^t[a^k,b^{-9}a^kb^9]^{200}a^{2t}\cdots a^{(r-1)t}[a^k,b^{-9}a^kb^9]^{200}\,a^t \overset{\Gamma^+}{=} 1.$$

From here, by Lemma 2.7, we obtain that $k=\pm d$ and $t=\pm 1$.

In the case $t=1$ we have $\phi(a)=a$ and $\phi(G_1) \overset{\prod_{i\in I}{}^n\langle a_i\rangle}{=} a^j[a^d,\ b^{-9}a^db^9]a^{-j}$. Applying ϕ to both sides of the relation (3.2), we obtain

$$(a^j[a^d,b^{-9}a^db^9]a^{-j})^{300}a(a^j[a^d,b^{-9}a^db^9]a^{-j})^{300}\cdots a^{(r-1)}(a^j[a^d,b^{-9}a^db^9]a^{-j})^{300}\,u^{-1}bu \in N.$$

Therefore,

$$[a^d,b^{-9}a^db^9]^{300}a[a^d,b^{-9}a^db^9]^{300}a^2\cdots a^{(r-1)}[a^d,b^{-9}a^db^9]^{300}\,a^{-j}u^{-1}bua^j \overset{\Gamma^+}{=} 1.$$

Using the last equality and (2.5), we immediately deduce that the equality $a=a^{-j}u^{-1}bua^j$ holds in the group Γ^+, that is $a \overset{\Gamma^+}{=} u^{-1}bu$. Thus, $\phi(a)\overset{\Gamma^+}{=}\phi(b)$ and hence $\phi(a^{-1}b)\in N$. Since $\phi(N)=N$, we obtain $a^{-1}b\in N$, which implies that Γ^+ is a finite cyclic group. This contradicts the infinity of Γ (see Lemma 2.2).

The case $t=-1$ can be excluded in a similar way, using the relations of the form (2.5). □

Proposition 3.3. *Suppose that $a,b\in\{a_i\}, i\in I$ and $\phi:\prod_{i\in I}{}^n\langle a_i\rangle \to \prod_{i\in I}{}^n\langle a_i\rangle$ is a normal automorphism satisfying $\phi(a)=a^t$, $\phi(b)=ub^tu^{-1}$. Let us fix an element Z such that $\phi(Z)=b^9$. If the commutators $[a^d,\ Z^{-1}a^dZ]$ and $[a^d,\ b^{-9}\,a^d\,b^{\,9}]$ are conjugate in the group $\prod_{i\in I}{}^n\langle a_i\rangle$, then for some integers p,s,l,r we have $Z=a^pb^9a^s$, $t=1$ and $u=b^la^r$.*

Proof. Since the commutators $[a^d,\ Z^{-1}a^dZ]$ and $[a^d,\ b^{-9}\,a^d\,b^{\,9}]$ are conjugate, by Lemma 2.3, we obtain that for some integers r and s either

$$Z^{-1}a^{-d}Z \overset{\prod_{i\in I}{}^n\langle a_i\rangle}{=} a^r\,b^{-9}a^{-d}b^9\,a^s$$

or

$$Za^{-d}Z^{-1} \overset{\prod_{i\in I}{}^n\langle a_i\rangle}{=} a^r\,b^{-9}\,a^db^9\,a^s.$$

Consider each of these cases:

A) Let

$$Za^{-d}Z^{-1} \overset{\prod_{i\in I}{}^{n}\langle a_i\rangle}{=} a^r\, b^{-9}\, a^d b^9\, a^s.$$

Then

$$a^s Za^{-d}Z^{-1}a^{-s} \overset{\prod_{i\in I}{}^{n}\langle a_i\rangle}{=} a^{s+r}\, b^{-9} a^d b^9.$$

If $s+r \not\equiv 0(mod\ r)$, then the word $a^{s+r}\, b^{-9}\, a^{-d} b^9$ is an elementary period of rank 2. Thus, the elementary period $a^{s+r}\, b^9\, a^{-d} b^{-9}$ of rank 2 is conjugate to some power of a, which contradicts Lemma 6.6 from [3]. If $s+r \equiv 0(mod\ r)$, we obtain that a^{-d} and a^d are conjugate, which contradicts Lemma 2.2. Therefore case A) is impossible.

B) Let $Z^{-1}a^{-d}Z \overset{\prod_{i\in I}{}^{n}\langle a_i\rangle}{=} a^r\, b^{-9}a^{-d}b^9\, a^s$. Repeating the reasoning of the previous case, we get $s+r \equiv 0(mod\ r)$ and

$$a^s Z^{-1}a^{-d}Za^{-s} \overset{\prod_{i\in I}{}^{n}\langle a_i\rangle}{=} b^{-9}a^{-d}b^9.$$

This means that the element $b^9 a^s Z^{-1}$ belongs to the centralizer of the element a^{-d} in the group $\prod_{i\in I}{}^{n}\langle a_i\rangle$. Applying Theorem 5 of [1], we get $Z = a^p b^9 a^s$ for some integer p.

Next we prove that $u = b^l a^r$. Applying ϕ to both sides of the equality $Z = a^p b^9 a^s$ we obtain $b^9 = a^{pt}u^{-1}b^{9t}ua^{st}$. Now applying the homomorphism $\alpha : F \to F$ defined by the formulae $\alpha(a) = a$, $\alpha(b) = 1$ to both sides of the last equality, we get $a^{pt+st} = 1$. Since $(t, r) = 1$, then $p \equiv -s(\mathrm{mod} r)$.

Thus, the element $a^p u^{-1}$ belongs to the normalizer of the element b^9. According to Lemma 2.2, this implies $u = b^l a^p$ for some integer l. It remains to show that $t = 1$.

Note that from the equalities $b^9 = a^{pt}u^{-1}b^{9t}ua^{-pt}$ and $u = b^l a^{pt}$ we have that $b^9 = b^{9t}$. Hence,

$$\phi(\left[a^d,\, b^{-9}a^d\, b^9\right]) \overset{B(m,n)}{=} a^{-pt}\left[a^k,\ b^{-9}\, a^k\, b^9\right] a^{pt},$$

for some $k \equiv d\cdot t\,(\mathrm{mod}\, r)$, $(k,\, r) = 1$ and $1 \le |\,k\,| \le \frac{r-1}{2}$.

Suppose that $\Gamma = F/N$ and $\Gamma \in \mathcal{K}'$. Applying the normal automorphism ϕ to the left part of the relation (2.5) and conjugating the obtained element by a^{pt}, we get

$$\left[a^k,\ b^{-9}\, a^k\, b^9\right]^{300} a^t \left[a^k,\ b^{-9}\, a^k\, b^9\right]^{300} a^{2t}\cdots a^{(r-1)t}\left[a^k,\, b^{-9}\, a^{-k}\, b^9\right]^{300}, a^t \in N.$$

From here, by Lemma 2.7, it follows that $k = \pm d$ and $t = \pm 1$. Comparing the equality $b^9 = b^{9t}$ with $t = \pm 1$ we deduce that $t = 1$. The lemma is proved. □

4. **Proof of the main theorem**

Let $\phi : \prod_{i\in I}{}^{n}\langle a_i\rangle \to \prod_{i\in I}{}^{n}\langle a_i\rangle$ be a normal automorphism and $a, b \in \{a_i\}, i \in I$, are such elements that $\phi(a) = a^t$, $\phi(b) = ub^t u^{-1}$. Fix an element Z with $\phi(Z) = b^9$. According to Lemma 2.6, the commutators $[a^d,\, Z^{-1}a^d Z]$ and $[a^d,\ b^{-9}\, a^d\, b^9]$ are not conjugate in the group $\prod_{i\in I}{}^{n}\langle a_i\rangle$. Thus, by Proposition 3.3 we get $Z = a^p b^9 a^s$, $t = 1$ and $u = b^l a^k$ for some integers p, s, l, k. This means that $\phi(a) = a$ and $\phi(b) = a^k ba^{-k}$. Suppose that a_j is one of the generators of the group F, different from

a and b. Arguing as above, we can state that $\phi(a_j) = a^s a_j a^{-s}$ for some $s \in \mathbb{Z}$. It remains to prove that $k \equiv s (mod\ r)$. Let us multiply the automorphism ϕ with inner automorphism generated by the element a^{-k}. We obtain a new normal automorphism ϕ_1, satisfying the conditions $\phi_1(a) = a$, $\phi(b) = b$ and $\phi_1(a_j) = a^{s-k} a_j a^{k-s}$. Applying Proposition 3.3 to the pair b, a_j, we obtain that for some integer m the relation $a^{s-k} a_j a^{-(s-k)} = b^m a_j b^{-m}$ holds in the group F. Finally using the last equality and Lemma 2.2, we obtain the equalities $a^{s-k} = b^m = a_j^l$ in F, for some integer l. But the latter is possible only if $s - k \equiv m \equiv l \equiv 0 (mod\ r)$. This completes the proof of Theorem 1.1.

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