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TOPOLOGICAL LOOPS WITH SOLVABLE MULTIPLICATION GROUPS OF DIMENSION AT MOST SIX ARE CENTRALLY NILPOTENT

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ABSTRACT. The main result of our consideration is the proof of the centrally nilpotency of class two property for connected topological proper loops L of dimension ≤ 3 which have an at most six-dimensional solvable indecomposable Lie group as their multiplication group. This theorem is obtained from our previous classification by the investigation of six-dimensional indecomposable solvable multiplication Lie groups having a five-dimensional nilradical. We determine the Lie algebras of these multiplication groups and the subalgebras of the corresponding inner mapping groups.

1. Introduction

The multiplication group $Mult(L)$ and the inner mapping group $Inn(L)$ of a loop L give important informations about the normal subloop structure of the loop L . If the group $Mult(L)$ is simple, then the loop L is simple and the subgroup $Inn(L)$ of $Mult(L)$ is maximal (cf. [1]). If the group $Mult(L)$ is nilpotent, then the loop L is centrally nilpotent and the group $Inn(L)$ is abelian (see [2]). If the loop L is finite, then the solvability of the group $Mult(L)$ implies that L is classically solvable (cf. [13]). An interesting question is to be analyzed that under which circumstances a group G is the multiplication

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group $Mult(L)$ of a loop L and to be found the groups which are realized as the group $Mult(L)$ of L . A purely group theoretical characterization of multiplication groups is given in [11].

Using the framework of the investigations of P. T. Nagy and K. Strambach in [10] we deal with topological loops L which can be considered as continuous sections $\sigma : G/H \rightarrow G$, where G is a connected Lie group and H is the stabilizer of the identity element $e \in L$ in G . In this case G is a Lie transformation group acting transitively and effectively on L . If L has dimension ≤ 3 and the multiplication group of L is solvable, then L is classically solvable (cf. [7, Theorem 7] and in [3, Theorem 1]). The purpose of this paper is to prove that if the Lie group $Mult(L)$ of L is solvable, indecomposable and has dimension ≤ 6 , then the loop L has nilpotency class 2.

In general the multiplication group $Mult(L)$ for a topological proper loop L has infinite dimension. This is the case if $\dim(L) = 1$ (cf. [10, Theorem 18.18]). In [3] we proved that each 2-dimensional connected topological proper loop L having a Lie group as the group $Mult(L)$ of L is centrally nilpotent of class 2 and precisely the elementary filiform nilpotent Lie groups \mathcal{F}_n of dimension $n \geq 4$ occur as the group $Mult(L)$ of simply connected loops L . In [4]-[7] we focused our attention to 3-dimensional topological loops L such that their multiplication group is a Lie group. We obtained in [4], respectively in [5] that all 3-dimensional connected topological proper loops L having a solvable Lie group of dimension ≤ 5 , respectively an at most 6-dimensional nilpotent Lie group as their multiplication group have nilpotency class 2. The 6-dimensional solvable indecomposable Lie algebras have 4 or 5-dimensional nilradical (cf. [9]). In [7] we showed that the centrally nilpotency of class two property is valid for the 3-dimensional connected topological loops L such that the Lie algebra of their group $Mult(L)$ is a 6-dimensional indecomposable solvable Lie algebra having a 4-dimensional nilradical. To prove that this common feature is relevant for the structure of the connected topological proper loops L such that $\dim(L) \leq 3$ and the group $Mult(L)$ is an at most 6-dimensional indecomposable solvable Lie group it remains an issue for us to investigate the 6-dimensional solvable indecomposable Lie algebras having 5-dimensional nilradical. These Lie algebras depend on at most 4 real parameters and they belong to nine classes according to the different types of their nilradicals. To assert that every connected topological loop L of dimension 3 such that the Lie algebra of the group $Mult(L)$ of L is a 6-dimensional indecomposable solvable Lie algebra with 5-dimensional nilradical has nilpotency class 2 we show that L has a 1-dimensional centre $Z(L)$ and the factor loop $L/Z(L)$ is the abelian group \mathbb{R}^2 (cf. Theorems 3.6, 3.7). Since the linear representations of the simply connected Lie groups for the solvable Lie algebras having a 4-dimensional nilradical are known, we obtained in [7] that only a one-parameter family of Lie groups with abelian nilradical can be represented as the group $Mult(L)$ of L . Using this procedure we could find a realization of the sets of the left translations of the loops L and hence of the loops L . For the solvable Lie algebras having a 5-dimensional nilradical the situation changes. The nilradical of the Lie algebra \mathfrak{g} for the groups $Mult(L)$ of L is not abelian (cf. Proposition 3.5). It is isomorphic

either to the direct sum of the 3-dimensional Heisenberg Lie algebra and \mathbb{R}^2 or to the direct sum of the 4-dimensional elementary filiform Lie algebra and \mathbb{R} or to the 5-dimensional indecomposable Lie algebra such that its 2-dimensional centre coincides with its commutator ideal. In Theorems 3.6, 3.7 we determine the Lie algebras \mathfrak{g} and their abelian subalgebras \mathfrak{k} of the multiplication groups $Mult(L)$ of L and of the corresponding inner mapping groups $Inn(L)$ of $Mult(L)$.

2. Preliminaries

A set L with a binary operation $(x, y) \mapsto x \cdot y$ is called a loop if there exists an element $e \in L$ such that $x = e \cdot x = x \cdot e$ holds for all $x \in L$ and the equations $a \cdot y = b$ and $x \cdot a = b$ have precisely one solution, which we denote by $y = a \setminus b$ and $x = b / a$. A loop L is proper if it is not a group.

The left and right translations $\lambda_a = y \mapsto a \cdot y : L \rightarrow L$ and $\rho_a = y \mapsto y \cdot a : L \rightarrow L$, $a \in L$, are permutations of L . The permutation group $Mult(L) = \langle \lambda_a, \rho_a; a \in L \rangle$ is called the multiplication group of L . The stabilizer of the identity element $e \in L$ in $Mult(L)$ is called the inner mapping group $Inn(L)$ of L . The core $Co_G(H)$ of the subgroup H in the group G is the largest normal subgroup of G contained in H .

The kernel of a homomorphism $\alpha : (L, \cdot) \rightarrow (L', *)$ of a loop L into a loop L' is a normal subloop N of L . The centre $Z(L)$ of a loop L consists of all elements z satisfying $zx \cdot y = z \cdot xy$, $x \cdot yz = xy \cdot z$, $xz \cdot y = x \cdot zy$, $zx = xz$ for all $x, y \in L$. Putting $Z_0 = e$, $Z_1 = Z(L)$ and $Z_i / Z_{i-1} = Z(L / Z_{i-1})$ we obtain a series of normal subloops of L . If Z_{n-1} is a proper subloop of L but $Z_n = L$, then L is centrally nilpotent of class n . A loop L is classically solvable if there exists a series of subloops of L of the form $\{e\} = L_0 \leq L_1 \leq \dots \leq L_n = L$ such that L_{i-1} is a normal subloop in L_i and L_i / L_{i-1} is an abelian group for all $i = 1, \dots, n$. The next assertion was proved in [1], Theorems 3, 4 and 5, in [2], IV.1, Lemma 1.3 and in [5], Lemma 2.3.

Lemma 2.1. *Let L be a loop with multiplication group $Mult(L)$ and identity element e .*

(i) *Let α be a homomorphism of the loop L onto the loop $\alpha(L)$ with kernel N . Then α induces a homomorphism of the group $Mult(L)$ onto the group $Mult(\alpha(L))$. Let $M(N)$ be the set $\{m \in Mult(L); xN = m(x)N \text{ for all } x \in L\}$. Then $M(N)$ is a normal subgroup of $Mult(L)$ containing the multiplication group $Mult(N)$ of the loop N and the multiplication group of the factor loop L/N is isomorphic to $Mult(L)/M(N)$.*

(ii) *For every normal subgroup \mathcal{N} of $Mult(L)$ the orbit $\mathcal{N}(e)$ is a normal subloop of L and $\mathcal{N} \leq M(\mathcal{N}(e))$.*

A loop L is called topological if L is a topological space and the binary operations $(x, y) \mapsto x \cdot y$, $(x, y) \mapsto x \setminus y$, $(x, y) \mapsto y / x : L \times L \rightarrow L$ are continuous. We often use the following lemma, which is proved in [8, IX.1], and in [5, Lemma 2.4].

Lemma 2.2. *For every connected topological loop there exists the universal covering loop. If L is a 3-dimensional connected simply connected topological loop having a solvable Lie group as its multiplication group, then it is homeomorphic to \mathbb{R}^3 .*

A Lie algebra is called indecomposable if it is not the direct sum of two proper ideals. In this paper we look for those 6-dimensional indecomposable solvable Lie algebras with 5-dimensional nilradical which are the Lie algebra of the multiplication group of a 3-dimensional topological loop. Among the 1-dimensional connected simply connected loops only the group \mathbb{R} has a Lie group as its multiplication group (cf. [10, Theorem 18.18]). The elementary filiform Lie group \mathcal{F}_n is the simply connected Lie group of dimension $n \geq 3$ such that its Lie algebra has a basis $\{e_1, \dots, e_n\}$ with $[e_1, e_i] = e_{i+1}$ for $2 \leq i \leq n - 1$. A 2-dimensional simply connected loop $L_{\mathcal{F}}$ is called an elementary filiform loop if its multiplication group is an elementary filiform group \mathcal{F}_n , $n \geq 4$ (cf. [3, p. 2]). Among the 2-dimensional connected simply connected loops only the 2-dimensional Lie groups \mathbb{R}^2 and \mathcal{L}_2 and the elementary filiform loops $L_{\mathcal{F}}$ have a Lie group as their multiplication group. The following lemma is proved in [11, Proposition 2.7].

Lemma 2.3. *Let L be a loop with multiplication group $Mult(L)$ and inner mapping group $Inn(L)$. Then the normalizer $N_{Mult(L)}(Inn(L))$ is the direct product $Inn(L) \times Z(Mult(L))$, where $Z(Mult(L))$ is the centre of the group $Mult(L)$ and the core of $Inn(L)$ in $Mult(L)$ is trivial.*

3. 3-dimensional loops with 6-dimensional indecomposable solvable multiplication groups have nilpotency class 2

This chapter is devoted to showing the following theorem:

Theorem 3.1. *Let L be a connected topological proper loop L of dimension ≤ 3 having an at most 6-dimensional solvable indecomposable Lie group as its multiplication group $Mult(L)$. Then L is centrally nilpotent of class 2.*

In order to prove this theorem we collect the cases which are obtained previously: There does not exist any proper 1-dimensional loop which have a Lie group as its multiplication group (cf. [10, Theorem 18.18, p. 248]). According to [3, Theorem 1. p. 420], every 2-dimensional connected topological proper loop having a Lie group as its multiplication group has nilpotency class 2. Each 3-dimensional connected topological proper loop L having an at most 5-dimensional solvable non-nilpotent Lie group as the group $Mult(L)$ is centrally nilpotent of class 2 (cf. [4, Proposition 17, Theorem 18]). In Theorem of [5] we obtained that the 3-dimensional connected topological proper loops which have an at most 6-dimensional nilpotent Lie group as their multiplication groups are centrally nilpotent of class 2. It is showed in [7, Theorems 14 and 16] that each 3-dimensional connected topological proper loop L such

that the Lie algebra of the group $Mult(L)$ of L is a 6-dimensional indecomposable solvable Lie algebra with 4-dimensional nilradical has nilpotency class 2. Since the nilradical of a solvable indecomposable Lie algebra has dimension 4 or 5 (cf. [9]) to achieve the assertion of Theorem 3.1 it remains to investigate the 6-dimensional indecomposable solvable Lie algebras having a 5-dimensional nilradical. From now on we deal with these Lie algebras. They have only one non-nilpotent basis element (cf. [9]). Hence they have no subalgebra and no factor Lie algebra isomorphic to the direct sum $\mathbf{l}_2 \oplus \mathbf{l}_2$, where \mathbf{l}_2 is the 2-dimensional non-abelian solvable Lie algebra. Therefore if L is a 3-dimensional connected simply connected topological loop having a 6-dimensional solvable indecomposable Lie algebra with 5-dimensional nilradical as the Lie algebra of its multiplication group, then L has no subloop and no factor loop isomorphic to the 2-dimensional non-abelian Lie group \mathcal{L}_2 . Using this fact and summarizing the results on the structure of the 6-dimensional solvable indecomposable Lie groups which are the multiplication group of a 3-dimensional topological loop (cf. [7, Lemma 5, Theorems 6, 9, 10, Proposition 8]) we obtain the following.

Lemma 3.2. *Let L be a 3-dimensional proper connected simply connected topological loop such that the Lie algebra of its multiplication group $Mult(L)$ is a 6-dimensional indecomposable solvable Lie algebra having a 5-dimensional nilradical.*

a) *Then L is classically solvable and it has a 1-dimensional connected normal subloop N . Every such subloop N of L is isomorphic to \mathbb{R} and lies in a 2-dimensional connected normal subloop M of L . The factor loop L/M is isomorphic to \mathbb{R} , whereas M and L/N are isomorphic either to the Lie group \mathbb{R}^2 or to an elementary filiform loop $L_{\mathcal{F}}$.*

b) *The centre Z of the group $Mult(L)$ is isomorphic to the centre $Z(L) = Z(e)$ of the loop L , where e is the identity of L . The centre Z is either discrete or it has dimension 1.*

c) *If $Mult(L)$ has discrete centre, then for every normal subloop $N \cong \mathbb{R}$ of L the factor loop L/N is isomorphic to a loop $L_{\mathcal{F}}$. The group $Mult(L)$ has a normal subgroup S containing $Mult(N) \cong \mathbb{R}$ such that the factor group $Mult(L)/S$ is isomorphic to an elementary filiform Lie group \mathcal{F}_n , $n \geq 4$.*

The loop L has a normal subloop M isomorphic either to \mathbb{R}^2 or to a loop $L_{\mathcal{F}}$ such that N lies in M . The group $Mult(L)$ has a normal subgroup V such that the orbit $V(e)$ is the loop M , $Mult(L)/V \cong \mathbb{R}$, V contains the inner mapping group $Inn(L)$ of L , the group $Mult(M)$ of M and the commutator subgroup of $Mult(L)$.

d) *If $Mult(L)$ has 1-dimensional centre Z , then for every normal subloop $N \cong \mathbb{R}$ of L one has the following possibilities:*

(i) *The factor loop L/N is isomorphic to \mathbb{R}^2 . Then L is centrally nilpotent of class 2, N coincides with the centre $Z(L)$ of L and the group $Mult(L)$ is a semidirect product of a group $Q \cong \mathbb{R}^2$ with the normal subgroup $P = Z \times Inn(L) \cong \mathbb{R}^4$ and one has $P(e) = N = Z(L)$.*

(ii) *The loop L/N is isomorphic to an elementary filiform loop $L_{\mathcal{F}}$. Then we have case c).*

In cases (i), (ii) the loop L has a 2-dimensional normal subloop M as well as the group $\text{Mult}(L)$ has a normal subgroup V as in case c).

The next Proposition is the main technical tool which we systematically use to exclude those Lie algebras which are not the Lie algebra of the multiplication group of a 3-dimensional topological loop.

Proposition 3.3. *Let L be a 3-dimensional connected simply connected topological loop having a 6-dimensional solvable indecomposable Lie algebra \mathfrak{g} with 5-dimensional nilradical $\mathfrak{n}_{\text{rad}}$ as the Lie algebra of its multiplication group.*

a) *For each 1-dimensional ideal \mathfrak{i} of \mathfrak{g} the orbit $I(e)$, where I is the simply connected Lie group of \mathfrak{i} and e is the identity element of L , is a normal subgroup of L isomorphic to \mathbb{R} . We have one of the following possibilities:*

(i) *The factor loop $L/I(e)$ is isomorphic to \mathbb{R}^2 . Then the nilradical contains the ideal $\mathfrak{p} = \mathfrak{z} \oplus \mathfrak{inn}(\mathbf{L}) \cong \mathbb{R}^4$ of \mathfrak{g} such that the commutator ideal \mathfrak{g}' of \mathfrak{g} lies in \mathfrak{p} . Here \mathfrak{z} is the 1-dimensional centre of \mathfrak{g} and $\mathfrak{inn}(\mathbf{L})$ is the Lie algebra of the inner mapping group $\text{Inn}(L)$.*

(ii) *The factor loop $L/I(e)$ is isomorphic to an elementary filiform loop $L_{\mathcal{F}}$. Then there exists an ideal \mathfrak{s} of \mathfrak{g} such that $\mathfrak{i} \leq \mathfrak{s}$ and the factor Lie algebra $\mathfrak{g}/\mathfrak{s}$ is isomorphic to an elementary filiform Lie algebra \mathfrak{f}_4 or \mathfrak{f}_5 .*

b) *Let \mathfrak{a} be an ideal of \mathfrak{g} such that $\dim(\mathfrak{a}) = 2$, $\mathfrak{a} \subseteq \mathfrak{g}'$ and the factor Lie algebra $\mathfrak{g}/\mathfrak{a}$ is not isomorphic to \mathfrak{f}_4 . Then the orbit $A(e)$, where A is the simply connected Lie group of \mathfrak{a} , is either a 2-dimensional connected normal subloop M of L or the factor loop $L/A(e)$ is isomorphic to \mathbb{R}^2 .*

If one has $A(e) = M$, then there exists a 5-dimensional ideal \mathfrak{v} of \mathfrak{g} containing the Lie algebra $\mathfrak{inn}(\mathbf{L})$, the Lie algebra $\mathfrak{mult}(M)$ of the multiplication group of M and the commutator ideal \mathfrak{g}' of \mathfrak{g} . Moreover, for all ideals \mathfrak{b} of \mathfrak{g} such that $\dim(\mathfrak{b}) \geq 3$, $\mathfrak{a} \subset \mathfrak{b} \subseteq \mathfrak{g}'$ the orbit $B(e)$, where B is the simply connected Lie group of \mathfrak{b} , coincides with M . One has $\mathfrak{a} \cap \mathfrak{inn}(\mathbf{L}) = \{0\}$ and the intersection $\mathfrak{b} \cap \mathfrak{inn}(\mathbf{L})$ has dimension $\dim(\mathfrak{b}) - 2$.

If the factor loop $L/A(e)$ is isomorphic to \mathbb{R}^2 , then we have case (i).

Proof. Each 1-dimensional ideal \mathfrak{i} of \mathfrak{g} lies in $\mathfrak{n}_{\text{rad}}$. The orbit $I(e)$ is a connected normal subloop of L (cf. [Lemma]brucklemma) and $I(e) \neq \{e\}$ otherwise I would be a subgroup of the core of $\text{Inn}(L)$ in $\text{Mult}(L)$ which contradicts Lemma 2.3. By Lemma 3.2 a) the orbit $I(e)$ is isomorphic to the group \mathbb{R} . If the factor loop $L/I(e)$ is isomorphic to \mathbb{R}^2 , then the orbit $I(e)$ coincides with the 1-dimensional centre $Z(L)$ of L (cf. Lemma 3.2 d) (i)). The Lie algebra \mathfrak{p} of the normal subgroup P in Lemma 3.2 d) (i) is a 4-dimensional abelian ideal $\mathfrak{p} = \mathfrak{z} \oplus \mathfrak{inn}(\mathbf{L})$ of \mathfrak{g} . As the factor Lie algebra $\mathfrak{g}/\mathfrak{p}$ is abelian (see Lemma 3.2 d) (i)) the commutator ideal \mathfrak{g}' of \mathfrak{g} lies in \mathfrak{p} . The ideal \mathfrak{p} is nilpotent hence one has $\mathfrak{p} \subset \mathfrak{n}_{\text{rad}}$. This proves assertion (i). Assertion (ii) follows from Lemma 3.2 c) and d) (ii).

As $\mathfrak{g}/\mathfrak{n}_{\text{rad}}$ is isomorphic to \mathbb{R} the commutator ideal \mathfrak{g}' of \mathfrak{g} lies in $\mathfrak{n}_{\text{rad}}$. Let \mathfrak{a} be an ideal of \mathfrak{g} as in assertion b). According to Lemmata 2.1, 2.3 the orbit $A(e)$ is a connected normal subloop of L of dimension ≥ 1 .

Firstly, let $A(e)$ be isomorphic to \mathbb{R} . Since $\mathfrak{g}/\mathfrak{a}$ is not isomorphic to \mathfrak{f}_4 , the factor loop $L/A(e)$ is isomorphic to \mathbb{R}^2 . According to Lemma 3.2 d) (i) we have case (i).

Let $A(e)$ be a 2-dimensional connected normal subloop M of L . The 5-dimensional ideal \mathfrak{v} is the Lie algebra of the normal subgroup V in Lemma 3.2 c), d) and \mathfrak{v} has the properties as in assertion b). In particular, one has $V(e) = M$. Let N be the simply connected Lie group of \mathfrak{g}' . As $\mathfrak{a} \subseteq \mathfrak{g}'$ one has $A(e) \subseteq N(e)$. Hence the orbit $N(e)$ is a normal subloop of L having dimension 2 or 3. Furthermore, $N(e)$ is either the subloop M or the loop L . As $\mathfrak{g}' \subseteq \mathfrak{v}$ we obtain that $N(e) = A(e) := M$. Since $\mathfrak{a} \subseteq \mathfrak{g}' \subseteq \mathfrak{n}_{\text{rad}}$, the ideal \mathfrak{a} is nilpotent. As $\dim(\mathfrak{a}) = 2$ the simply connected Lie group A of \mathfrak{a} is \mathbb{R}^2 and it acts sharply transitively on $A(e)$. Hence one has $A \cap \text{Inn}(L) = \{1\}$. As $\mathfrak{a} \subset \mathfrak{b} \subseteq \mathfrak{g}'$ one has $B(e) = M$. Since $\dim(\mathfrak{b}) \geq 3$ and $\dim(B(e)) = 2$ there is a subgroup of B of dimension $\dim(\mathfrak{b}) - 2$, which fixes the identity element e of L . This proves the assertion. \square

Proposition 3.4. *There does not exist any 3-dimensional connected topological loop L such that the Lie algebra of the group $\text{Mult}(L)$ is an indecomposable solvable 6-dimensional Lie algebra having one of the following nilradicals: (a) $[e_2, e_4] = e_3, [e_2, e_5] = e_1, [e_4, e_5] = e_2$; (b) $[e_2, e_4] = e_1, [e_3, e_5] = e_1$; (c) $[e_3, e_4] = e_1, [e_2, e_5] = e_1, [e_3, e_5] = e_2$; (d) $[e_3, e_4] = e_1, [e_2, e_5] = e_1, [e_3, e_5] = e_2, [e_4, e_5] = e_3$.*

Proof. We may assume that L is simply connected and hence it is homeomorphic to \mathbb{R}^3 (cf. Lemma 2.2). The 6-dimensional solvable indecomposable Lie algebras having nilradical as in cases (a) to (d) of the assertion are the Lie algebras $\mathfrak{g}_{6,i}, i = 76, \dots, 99$, in [12], pp. 40-41. The Lie algebras $\mathfrak{g}_{6,i}, i \in \{76, \dots, 99\} \setminus \{80, 81\}$, have the 1-dimensional ideal $\mathfrak{i} = \langle e_1 \rangle$. There does not exist any ideal \mathfrak{s} of $\mathfrak{g}_{6,i}$ such that $\mathfrak{i} \leq \mathfrak{s}$ and the factor Lie algebras $\mathfrak{g}_{6,i}/\mathfrak{s}$ are isomorphic to an elementary filiform Lie algebra $\mathfrak{f}_n, n \in \{4, 5\}$. If $\mathfrak{g}_{6,i}, i \in \{76, \dots, 99\} \setminus \{80, 81\}$, would be the Lie algebra of the multiplication group of L , then by Proposition 3.3 a) the orbit $I(e)$ is isomorphic to \mathbb{R} and the factor loop $L/I(e)$ is isomorphic to \mathbb{R}^2 . In this case the nilradical would contain a 4-dimensional abelian ideal of $\mathfrak{g}_{6,i}$. None of the Lie algebras $\mathfrak{g}_{6,i}, i = 76, \dots, 99$, have a 4-dimensional abelian ideal in their nilradical. Hence these Lie algebras are excluded.

The Lie algebras $\mathfrak{g}_{6,i}^{***}, i \in \{80, 81\}$, have trivial centre and the unique minimal ideal $\mathfrak{s} = \langle e_1, e_3 \rangle$. Let S be the simply connected Lie group of \mathfrak{s} . By Lemma 3.2 a) and c) the orbit $S(e)$ is a normal subgroup of L isomorphic to \mathbb{R} such that the factor loop $L/S(e)$ is isomorphic to a loop $L_{\mathcal{F}}$. Since the factor Lie algebras $\mathfrak{g}_{6,i}^{***}/\mathfrak{s}, i = 80, 81$, are not isomorphic to the elementary filiform Lie algebra \mathfrak{f}_4 we obtain a contradiction. Hence these Lie algebras cannot be the Lie algebra of the group $\text{Mult}(L)$ of L . This yields the assertion. \square

Proposition 3.5. *The solvable indecomposable 6-dimensional Lie algebras having a 5-dimensional abelian nilradical are not the Lie algebra of the multiplication group of a 3-dimensional connected topological loop L .*

Proof. We may assume that L is homeomorphic to \mathbb{R}^3 (cf. Lemma 2.2). The 6-dimensional solvable indecomposable Lie algebras having 5-dimensional abelian nilradical are given in [12], p. 37. All these Lie algebras $\mathfrak{g}_{6,i}$, $i = 1, \dots, 12$ have the 1-dimensional ideal $\mathfrak{i} = \langle e_1 \rangle$. With the exception of the Lie algebra $\mathfrak{g}_{6,4}^{a=0}$ there does not exist any ideal \mathfrak{s} of $\mathfrak{g}_{6,i}$ containing \mathfrak{i} such that the factor Lie algebras $\mathfrak{g}_{6,i}/\mathfrak{s}$ are isomorphic to an elementary filiform Lie algebra \mathfrak{f}_n , $n = 4, 5$. Let I be the simply connected Lie group of the ideal \mathfrak{i} . If $\mathfrak{g}_{6,i}$, $i = 1, \dots, 12$, would be the Lie algebra of the group $Mult(L)$ of L , then the orbit $I(e)$ is isomorphic to \mathbb{R} and the factor loop $L/I(e)$ is isomorphic to \mathbb{R}^2 (cf. Proposition 3.3 a). By Lemma 3.2 d) (i) $I(e)$ coincides with the 1-dimensional centre $Z(L)$ of L . By Proposition 3.3 a) the Lie algebra $\mathbf{inn}(\mathbf{L})$ of the group $Inn(L)$ lies in the 5-dimensional abelian nilradical of $\mathfrak{g}_{6,i}$ which contains the ideal $\mathfrak{p} = \mathfrak{z} \oplus \mathbf{inn}(\mathbf{L}) \cong \mathbb{R}^4$ of $\mathfrak{g}_{6,i}$, where \mathfrak{z} is the 1-dimensional centre of \mathfrak{g} . Then the normalizer $N_{\mathfrak{g}_{6,i}}(\mathbf{inn}(\mathbf{L}))$, $i = 1, \dots, 12$, is the nilradical of $\mathfrak{g}_{6,i}$ which contradicts Lemma 2.3.

If the Lie algebra $\mathfrak{g}_{6,4}^{a=0}$ would be the Lie algebra of the multiplication group of a 3-dimensional loop L , then from the above discussion it follows that the factor loop $L/Z(L)$ is isomorphic to a loop $L_{\mathcal{F}}$. In fact, for the ideal $\mathfrak{s} = \langle e_1, e_5 \rangle$ the factor Lie algebra $\mathfrak{g}_{6,4}^{a=0}/\mathfrak{s}$ is isomorphic to the elementary filiform Lie algebra \mathfrak{f}_4 . Since the orbit $S(e)$, where $S = \exp(\mathfrak{s})$, has dimension 1 we obtain that $\dim(\mathfrak{s} \cap \mathbf{inn}(\mathbf{L})) = 1$. For the simply connected Lie group $I_2 = \{\exp(te_5); t \in \mathbb{R}\}$ of the ideal $\mathfrak{i}_2 = \langle e_5 \rangle$ we obtain that the orbit $I_2(e)$ is a normal subgroup of L isomorphic to \mathbb{R} . Hence one has $\mathfrak{i}_2 \cap \mathbf{inn}(\mathbf{L}) = 0$. The abelian ideals $\mathfrak{a} = \langle e_1, e_2 \rangle$, $\mathfrak{b} = \langle e_1, e_2, e_3 \rangle$, $\mathfrak{g}_{6,4}^{a=0} = \langle e_1, e_2, e_3, e_5 \rangle$ of $\mathfrak{g}_{6,4}^{a=0}$ satisfy the conditions of Proposition 3.3 b). Let A , B and N be the simply connected Lie group of \mathfrak{a} , \mathfrak{b} and $\mathfrak{g}_{6,4}'$. Since $\langle e_1 \rangle = \mathfrak{z} \subset \mathfrak{a}$ the orbit $A(e)$ contains the centre $Z(L)$ of L . If $\dim(A(e)) = 1$, then $A(e) = Z(L)$. As the factor Lie algebra $\mathfrak{g}_{6,4}^{a=0}/\mathfrak{a}$ is not isomorphic to the elementary filiform Lie algebra \mathfrak{f}_4 the factor loop $L/Z(L)$ is not isomorphic to a loop $L_{\mathcal{F}}$.

According to Proposition 3.3 b) the orbit $A(e)$ is a 2-dimensional connected normal subloop M of L containing $Z(L)$ and the orbits $B(e)$ and $N(e)$ coincide with M . Therefore the Lie algebra $\mathbf{inn}(\mathbf{L})$ contains the subalgebra $\langle e_3 + a_1 e_1 + a_2 e_2, e_5 + b_1 e_1 \rangle$, $a_i, b_1 \in \mathbb{R}$, $i = 1, 2$, $b_1 \neq 0$. The ideal \mathfrak{v} in Proposition 3.3 b) has one of the following forms: $\mathfrak{v}_{1,k} = \langle e_1, e_2, e_3, e_5, e_4 + k e_6 \rangle$, $k \in \mathbb{R}$, $\mathfrak{v}_2 = \langle e_1, e_2, e_3, e_5, e_6 \rangle$. Therefore the Lie algebra $\mathbf{inn}(\mathbf{L})$ has as generator either $e_4 + k e_6 + c_1 e_1 + c_2 e_2$ or $e_6 + c_1 e_1 + c_2 e_2$, $k, c_1, c_2 \in \mathbb{R}$. Only the subspace $\langle e_3 + a_1 e_1 + a_2 e_2, e_4 + c_1 e_1 + c_2 e_2, e_5 + b_1 e_1 \rangle \subset \mathfrak{n}_{\text{rad}}$ is a 3-dimensional Lie algebra. Hence it would be the Lie algebra $\mathbf{inn}(\mathbf{L})$. The normalizer $N_{\mathfrak{g}_{6,4}^{a=0}}(\mathbf{inn}(\mathbf{L}))$ equals to $\mathfrak{n}_{\text{rad}}$ which contains $\mathfrak{z} \oplus \mathbf{inn}(\mathbf{L})$ as a proper ideal. This contradiction to Lemma 2.3 yields the assertion. \square

Theorem 3.6. *Let L be a connected topological loop of dimension 3 such that the Lie algebra \mathfrak{g} of its multiplication group is a 6-dimensional indecomposable solvable Lie algebra having one of the following nilradicals: (a) the direct sum $H \oplus \mathbb{R}^2$, where H is the 3-dimensional Heisenberg Lie algebra, (b) the 5-dimensional elementary filiform Lie algebra \mathfrak{f}_5 .*

Then L is centrally nilpotent of class 2 and the nilradical of \mathfrak{g} is $H \oplus \mathbb{R}^2$. The following Lie algebra pairs can occur as the Lie algebra \mathfrak{g} of the group $Mult(L)$ and the subalgebra \mathfrak{k} of the subgroup $Inn(L)$:

$$\mathfrak{g}_1 := \mathfrak{g}_{6,21}^{a=0} : [e_2, e_3] = e_1, [e_2, e_6] = e_3, [e_4, e_6] = e_4, [e_5, e_6] = be_5, 0 < |b| \leq 1, \mathfrak{k}_1 = \langle e_3, e_4 + e_1, e_5 + e_1 \rangle,$$

$$\mathfrak{g}_2 := \mathfrak{g}_{6,22}^{a=0} : [e_2, e_3] = e_1 = [e_5, e_6], [e_2, e_6] = e_3, [e_4, e_6] = e_4, \mathfrak{k}_2 = \langle e_3, e_4 + e_1, e_5 \rangle,$$

$$\mathfrak{g}_3 := \mathfrak{g}_{6,24} : [e_2, e_3] = e_1 = [e_4, e_6], [e_2, e_6] = e_3, [e_3, e_6] = e_4, [e_5, e_6] = e_5, \mathfrak{k}_3 = \langle e_3, e_4, e_5 + e_1 \rangle,$$

$$\mathfrak{g}_4 := \mathfrak{g}_{6,30} : [e_2, e_3] = e_1, [e_2, e_6] = e_3, [e_4, e_6] = e_4 + e_5, [e_5, e_6] = e_5, \mathfrak{k}_4 = \langle e_3, e_4 + a_2e_1, e_5 + e_1 \rangle, a_2 \in \mathbb{R},$$

$$\mathfrak{g}_5 := \mathfrak{g}_{6,36}^{a=0} : [e_2, e_3] = e_1, [e_2, e_6] = e_3, [e_4, e_6] = be_4 + e_5, [e_5, e_6] = be_5 - e_4, b \geq 0, \mathfrak{k}_5 = \langle e_3, e_4, e_5 + e_1 \rangle$$

or $\mathfrak{k}_{5,a_3} = \langle e_3, e_4 + e_1, e_5 + a_3e_1 \rangle, a_3 \in \mathbb{R},$

$$\mathfrak{g}_6 := \mathfrak{g}_{6,14}^{a=b=0} : [e_2, e_3] = e_1 = [e_5, e_6], [e_4, e_6] = e_4, \mathfrak{k}_{6,1} = \langle e_2, e_4 + e_1, e_5 \rangle, \mathfrak{k}_{6,2} = \langle e_3, e_4 + e_1, e_5 \rangle,$$

$$\mathfrak{g}_7 := \mathfrak{g}_{6,17}^{\delta=1, a=\varepsilon=0} : [e_2, e_3] = e_1 = [e_4, e_6], [e_3, e_6] = e_4, [e_5, e_6] = e_5, \mathfrak{k}_{7,1} = \langle e_2, e_4, e_5 + e_1 \rangle,$$

$$\mathfrak{k}_{7,2} = \langle e_3, e_4, e_5 + e_1 \rangle,$$

$$\mathfrak{g}_8 := \mathfrak{g}_{6,15}^{a=0} : [e_2, e_3] = e_1 = [e_1, e_6], [e_2, e_6] = e_2 + e_4, [e_3, e_6] = e_5, [e_4, e_6] = e_4,$$

$$\mathfrak{k}_8 = \langle e_1 + e_5, e_2 + a_2e_5, e_4 + a_3e_5 \rangle, a_3 \in \mathbb{R} \setminus \{0\}, a_2 \in \mathbb{R},$$

$$\mathfrak{g}_9 := \mathfrak{g}_{6,16} : [e_2, e_3] = e_1 = [e_1, e_6], [e_2, e_6] = e_2 + e_4, [e_3, e_6] = e_5, [e_4, e_6] = e_1 + e_4,$$

$$\mathfrak{k}_{9,1} = \langle e_1 + e_5, e_2 + a_2e_5, e_4 + a_3e_5 \rangle, a_2, a_3 \in \mathbb{R},$$

$$\mathfrak{g}_{10} := \mathfrak{g}_{6,17}^{\delta=\varepsilon=0, a \neq 0} : [e_2, e_3] = e_1, [e_1, e_6] = ae_1, [e_2, e_6] = ae_2, [e_3, e_6] = e_4, [e_5, e_6] = e_5,$$

$$\mathfrak{k}_{10} = \langle e_1 + e_4, e_2 + a_2e_4, e_5 + e_4 \rangle, a_2 \in \mathbb{R},$$

$$\mathfrak{g}_{11} := \mathfrak{g}_{6,17}^{\delta=0, a=\varepsilon=1} : [e_2, e_3] = e_1 = [e_1, e_6], [e_2, e_6] = e_2, [e_3, e_6] = e_4, [e_5, e_6] = e_1 + e_5,$$

$$\mathfrak{k}_{11} = \langle e_1 + e_4, e_2 + a_2e_4, e_5 + a_3e_4 \rangle, a_2, a_3 \in \mathbb{R},$$

$$\mathfrak{g}_{12} := \mathfrak{g}_{6,25}^{a=b=0} : [e_2, e_3] = e_1 = [e_1, e_6], [e_2, e_6] = e_2, [e_4, e_6] = e_5, \mathfrak{k}_{12} = \langle e_1 + e_5, e_2 + \varepsilon e_5, e_4 \rangle, \varepsilon = 0, 1,$$

$$\mathfrak{g}_{13} := \mathfrak{g}_{6,27}^{a=1, b=\delta=0} : [e_2, e_3] = e_1 = [e_1, e_6], [e_2, e_6] = e_2, [e_3, e_6] = e_4, [e_4, e_6] = e_5,$$

$$\mathfrak{k}_{13} = \langle e_1 + e_5, e_2 + a_2e_5, e_4 \rangle, a_2 \in \mathbb{R}.$$

Proof. By Lemma 2.2 we may assume that L is homeomorphic to \mathbb{R}^3 . The 6-dimensional solvable Lie algebras having nilradical as in the assertion are in [12], p. 38 and p. 40. The Lie algebra $\mathfrak{g}_{6,27}^{a=1, b=\delta=0}$ has the centre $\mathfrak{i} = \langle e_5 \rangle$. For all other Lie algebras $\mathfrak{g}_{6,i}, i = 13, \dots, 38$ and $71, \dots, 75$ we consider the ideal $\mathfrak{i} = \langle e_1 \rangle$. With exception of the Lie algebras $\mathfrak{g}_{6,23}^{\delta=0}, \mathfrak{g}_{6,24}$ there does not exist any ideal \mathfrak{s} of $\mathfrak{g}_{6,i}$ such that $\mathfrak{i} \subseteq \mathfrak{s}$ and the factor Lie algebras $\mathfrak{g}_{6,i}/\mathfrak{s}$ are isomorphic to an elementary filiform Lie algebra $\mathfrak{f}_n, n = 4, 5$. Let I be the simply connected Lie group of the ideal \mathfrak{i} . If the Lie algebras $\mathfrak{g}_{6,i}, i = 13, \dots, 38$ and $71, \dots, 75$, are the Lie algebra of the group $Mult(L)$ of L , then the orbit $I(e)$ is a normal subloop of L isomorphic to \mathbb{R} , the factor loop $L/I(e)$ is isomorphic to \mathbb{R}^2 and $I(e) = Z(L)$ (cf. Proposition 3.3

a) (i) and Lemma 3.2 d) (i)). Hence the simply connected loop L is a central extension of the group \mathbb{R} by the group \mathbb{R}^2 . This means it is centrally nilpotent of class 2. By Proposition 3.3 a) (i) the Lie algebra $\mathfrak{g}_{6,i}$ has a 4-dimensional abelian ideal $\mathfrak{p} = \mathfrak{z} \oplus \mathfrak{k}$, where \mathfrak{z} is the 1-dimensional centre of $\mathfrak{g}_{6,i}$ and \mathfrak{k} is the Lie algebra of the group $\text{Inn}(L)$ and $\mathfrak{g}'_{6,i} \subset \mathfrak{p}$. Then the nilradical of $\mathfrak{g}_{6,i}$ is the direct sum of the 3-dimensional Heisenberg Lie algebra and \mathbb{R}^2 . According to Lemma 2.3 the subalgebra \mathfrak{k} does not contain any non-zero ideal of $\mathfrak{g}_{6,i}$ and the normalizer $N_{\mathfrak{g}_{6,i}}(\mathfrak{k})$ of \mathfrak{k} in $\mathfrak{g}_{6,i}$ is \mathfrak{p} . Then for the triples $(\mathfrak{g}_{6,i}, \mathfrak{p}, \mathfrak{k})$ we obtain:

(a) The Lie algebras $\mathfrak{g}_{6,i}^{a=0}$, $i = 21, 22, 36$, and $\mathfrak{g}_{6,j}$, $j = 24, 30$, have the centre $\mathfrak{z} = \langle e_1 \rangle$ and \mathfrak{p} is $\langle e_1, e_3, e_4, e_5 \rangle$. The subalgebra \mathfrak{k} has the form: $\mathfrak{k}_{a_1, a_2, a_3} = \langle e_3 + a_1 e_1, e_4 + a_2 e_1, e_5 + a_3 e_1 \rangle$, $a_i \in \mathbb{R}$, $i = 1, 2, 3$, such that in the case $\mathfrak{g}_{6,21}^{a=0}$: $a_2 \neq 0, a_3 \neq 0$ since $\langle e_4 \rangle$ and $\langle e_5 \rangle$ are ideals of $\mathfrak{g}_{6,21}^{a=0}$, in the case $\mathfrak{g}_{6,22}^{a=0}$: $a_2 \neq 0$ because $\langle e_4 \rangle$ is an ideal of $\mathfrak{g}_{6,22}^{a=0}$,

in the cases $\mathfrak{g}_{6,i}$, $i = 24, 30$: $a_3 \neq 0$ since $\langle e_5 \rangle$ is an ideal of $\mathfrak{g}_{6,i}$,

in the case $\mathfrak{g}_{6,36}^{a=0}$: $a_2 \neq 0$ or $a_3 \neq 0$ because $\langle e_4, e_5 \rangle$ is an ideal of $\mathfrak{g}_{6,36}^{a=0}$. Using the automorphism $\alpha(e_i) = e_i$, $i = 1, 2$, $\alpha(e_3) = e_3 - a_1 e_1$, $\alpha(e_4) = a_2 e_4$, $\alpha(e_5) = a_3 e_5$, $\alpha(e_6) = e_6 - a_1 e_3$ for $\mathfrak{g}_{6,21}^{a=0}$, respectively $\alpha(e_5) = e_5 - a_3 e_1$ for $\mathfrak{g}_{6,22}^{a=0}$, respectively $\alpha(e_4) = e_4 - a_2 e_1$, $\alpha(e_6) = e_6 + a_2 e_2 - a_1 e_3$ for $\mathfrak{g}_{6,24}$, respectively $\alpha(e_j) = a_3 e_j$, $j = 4, 5$, for $\mathfrak{g}_{6,30}$, the Lie algebra $\mathfrak{k}_{a_1, a_2, a_3}$ reduces to $\mathfrak{k} = \langle e_3, e_4 + e_1, e_5 + e_1 \rangle$, respectively $\mathfrak{k} = \langle e_3, e_4 + e_1, e_5 \rangle$, respectively $\mathfrak{k} = \langle e_3, e_4, e_5 + e_1 \rangle$, respectively $\mathfrak{k}_{a_2} = \langle e_3, e_4 + a_2 e_1, e_5 + e_1 \rangle$, $a_2 \in \mathbb{R}$. Applying the automorphism $\alpha(e_i) = e_i$, $i = 1, 2$, $\alpha(e_3) = e_3 - a_1 e_1$, $\alpha(e_j) = a_2 e_j$, $j = 4, 5$, $\alpha(e_6) = e_6 - a_1 e_3$ for the Lie algebra $\mathfrak{g}_{6,36}^{a=0}$ if $a_2 \neq 0$, respectively $\alpha(e_j) = a_3 e_j$, $j = 4, 5$, if $a_2 = 0$ and $a_3 \neq 0$ we can reduce $\mathfrak{k}_{a_1, a_2, a_3}$ to $\mathfrak{k}_{a_3} = \langle e_3, e_4 + e_1, e_5 + a_3 e_1 \rangle$, $a_3 \in \mathbb{R}$, respectively $\mathfrak{k}_{a_1, 0, a_3}$ to $\mathfrak{k} = \langle e_3, e_4, e_5 + e_1 \rangle$.

(b) The Lie algebras $\mathfrak{g}_{6,14}^{a=b=0}$ and $\mathfrak{g}_{6,17}^{\delta=1, a=\varepsilon=0}$ have the centre $\mathfrak{z} = \langle e_1 \rangle$ and the ideal \mathfrak{p} has one of the forms: $\mathfrak{p}_{1,k} = \langle e_1, e_2 + k e_3, e_4, e_5 \rangle$, $k \in \mathbb{R}$, and $\mathfrak{p}_2 = \langle e_1, e_3, e_4, e_5 \rangle$. With respect to the ideals $\mathfrak{p}_{1,k}$, \mathfrak{p}_2 we obtain the subalgebras $\mathfrak{k}_{1,k} = \langle e_2 + k e_3 + a_1 e_1, e_4 + a_2 e_1, e_5 + a_3 e_1 \rangle$, $k \in \mathbb{R}$, $\mathfrak{k}_2 = \langle e_3 + a_1 e_1, e_4 + a_2 e_1, e_5 + a_3 e_1 \rangle$, $a_i \in \mathbb{R}$, $i = 1, 2, 3$, such that for $\mathfrak{g}_{6,14}^{a=b=0}$ one has $a_2 \neq 0$ since $\langle e_4 \rangle$ is an ideal of $\mathfrak{g}_{6,14}^{a=b=0}$ and for $\mathfrak{g}_{6,17}^{\delta=1, a=\varepsilon=0}$ we get $a_3 \neq 0$ because $\langle e_5 \rangle$ is an ideal of $\mathfrak{g}_{6,17}^{\delta=1, a=\varepsilon=0}$. The automorphism $\alpha(e_i) = e_i$, $i = 1, 6$, $\alpha(e_4) = a_2 e_4$, $\alpha(e_5) = e_5 - a_3 e_1$, $\alpha(e_2) = e_2 - k e_3 - a_1 e_1$, $\alpha(e_3) = e_3$, respectively $\alpha(e_2) = e_2$, $\alpha(e_3) = e_3 - a_1 e_1$ of $\mathfrak{g}_{6,14}^{a=b=0}$ maps the subalgebra $\mathfrak{k}_{1,k}$ onto $\mathfrak{k} = \langle e_2, e_4 + e_1, e_5 \rangle$, respectively \mathfrak{k}_2 onto $\mathfrak{k} = \langle e_3, e_4 + e_1, e_5 \rangle$. The automorphism $\alpha(e_1) = e_1$, $\alpha(e_4) = e_4 - a_2 e_1$, $\alpha(e_5) = a_3 e_5$, $\alpha(e_6) = e_6 + a_2 e_2$, $\alpha(e_2) = e_2 - a_1 e_1 - k e_3$, $\alpha(e_3) = e_3$, respectively $\alpha(e_3) = e_3 - a_1 e_1$, $\alpha(e_2) = e_2$ of $\mathfrak{g}_{6,17}^{\delta=1, a=\varepsilon=0}$ maps the subalgebra $\mathfrak{k}_{1,k}$ onto $\mathfrak{k} = \langle e_2, e_4, e_5 + e_1 \rangle$, respectively \mathfrak{k}_2 onto $\mathfrak{k} = \langle e_3, e_4, e_5 + e_1 \rangle$.

(c) The Lie algebras $\mathfrak{g}_{6,17}^{\delta=\varepsilon=0, a \neq 0}$ and $\mathfrak{g}_{6,17}^{\delta=0, a=\varepsilon=1}$ have the centre $\mathfrak{z} = \langle e_4 \rangle$ and the ideal \mathfrak{p} equals to $\langle e_1, e_2, e_4, e_5 \rangle$. Hence the subalgebra \mathfrak{k} has the form $\mathfrak{k}_{a_1, a_2, a_3} = \langle e_1 + a_1 e_4, e_2 + a_2 e_4, e_5 + a_3 e_4 \rangle$, $a_i \in \mathbb{R}$, $i = 1, 2, 3$ such that for $\mathfrak{g}_{6,17}^{\delta=\varepsilon=0, a \neq 0}$ we have $a_1 \neq 0, a_3 \neq 0$ since it has the ideals $\langle e_1 \rangle$, $\langle e_5 \rangle$ and for $\mathfrak{g}_{6,17}^{\delta=0, a=\varepsilon=1}$ one obtains $a_1 \neq 0$ because it has the ideal $\langle e_1 \rangle$. With the automorphism $\alpha(e_i) = a_1 e_i$, $i = 1, 2$, $\alpha(e_j) = e_j$, $j = 3, 4, 6$, $\alpha(e_5) = a_3 e_5$ of $\mathfrak{g}_{6,17}^{\delta=\varepsilon=0, a \neq 0}$, respectively $\alpha(e_5) = a_1 e_5$ for $\mathfrak{g}_{6,17}^{\delta=0, a=\varepsilon=1}$

we can change the subalgebra $\mathbf{k}_{a_1, a_2, a_3}$ onto $\mathbf{k}_{a_2} = \langle e_1 + e_4, e_2 + a_2 e_4, e_5 + e_4 \rangle$, $a_2 \in \mathbb{R}$, respectively $\mathbf{k}_{a_2, a_3} = \langle e_1 + e_4, e_2 + a_2 e_4, e_5 + a_3 e_4 \rangle$, $a_2, a_3 \in \mathbb{R}$.

(d) The Lie algebras $\mathfrak{g}_{6,15}^{a=0}$, $\mathfrak{g}_{6,16}$, $\mathfrak{g}_{6,25}^{a=b=0}$, $\mathfrak{g}_{6,27}^{a=1, b=\delta=0}$ have the centre $\mathbf{z} = \langle e_5 \rangle$ and their ideal \mathbf{p} is $\langle e_1, e_2, e_4, e_5 \rangle$. Therefore the subalgebra \mathbf{k} has the form $\mathbf{k}_{a_1, a_2, a_3} = \langle e_1 + a_1 e_5, e_2 + a_2 e_5, e_4 + a_3 e_5 \rangle$, $a_i \in \mathbb{R}$, $i = 1, 2, 3$. For $\mathfrak{g}_{6,15}^{a=0}$ one has $a_1 \neq 0$, $a_3 \neq 0$ since $\langle e_1 \rangle$, $\langle e_4 \rangle$ are ideals of $\mathfrak{g}_{6,15}^{a=0}$. For $\mathfrak{g}_{6,k}$, $k = 16, 25, 27$, we have $a_1 \neq 0$ because $\langle e_1 \rangle$ is an ideal of $\mathfrak{g}_{6,k}$. For $\mathfrak{g}_{6,15}^{a=0}$ and $\mathfrak{g}_{6,16}$ using the automorphism $\alpha(e_i) = a_1 e_i$, $i = 1, 2, 4$, $\alpha(e_j) = e_j$, $j = 3, 5, 6$, the subalgebra $\mathbf{k}_{a_1, a_2, a_3}$ reduces to $\mathbf{k}_{a_2} = \langle e_1 + e_5, e_2 + a_2 e_5, e_4 + a_3 e_5 \rangle$. For $\mathfrak{g}_{6,25}^{a=b=0}$ applying the automorphism $\alpha(e_i) = e_i$, $i = 5, 6$, $\alpha(e_1) = a_1 e_1$, $\alpha(e_4) = e_4 - a_3 e_5$, $\alpha(e_2) = a_2 e_2$, $\alpha(e_3) = \frac{a_1}{a_2} e_3$ if $a_2 \neq 0$, respectively $\alpha(e_2) = e_2$, $\alpha(e_3) = a_1 e_3$ if $a_2 = 0$, we can change the subalgebra $\mathbf{k}_{a_1, a_2, a_3}$ to $\mathbf{k} = \langle e_1 + e_5, e_2 + e_5, e_4 \rangle$, respectively $\mathbf{k}_{a_1, 0, a_3}$ to $\mathbf{k} = \langle e_1 + e_5, e_2, e_4 \rangle$. The automorphism $\alpha(e_i) = e_i$, $i = 5, 6$, $\alpha(e_j) = a_1 e_j$, $j = 1, 2$, $\alpha(e_3) = e_3 - a_3 e_4$, $\alpha(e_4) = e_4 - a_3 e_5$ of $\mathfrak{g}_{6,27}^{a=1, b=\delta=0}$ maps $\mathbf{k}_{a_1, a_2, a_3}$ onto $\mathbf{k}_{a_2} = \langle e_1 + e_5, e_2 + a_2 e_5, e_4 \rangle$.

The Lie algebra $\mathfrak{g}_{6,23}^{\delta=0}$ has 2-dimensional centre. Hence it is excluded by Lemma 3.2 b).

The Lie algebra $\mathfrak{g}_{6,24}$ has the centre $\mathbf{z} = \langle e_1 \rangle$ and the ideals $\mathbf{i}_2 = \langle e_5 \rangle$, $\mathbf{s} = \langle e_1, e_5 \rangle$, $\mathbf{a} = \langle e_1, e_4 \rangle$, $\mathbf{b} = \langle e_1, e_3, e_4 \rangle$, $\mathfrak{g}'_{6,24} = \langle e_1, e_3, e_4, e_5 \rangle$. Let Z, I_2, S, A, B, N be the simply connected Lie groups of $\mathbf{z}, \mathbf{i}_2, \mathbf{s}, \mathbf{a}, \mathbf{b}, \mathfrak{g}'_{6,24}$ in this order. The factor Lie algebra instead of The factor Lie algebras $\mathfrak{g}_{6,24}/\mathbf{s}$ is isomorphic to the elementary filiform Lie algebra \mathbf{f}_4 . If $\mathfrak{g}_{6,24}$ is the Lie algebra of the group $Mult(L)$ of L , then from the above discussion it follows that the factor loop $L/Z(e) = L/S(e)$ is isomorphic to a loop $L_{\mathcal{F}}$. Since $Z(e) = \mathbb{R} = S(e)$ one has $\dim(\mathbf{s} \cap \mathbf{inn}(\mathbf{L})) = 1$. The orbit $I_2(e)$ is a normal subgroup of L isomorphic to \mathbb{R} (cf. Proposition 3.3 a). As $\mathbf{i}_2 \subset \mathbf{s}$ we have $I_2(e) = S(e)$ and $\mathbf{i}_2 \cap \mathbf{inn}(\mathbf{L}) = 0$. For the ideals $\mathbf{a}, \mathbf{b}, \mathfrak{g}'_{6,24}$ the conditions of Proposition 3.3 b) are satisfied. Since $\mathbf{z} \subset \mathbf{a}$ the orbit $A(e)$ contains the centre $Z(e)$ of L . If $\dim(A(e)) = 1$, then $A(e) = Z(e)$. As the factor Lie algebra $\mathfrak{g}_{6,24}/\mathbf{a}$ is not isomorphic to the elementary filiform Lie algebra \mathbf{f}_4 the factor loop $L/Z(e)$ cannot be isomorphic to a loop $L_{\mathcal{F}}$.

According to Proposition 3.3 b) we obtain that $A(e) = B(e) = N(e) = M$, where M is a 2-dimensional connected normal subloop of L such that $\mathbf{a} \cap \mathbf{inn}(\mathbf{L}) = 0$, $\mathbf{b} \cap \mathbf{inn}(\mathbf{L})$ has dimension 1 whereas $\mathfrak{g}'_{6,24} \cap \mathbf{inn}(\mathbf{L})$ has dimension 2 and $Z(e) \subset M$. For the ideal \mathbf{v} in Proposition 3.3 b) we obtain one of the following forms: $\mathbf{v}_{1,k} = \langle e_1, e_3, e_4, e_5, e_2 + k e_6 \rangle$, $k \in \mathbb{R}$, $\mathbf{v}_2 = \langle e_1, e_3, e_4, e_5, e_6 \rangle$. Hence the Lie algebra $\mathbf{inn}(\mathbf{L})$ has either the generators $b_1 = e_3 + a_1 e_1 + a_2 e_4$, $b_2 = e_5 + b_1 e_1$, $b_{3,k} = e_2 + k e_6 + c_1 e_1 + c_2 e_4$ or $b_1, b_2, b_3 = e_6 + c_1 e_1 + c_2 e_4$, $a_i, b_1, k, c_i \in \mathbb{R}$, $i = 1, 2$, $b_1 \neq 0$. None of the vector spaces $\langle b_1, b_2, b_{3,k} \rangle$, $\langle b_1, b_2, b_3 \rangle$ are 3-dimensional Lie algebras. Hence we get a contradiction and the assertion is proved. \square

Theorem 3.7. *Let L be a connected topological loop of dimension 3 such that the Lie algebra \mathfrak{g} of the group $Mult(L)$ is a 6-dimensional solvable indecomposable Lie algebra having nilradical isomorphic either to the direct sum of the 4-dimensional elementary filiform Lie algebra \mathbf{f}_4 and \mathbb{R} or to the Lie algebra defined by $[e_3, e_5] = e_1$, $[e_4, e_5] = e_2$. Then the loop L is centrally nilpotent of class 2 and*

the following pairs can occur as Lie algebra \mathfrak{g} of the group $Mult(L)$ and the subalgebra \mathfrak{k} of the group $Inn(L)$:

$$\mathfrak{g}_1 := \mathfrak{g}_{6,49} : [e_1, e_5] = e_2 = [e_2, e_6], [e_4, e_5] = e_1 = [e_1, e_6], [e_4, e_6] = \varepsilon e_2 + e_4, [e_5, e_6] = e_3, \varepsilon = 0, \pm 1, \\ \mathfrak{k}_1 = \langle e_1 + a_1 e_3, e_2 + e_3, e_4 + a_3 e_3 \rangle, a_1, a_3 \in \mathbb{R},$$

$$\mathfrak{g}_2 := \mathfrak{g}_{6,51} : [e_1, e_5] = e_2, [e_4, e_5] = e_1, [e_3, e_6] = e_3, [e_4, e_6] = \varepsilon e_2, \varepsilon = \pm 1, \mathfrak{g}_3 := \mathfrak{g}_{6,52} : [e_1, e_5] = e_2, \\ [e_4, e_5] = e_1, [e_3, e_6] = e_3, [e_4, e_6] = \varepsilon e_2, [e_5, e_6] = e_4, \varepsilon = 0, \pm 1, \mathfrak{k}_2 = \mathfrak{k}_3 = \langle e_1 + a_1 e_2, e_3 + e_2, e_4 \rangle, \\ a_1 \in \mathbb{R},$$

$$\mathfrak{g}_4 := \mathfrak{g}_{6,54}^{a=b=0} : [e_3, e_5] = e_1 = [e_1, e_6], [e_4, e_5] = e_2, [e_3, e_6] = e_3, \mathfrak{g}_5 := \mathfrak{g}_{6,57}^{a=0} : [e_3, e_5] = e_1 = [e_1, e_6], \\ [e_4, e_5] = e_2, [e_3, e_6] = e_3, [e_5, e_6] = e_4, \mathfrak{g}_6 := \mathfrak{g}_{6,59} : [e_3, e_5] = e_1 = [e_1, e_6], [e_4, e_5] = e_2 = [e_4, e_6], \\ [e_3, e_6] = e_3, [e_5, e_6] = \delta e_4, \delta = 0, 1, \mathfrak{g}_7 := \mathfrak{g}_{6,63}^{a=0} : [e_3, e_5] = e_1 = [e_1, e_6], [e_3, e_6] = e_3, [e_4, e_5] = e_2 = \\ [e_4, e_6], \mathfrak{k}_4 = \mathfrak{k}_5 = \mathfrak{k}_6 = \mathfrak{k}_7 = \langle e_1 + e_2, e_3 + a_2 e_2, e_4 \rangle, a_2 \in \mathbb{R}.$$

Proof. We may assume that L is homeomorphic to \mathbb{R}^3 (cf. Lemma 2.2). The 6-dimensional solvable indecomposable Lie algebras having nilradical as in the assertion are the Lie algebras $\mathfrak{g}_{6,i}$, $i = 39, \dots, 70$, in [12], p. 39.

The Lie algebra $\mathfrak{g}_{6,70}$ has trivial centre and the unique minimal ideal $\mathfrak{i} = \langle e_1, e_2 \rangle$. Let I be the simply connected Lie group of \mathfrak{i} . By Lemma 3.2 a) and c) the orbit $I(e)$ is a 1-dimensional normal subloop of L such that the factor loop $L/I(e)$ is isomorphic to a loop $L_{\mathcal{F}}$. The factor Lie algebra $\mathfrak{g}_{6,70}/\mathfrak{i}$ is not isomorphic to the elementary filiform Lie algebra \mathfrak{f}_4 . Hence the Lie algebra \mathfrak{g}_{70} is not the group $Mult(L)$ of L .

All other Lie algebras have the ideal $\mathfrak{i} = \langle e_2 \rangle$. With the exception of the Lie algebra $\mathfrak{g}_{6,52}$ there does not exist any ideal \mathfrak{s} of $\mathfrak{g}_{6,i}$, $i = 39, \dots, 69$, such that $\mathfrak{i} \leq \mathfrak{s}$ and the factor Lie algebras $\mathfrak{g}_{6,i}/\mathfrak{s}$ are isomorphic to an elementary filiform Lie algebra \mathfrak{f}_n , $n \in \{4, 5\}$. If $\mathfrak{g}_{6,i}$, $i = 39, \dots, 69$, would be the Lie algebra of the group $Mult(L)$ of L , then the simply connected loop L has a 1-dimensional centre $Z(L) = I(e) \cong \mathbb{R}$, where $I = \exp \mathfrak{i}$, and the factor loop $L/I(e)$ is isomorphic to \mathbb{R}^2 (cf. Lemma 3.2 a) and d) (i). Hence L is centrally nilpotent of class 2. According to Proposition 3.3 a) (i) we seek for Lie algebras $\mathfrak{g}_{6,i}$ such that the nilradical of $\mathfrak{g}_{6,i}$ contains an ideal $\mathfrak{p} = \mathfrak{z} \oplus \mathfrak{inn}(\mathbf{L}) \cong \mathbb{R}^4$ of $\mathfrak{g}_{6,i}$ and the commutator ideal $\mathfrak{g}'_{6,i}$ of $\mathfrak{g}_{6,i}$ lies in \mathfrak{p} . Here \mathfrak{z} is the 1-dimensional centre of $\mathfrak{g}_{6,i}$ and $\mathfrak{inn}(\mathbf{L})$ is the Lie algebra of the group $Inn(L)$. By Lemma 2.3 the Lie algebra \mathfrak{k} does not contain any non-zero ideal of $\mathfrak{g}_{6,i}$ and the normalizer $N_{\mathfrak{g}_{6,i}}(\mathfrak{k})$ of \mathfrak{k} in $\mathfrak{g}_{6,i}$ is \mathfrak{p} . The following pairs $(\mathfrak{g}_{6,i}, \mathfrak{k})$ have the above properties:

(a) The Lie algebra $\mathfrak{g}_{6,49}$ has the centre $\mathfrak{z} = \langle e_3 \rangle$ and the ideal \mathfrak{p} is $\langle e_1, e_2, e_3, e_4 \rangle$. Hence for the subalgebra \mathfrak{k} we obtain $\mathfrak{k}_{a_1, a_2, a_3} = \langle e_1 + a_1 e_3, e_2 + a_2 e_3, e_4 + a_3 e_3 \rangle$, $a_2 \neq 0$, because $\langle e_2 \rangle$ is an ideal of $\mathfrak{g}_{6,49}$ and $a_1, a_3 \in \mathbb{R}$. The automorphism $\alpha(e_i) = a_2 e_i$, $i = 1, 2, 4$, $\alpha(e_j) = e_j$, $j = 3, 5, 6$, maps the subalgebra $\mathfrak{k}_{a_1, a_2, a_3}$ onto $\mathfrak{k}_{a_1, a_3} = \langle e_1 + a_1 e_3, e_2 + e_3, e_4 + a_3 e_3 \rangle$.

(b) The Lie algebras $\mathfrak{g}_{6,k}$, $k = 51, 52$, $\mathfrak{g}_{6,54}^{a=b=0}$, $\mathfrak{g}_{6,57}^{a=0}$, $\mathfrak{g}_{6,59}$, $\mathfrak{g}_{6,63}^{a=0}$ have the centre $\mathfrak{z} = \langle e_2 \rangle$ and the ideal \mathfrak{p} equals to $\langle e_1, e_2, e_3, e_4 \rangle$. Hence the Lie algebra \mathfrak{k} has the form $\mathfrak{k}_{a_1, a_2, a_3} = \langle e_1 + a_1 e_2, e_3 + a_2 e_2, e_4 + a_3 e_2 \rangle$,

$a_i \in \mathbb{R}$, $i = 1, 2, 3$, such that $a_2 \neq 0$ for the Lie algebras $\mathfrak{g}_{6,k}$, $k = 51, 52$, since $\langle e_3 \rangle$ is their ideal and $a_1 \neq 0$ for the Lie algebras $\mathfrak{g}_{6,k}$, $k = 54, 57, 59, 63$, because $\langle e_1 \rangle$ is their ideal. Applying the automorphism $\alpha(e_i) = e_i$, $i = 1, 2, 5$, $\alpha(e_3) = a_2 e_3$, $\alpha(e_4) = e_4 - a_3 e_2$, $\alpha(e_6) = e_6$ for $\mathfrak{g}_{6,51}$, respectively $\alpha(e_6) = e_6 + a_3 e_1$ for $\mathfrak{g}_{6,52}$ the subalgebra $\mathfrak{k}_{a_1, a_2, a_3}$ reduces to $\mathfrak{k}_{a_1} = \langle e_1 + a_1 e_2, e_3 + e_2, e_4 \rangle$. The automorphism $\alpha(e_i) = e_i$, $i = 2, 5, 6$, $\alpha(e_j) = a_1 e_j$, $j = 1, 3$, $\alpha(e_4) = e_4 - a_3 e_2$ for $\mathfrak{g}_{6,k}$, $k = 54, 63$, respectively $\alpha(e_6) = e_6 + a_3 e_4$ for $\mathfrak{g}_{6,l}$, $l = 57, 59$, maps $\mathfrak{k}_{a_1, a_2, a_3}$ onto $\mathfrak{k}_{a_2} = \langle e_1 + e_2, e_3 + a_2 e_2, e_4 \rangle$. The Lie algebra $\mathfrak{g}_{6,52}$ has the centre $\mathfrak{z} = \langle e_2 \rangle$ and the ideals $\mathfrak{i}_2 = \langle e_3 \rangle$, $\mathfrak{s} = \langle e_2, e_3 \rangle$, $\mathfrak{a} = \langle e_1, e_2 \rangle$, $\mathfrak{b}_1 = \langle e_1, e_2, e_3 \rangle$, $\mathfrak{b}_2 = \langle e_1, e_2, e_4 \rangle$, $\mathfrak{g}'_{6,52} = \langle e_1, e_2, e_3, e_4 \rangle$. Denote by Z , I_2 , S , A , B_i , $i = 1, 2$, and N the simply connected Lie group of \mathfrak{z} , \mathfrak{i}_2 , \mathfrak{s} , \mathfrak{a} , \mathfrak{b}_i , $i = 1, 2$, and $\mathfrak{g}'_{6,52}$. The factor Lie algebra $\mathfrak{g}_{6,52}/\mathfrak{s}$ is isomorphic to the elementary filiform Lie algebra \mathfrak{f}_4 . If $\mathfrak{g}_{6,52}$ would be the Lie algebra of the group $Mult(L)$ of L , then the above discussion yields that the factor loop $L/Z(e) = L/S(e)$ is isomorphic to a loop $L_{\mathcal{F}}$. As $Z(e) = \mathbb{R} = S(e)$ we have $\dim(\mathfrak{s} \cap \mathbf{inn}(\mathbf{L})) = 1$. Since the orbit $I_2(e)$ is a normal subgroup of L isomorphic to \mathbb{R} and $\mathfrak{i}_2 \subset \mathfrak{s}$ we obtain $I_2(e) = S(e)$ and $\mathfrak{i}_2 \cap \mathbf{inn}(\mathbf{L}) = 0$. The ideals \mathfrak{a} , \mathfrak{b}_i , $i = 1, 2$, and $\mathfrak{g}'_{6,52}$ have the properties as in Proposition 3.3 b). Since $\mathfrak{z} \subset \mathfrak{a}$ one has $Z(e) \subset A(e)$. If $\dim(A(e)) = 1$, then we get $A(e) = Z(e)$. Since the factor Lie algebra $\mathfrak{g}_{6,52}/\mathfrak{a}$ is not isomorphic to the elementary filiform Lie algebra \mathfrak{f}_4 the factor loop $L/Z(e)$ is not isomorphic to a loop $L_{\mathcal{F}}$.

Hence the orbit $A(e)$ is a 2-dimensional connected normal subloop M of L and $B_1(e) = B_2(e) = N(e) = M$ (cf. Proposition 3.3 b) such that $Z(e) \subset A(e)$. The ideal \mathfrak{v} in Proposition 3.3 b) has one of the following forms: $\mathfrak{v}_{1,k} = \langle e_1, e_2, e_3, e_4, e_5 + k e_6 \rangle$, $k \in \mathbb{R}$, $\mathfrak{v}_2 = \langle e_1, e_2, e_3, e_4, e_6 \rangle$. Therefore the Lie algebra $\mathbf{inn}(\mathbf{L})$ has either the basis elements $b_1 = e_3 + a_1 e_2$, $b_2 = e_4 + b_1 e_1 + b_2 e_2$, $b_{3,k} = e_5 + k e_6 + c_1 e_1 + c_2 e_2$ or $b_1, b_2, b_3 = e_6 + c_1 e_1 + c_2 e_2$, $a_1, b_i, k, c_i \in \mathbb{R}$, $i = 1, 2$, $a_1 \neq 0$. The vector spaces $\langle b_1, b_2, b_{3,k} \rangle$, $\langle b_1, b_2, b_3 \rangle$ are not 3-dimensional Lie algebras. This contradiction proves the assertion. \square

If the loop L in Theorems 3.6, 3.7 is not simply connected, then in the universal covering loop \tilde{L} of L there exists a discrete central normal subgroup Z such that L is isomorphic to \tilde{L}/Z . Since every element of Z associates and commutes with any element of \tilde{L} we have $\lambda_z \lambda_g(k) = \lambda_g \lambda_z(k)$ and $\lambda_z \rho_g(k) = \rho_g \lambda_z(k)$ for any $z \in Z$, $g, k \in \tilde{L}$. Hence the set $\tilde{Z} = \{\lambda_z; z \in Z\}$ is a discrete central subgroup of the group $Mult(\tilde{L})$. The centre of $Mult(\tilde{L})$ is isomorphic to \mathbb{R} (cf. Theorems 3.6, 3.7) and hence \tilde{Z} is isomorphic to \mathbb{Z} . Hence the fundamental group of L is isomorphic to \mathbb{Z} . Together with Theorems 14 and 16 in [7] and Theorem in [5] we obtain:

Corollary 3.8. *Every connected not simply connected 3-dimensional topological loop L having an indecomposable solvable Lie group of dimension 6 as the group $Mult(L)$ of L is homeomorphic to $\mathbb{R}^2 \times \mathbb{R}/\mathbb{Z}$ and the centre of the group $Mult(L)$ is isomorphic to the group $SO_2(\mathbb{R})$.*

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