www.theoryofgroups.ir

International Journal of Group Theory
ISSN (print): 2251-7650, ISSN (on-line): 2251-7669
Vol. x No. x (201x), pp. xx-xx.

www.ui.ac.ir

Communicated by

TOPOLOGICAL LOOPS WITH SOLVABLE MULTIPLICATION GROUPS OF DIMENSION AT MOST SIX ARE CENTRALLY NILPOTENT

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Abstract. The main result of our consideration is the proof of the centrally nilpotency of class two property for connected topological proper loops L of dimension ≤ 3 which have an at most six-dimensional solvable indecomposable Lie group as their multiplication group. This theorem is obtained from our previous classification by the investigation of six-dimensional indecomposable solvable multiplication Lie groups having a five-dimensional nilradical. We determine the Lie algebras of these multiplication groups and the subalgebras of the corresponding inner mapping groups.

1. Introduction

The multiplication group $Mult(L)$ and the inner mapping group $Inn(L)$ of a loop L give important informations about the normal subloop structure of the loop L. If the group $Mult(L)$ is simple, then the loop L is simple and the subgroup $Inn(L)$ of $Mult(L)$ is maximal (cf. [1]). If the group $Mult(L)$ is nilpotent, then the loop L is centrally nilpotent and the group $Inn(L)$ is abelian (see [2]). If the loop L is finite, then the solvability of the group $Mult(L)$ implies that L is classically solvable (cf. [13]). An interesting question is to be analyzed that under which circumstances a group G is the multiplication group $Mult(L)$ of a loop L and to be found the groups which are realized as the group $Mult(L)$ of L. A purely group theoretical characterization of multiplication groups is given in [11].

Using the framework of the investigations of P. T. Nagy and K. Strambach in [10] we deal with topological loops L which can be considered as continuous sections $\sigma : G/H \to G$, where G is a connected Lie group and H is the stabilizer of the identity element $e \in L$ in G. In this case G is a Lie transformation group acting transitively and effectively on L. If L has dimension ≤ 3 and the

MSC(2010): 57S20; Secondary: 57M60, 20N05, 17B30, 22E60.

Keywords: Multiplication group and inner mapping group of topological loops, topological transformation group, solvable Lie algebras, centrally nilpotent loops.

Received: dd mmmm yyyy, Accepted: dd mmmm yyyy.

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multiplication group of L is solvable, then L is classically solvable (cf. Theorem 7 in [7] and Theorem 1 in [3]). The purpose of this paper is to prove that if the Lie group $Mult(L)$ of L is solvable, indecomposable and has dimension ≤ 6, then the loop L has nilpotency class 2.

In general the multiplication group $Mult(L)$ for a topological proper loop L has infinite dimension. This is the case if $\dim(L) = 1$ (cf. Theorem 18.18 in [10]). In [3] we proved that each 2-dimensional connected topological proper loop L having a Lie group as the group $Mult(L)$ of L is centrally nilpotent of class 2 and precisely the elementary filiform nilpotent Lie groups $\mathcal{F}_n$ of dimension $n \geq 4$ occur as the group $Mult(L)$ of simply connected loops L. In [4]-[7] we focused our attention to 3-dimensional topological loops L such that their multiplication group is a Lie group. We obtained in [4], respectively in [5] that all 3-dimensional connected topological proper loops L having a solvable Lie group of dimension ≤ 5, respectively an at most 6-dimensional nilpotent Lie group as their multiplication group have nilpotency class 2. The 6-dimensional solvable indecomposable Lie algebras have 4 or 5-dimensional nilradical (cf. [9]). In [7] we showed that the centrally nilpotency of class two property is valid for the 3-dimensional connected topological loops L such that the Lie algebra of their group $Mult(L)$ is a 6-dimensional indecomposable solvable Lie algebra having a 4-dimensional nilradical. To prove that this common feature is relevant for the structure of the connected topological proper loops L such that $\dim(L) \leq 3$ and the group $Mult(L)$ is an at most 6-dimensional indecomposable solvable Lie group it remains an issue for us to investigate the 6-dimensional solvable indecomposable Lie algebras having 5-dimensional nilradical. These Lie algebras depend on at most 4 real parameters and they belong to nine classes according to the different types of their nilradicals. To assert that every connected topological loop L of dimension 3 such that the Lie algebra of the group $Mult(L)$ of L is a 6-dimensional indecomposable solvable Lie algebra with 5-dimensional nilradical has nilpotency class 2 we show that L has a 1-dimensional centre $Z(L)$ and the factor loop $L/Z(L)$ is the abelian group $\mathbb{R}^2$ (cf. Theorems 3.6, 3.7). Since the linear representations of the simply connected Lie groups for the solvable Lie algebras having a 4-dimensional nilradical are known, we obtained in [7] that only a one-parameter family of Lie groups with abelian nilradical can be represented as the group $Mult(L)$ of L. Using this procedure we could find a realization of the sets of the left translations of the loops L and hence of the loops L. For the solvable Lie algebras having a 5-dimensional nilradical the situation changes. The nilradical of the Lie algebra $\mathbf{g}$ for the groups $Mult(L)$ of L is not abelian (cf. Proposition 3.5). It is isomorphic either to the direct sum of the 3-dimensional Heisenberg Lie algebra and $\mathbb{R}^2$ or to the direct sum of the 4-dimensional elementary filiform Lie algebra and $\mathbb{R}$ or to the 5-dimensional indecomposable Lie algebra such that its 2-dimensional centre coincides with its commutator ideal. In Theorems 3.6, 3.7 we determine the Lie algebras $\mathbf{g}$ and their abelian subalgebras $\mathbf{k}$ of the multiplication groups $Mult(L)$ of L and of the corresponding inner mapping groups $Inn(L)$ of $Mult(L)$.

2. **Preliminaries**

A set L with a binary operation $(x, y) \mapsto x \cdot y$ is called a loop if there exists an element $e \in L$ such that $x = e \cdot x = x \cdot e$ holds for all $x \in L$ and the equations $a \cdot y = b$ and $x \cdot a = b$ have precisely one solution, which we denote by $y = a \backslash b$ and $x = b/a$. A loop L is proper if it is not a group.
The left and right translations $\lambda_a = y \mapsto a \cdot y : L \to L$ and $\rho_a : y \mapsto y \cdot a : L \to L$, $a \in L$, are permutations of L. The permutation group $Mult(L) = \langle \lambda_a, \rho_a;\ a \in L \rangle$ is called the multiplication group of L. The stabilizer of the identity element $e \in L$ in $Mult(L)$ is called the inner mapping group $Inn(L)$ of L. The core $Co_G(H)$ of the subgroup H in the group G is the largest normal subgroup of G contained in H.

The kernel of a homomorphism $\alpha : (L, \cdot) \to (L', *)$ of a loop L into a loop L' is a normal subloop N of L. The centre $Z(L)$ of a loop L consists of all elements z satisfying $zx \cdot y = z \cdot xy$, $x \cdot yz = xy \cdot z$, $xz \cdot y = x \cdot zy$, $zx = xz$ for all $x, y \in L$. Putting $Z_0 = e$, $Z_1 = Z(L)$ and $Z_i/Z_{i-1} = Z(L/Z_{i-1})$ we obtain a series of normal subloops of L. If Z_{n-1} is a proper subloop of L but $Z_n = L$, then L is centrally nilpotent of class n. A loop L is classically solvable if there exists a series of subloops of L of the form $\{e\} = L_0 \leq L_1 \leq \cdots \leq L_n = L$ such that L_{i-1} is a normal subloop in L_i and L_i/L_{i-1} is an abelian group for all $i = 1, \ldots, n$. The next assertion was proved in [1], Theorems 3, 4 and 5, in [2], IV.1, Lemma 1.3 and in [5], Lemma 2.3.

Lemma 2.1. *Let L be a loop with multiplication group $Mult(L)$ and identity element e.*
(i) Let α be a homomorphism of the loop L onto the loop $\alpha(L)$ with kernel N. Then α induces a homomorphism of the group $Mult(L)$ onto the group $Mult(\alpha(L))$. Let $M(N)$ be the set $\{m \in Mult(L);\ xN = m(x)N \text{ for all } x \in L\}$. Then $M(N)$ is a normal subgroup of $Mult(L)$ containing the multiplication group $Mult(N)$ of the loop N and the multiplication group of the factor loop L/N is isomorphic to $Mult(L)/M(N)$.
(ii) For every normal subgroup $\mathcal{N}$ of $Mult(L)$ the orbit $\mathcal{N}(e)$ is a normal subloop of L and $\mathcal{N} \leq M(\mathcal{N}(e))$.

A loop L is called topological if L is a topological space and the binary operations $(x, y) \mapsto x \cdot y$, $(x, y) \mapsto x \backslash y$, $(x, y) \mapsto y/x : L \times L \to L$ are continuous. We often use the following lemma, which is proved in [8], IX.1, and in [5], Lemma 2.4.

Lemma 2.2. *For every connected topological loop there exists the universal covering loop. If L is a 3-dimensional connected simply connected topological loop having a solvable Lie group as its multiplication group, then it is homeomorphic to $\mathbb{R}^3$.*

A Lie algebra is called indecomposable if it is not the direct sum of two proper ideals. In this paper we look for those 6-dimensional indecomposable solvable Lie algebras with 5-dimensional nilradical which are the Lie algebra of the multiplication group of a 3-dimensional topological loop. Among the 1-dimensional connected simply connected loops only the group $\mathbb{R}$ has a Lie group as its multiplication group (cf. [10], Theorem 18.18). The elementary filiform Lie group $\mathcal{F}_n$ is the simply connected Lie group of dimension $n \geq 3$ such that its Lie algebra has a basis $\{e_1, \cdots, e_n\}$ with $[e_1, e_i] = e_{i+1}$ for $2 \leq i \leq n-1$. A 2-dimensional simply connected loop $L_{\mathcal{F}}$ is called an elementary filiform loop if its multiplication group is an elementary filiform group $\mathcal{F}_n$, $n \geq 4$ (cf. [3], p. 2). Among the 2-dimensional connected simply connected loops only the 2-dimensional Lie groups $\mathbb{R}^2$ and $\mathcal{L}_2$ and the elementary filiform loops $L_{\mathcal{F}}$ have a Lie group as their multiplication group. The following lemma is proved in [11], Proposition 2.7.

Lemma 2.3. *Let L be a loop with multiplication group $Mult(L)$ and inner mapping group $Inn(L)$. Then the normalizer $N_{Mult(L)}(Inn(L))$ is the direct product $Inn(L) \times Z(Mult(L))$, where $Z(Mult(L))$ is the centre of the group $Mult(L)$ and the core of $Inn(L)$ in $Mult(L)$ is trivial.*

3. **3-dimensional loops with 6-dimensional indecomposable solvable multiplication groups have nilpotency class 2**

This chapter is devoted to showing the following theorem:

Theorem 3.1. *Let L be a connected topological proper loop L of dimension ≤ 3 having an at most 6-dimensional solvable indecomposable Lie group as its multiplication group $Mult(L)$. Then L is centrally nilpotent of class 2.*

In order to prove this theorem we collect the cases which are obtained previously: There does not exists any proper 1-dimensional loop which have a Lie group as its multiplication group (cf. [10], Theorem 18.18, p. 248). According to Theorem 1 in [3], p. 420, every 2-dimensional connected topological proper loop having a Lie group as its multiplication group has nilpotency class 2. Each 3-dimensional connected topological proper loop L having an at most 5-dimensional solvable non-nilpotent Lie group as the group $Mult(L)$ is centrally nilpotent of class 2 (cf. Proposition 17, Theorem 18 in [4]). In Theorem of [5] we obtained that the 3-dimensional connected topological proper loops which have an at most 6-dimensional nilpotent Lie group as their multiplication groups are centrally nilpotent of class 2. It is showed in Theorems 14 and 16 of [7] that each 3-dimensional connected topological proper loop L such that the Lie algebra of the group $Mult(L)$ of L is a 6-dimensional indecomposable solvable Lie algebra with 4-dimensional nilradical has nilpotency class 2. Since the nilradical of a solvable indecomposable Lie algebra has dimension 4 or 5 (cf. [9]) to achieve the assertion of Theorem 3.1 it remains to investigate the 6-dimensional indecomposable solvable Lie algebras having a 5-dimensional nilradical. From now on we deal with these Lie algebras. They have only one non-nilpotent basis element (cf. [9]). Hence they have no subalgebra and no factor Lie algebra isomorphic to the direct sum $\mathbf{l_2} \oplus \mathbf{l_2}$, where $\mathbf{l_2}$ is the 2-dimensional non-abelian solvable Lie algebra. Therefore if L is a 3-dimensional connected simply connected topological loop having a 6-dimensional solvable indecomposable Lie algebra with 5-dimensional nilradical as the Lie algebra of its multiplication group, then L has no subloop and no factor loop isomorphic to the 2-dimensional non-abelian Lie group $\mathcal{L}_2$. Using this fact and summarizing the results on the structure of the 6-dimensional solvable indecomposable Lie groups which are the multiplication group of a 3-dimensional topological loop (cf. Lemma 5, Theorems 6, 9, 10, Proposition 8 in [7]) we obtain the following.

Lemma 3.2. *Let L be a 3-dimensional proper connected simply connected topological loop such that the Lie algebra of its multiplication group $Mult(L)$ is a 6-dimensional indecomposable solvable Lie algebra having a 5-dimensional nilradical.*

a) Then L is classically solvable and it has a 1-dimensional connected normal subloop N. Every such subloop N of L is isomorphic to $\mathbb{R}$ and lies in a 2-dimensional connected normal subloop M of L. The factor loop L/M is isomorphic to $\mathbb{R}$, whereas M and L/N are isomorphic either to the Lie group $\mathbb{R}^2$ or to an elementary filiform loop $L_{\mathcal{F}}$.

b) The centre Z of the group $Mult(L)$ is isomorphic to the centre $Z(L) = Z(e)$ of the loop L, where e is the identity of L. The centre Z is either discrete or it has dimension 1.

c) If $Mult(L)$ has discrete centre, then for every normal subloop $N \cong \mathbb{R}$ of L the factor loop L/N is isomorphic to a loop $L_{\mathcal{F}}$. The group $Mult(L)$ has a normal subgroup S containing $Mult(N) \cong \mathbb{R}$ such that the factor group $Mult(L)/S$ is isomorphic to an elementary filiform Lie group $\mathcal{F}_n$, $n \geq 4$.

The loop L has a normal subloop M isomorphic either to $\mathbb{R}^2$ or to a loop $L_{\mathcal{F}}$ such that N lies in M. The group $Mult(L)$ has a normal subgroup V such that the orbit $V(e)$ is the loop M, $Mult(L)/V \cong \mathbb{R}$, V contains the inner mapping group $Inn(L)$ of L, the group $Mult(M)$ of M and the commutator subgroup of $Mult(L)$.

d) If $Mult(L)$ has 1-dimensional centre Z, then for every normal subloop $N \cong \mathbb{R}$ of L one has the following possibilities:

(i) The factor loop L/N is isomorphic to $\mathbb{R}^2$. Then L is centrally nilpotent of class 2, N coincides with the centre $Z(L)$ of L and the group $Mult(L)$ is a semidirect product of a group $Q \cong \mathbb{R}^2$ with the normal subgroup $P = Z \times Inn(L) \cong \mathbb{R}^4$ and one has $P(e) = N = Z(L)$.

(ii) The loop L/N is isomorphic to an elementary filiform loop $L_{\mathcal{F}}$. Then we have case c).

In cases (i), (ii) the loop L has a 2-dimensional normal subloop M as well as the group $Mult(L)$ has a normal subgroup V as in case c).

The next Proposition is the main technical tool which we systematically use to exclude those Lie algebras which are not the Lie algebra of the multiplication group of a 3-dimensional topological loop.

Proposition 3.3. *Let L be a 3-dimensional connected simply connected topological loop having a 6-dimensional solvable indecomposable Lie algebra $\mathbf{g}$ with 5-dimensional nilradical $\mathbf{n}_{rad}$ as the Lie algebra of its multiplication group.*

a) For each 1-dimensional ideal $\mathbf{i}$ of $\mathbf{g}$ the orbit $I(e)$, where I is the simply connected Lie group of $\mathbf{i}$ and e is the identity element of L, is a normal subgroup of L isomorphic to $\mathbb{R}$. We have one of the following possibilities:

(i) The factor loop $L/I(e)$ is isomorphic to $\mathbb{R}^2$. Then the nilradical contains the ideal $\mathbf{p} = \mathbf{z} \oplus \mathbf{inn}(\mathbf{L}) \cong \mathbb{R}^4$ of $\mathbf{g}$ such that the commutator ideal $\mathbf{g}'$ of $\mathbf{g}$ lies in $\mathbf{p}$. Here $\mathbf{z}$ is the 1-dimensional centre of $\mathbf{g}$ and $\mathbf{inn}(\mathbf{L})$ is the Lie algebra of the inner mapping group $Inn(L)$.

(ii) The factor loop $L/I(e)$ is isomorphic to an elementary filiform loop $L_{\mathcal{F}}$. Then there exists an

ideal $\mathbf{s}$ *of* $\mathbf{g}$ *such that* $\mathbf{i} \leq \mathbf{s}$ *and the factor Lie algebra* $\mathbf{g}/\mathbf{s}$ *is isomorphic to an elementary filiform Lie algebra* $\mathbf{f}_4$ *or* $\mathbf{f}_5$*.*
b) Let $\mathbf{a}$ *be an ideal of* $\mathbf{g}$ *such that dim*$(\mathbf{a}) = 2$*,* $\mathbf{a} \subseteq \mathbf{g}'$ *and the factor Lie algebra* $\mathbf{g}/\mathbf{a}$ *is not isomorphic to* $\mathbf{f}_4$*. Then the orbit* $A(e)$*, where* A *is the simply connected Lie group of* $\mathbf{a}$*, is either a 2-dimensional connected normal subloop* M *of* L *or the factor loop* $L/A(e)$ *is isomorphic to* $\mathbb{R}^2$*.*

If one has $A(e) = M$*, then there exists a 5-dimensional ideal* $\mathbf{v}$ *of* $\mathbf{g}$ *containing the Lie algebra* $\mathbf{inn}(\mathbf{L})$*, the Lie algebra* $\mathbf{mult}(M)$ *of the multiplication group of* M *and the commutator ideal* $\mathbf{g}'$ *of* $\mathbf{g}$*. Moreover, for all ideals* $\mathbf{b}$ *of* $\mathbf{g}$ *such that dim*$(\mathbf{b}) \geq 3$*,* $\mathbf{a} \subset \mathbf{b} \subseteq \mathbf{g}'$ *the orbit* $B(e)$*, where* B *is the simply connected Lie group of* $\mathbf{b}$*, coincides with* M*. One has* $\mathbf{a} \cap \mathbf{inn}(\mathbf{L}) = \{0\}$ *and the intersection* $\mathbf{b} \cap \mathbf{inn}(\mathbf{L})$ *has dimension dim*$(\mathbf{b}) - 2$*.*

If the factor loop $L/A(e)$ *is isomorphic to* $\mathbb{R}^2$*, then we have case (i).*

Proof. Each 1-dimensional ideal $\mathbf{i}$ of $\mathbf{g}$ lies in $\mathbf{n}_{\mathrm{rad}}$. The orbit $I(e)$ is a connected normal subloop of L (cf. Lemma 2.1) and $I(e) \neq \{e\}$ otherwise I would be a subgroup of the core of $Inn(L)$ in $Mult(L)$ which contradicts Lemma 2.3. By Lemma 3.2 a) the orbit $I(e)$ is isomorphic to the group $\mathbb{R}$. If the factor loop $L/I(e)$ is isomorphic to $\mathbb{R}^2$, then the orbit $I(e)$ coincides with the 1-dimensional centre $Z(L)$ of L (cf. Lemma 3.2 d) (i)). The Lie algebra $\mathbf{p}$ of the normal subgroup P in Lemma 3.2 d) (i) is a 4-dimensional abelian ideal $\mathbf{p} = \mathbf{z} \oplus \mathbf{inn}(\mathbf{L})$ of $\mathbf{g}$. As the factor Lie algebra $\mathbf{g}/\mathbf{p}$ is abelian (see Lemma 3.2 d) (i)) the commutator ideal $\mathbf{g}'$ of $\mathbf{g}$ lies in $\mathbf{p}$. The ideal $\mathbf{p}$ is nilpotent hence one has $\mathbf{p} \subset \mathbf{n}_{\mathrm{rad}}$. This proves assertion (i). Assertion (ii) follows from Lemma 3.2 c) and d) (ii).
As $\mathbf{g}/\mathbf{n}_{\mathrm{rad}}$ is isomorphic to $\mathbb{R}$ the commutator ideal $\mathbf{g}'$ of $\mathbf{g}$ lies in $\mathbf{n}_{\mathrm{rad}}$. Let $\mathbf{a}$ be an ideal of $\mathbf{g}$ as in assertion b). According to Lemmata 2.1, 2.3 the orbit $A(e)$ is a connected normal subloop of L of dimension ≥ 1.
Firstly, let $A(e)$ be isomorphic to $\mathbb{R}$. Since $\mathbf{g}/\mathbf{a}$ is not isomorphic to $\mathbf{f}_4$, the factor loop $L/A(e)$ is isomorphic to $\mathbb{R}^2$. According to Lemma 3.2 d) (i) we have case (i).
Let $A(e)$ be a 2-dimensional connected normal subloop M of L. The 5-dimensional ideal $\mathbf{v}$ is the Lie algebra of the normal subgroup V in Lemma 3.2 c), d) and $\mathbf{v}$ has the properties as in assertion b). In particular, one has $V(e) = M$. Let N be the simply connected Lie group of $\mathbf{g}'$. As $\mathbf{a} \subseteq \mathbf{g}'$ one has $A(e) \subseteq N(e)$. Hence the orbit $N(e)$ is a normal subloop of L having dimension 2 or 3. Furthermore, $N(e)$ is either the subloop M or the loop L. As $\mathbf{g}' \subseteq \mathbf{v}$ we obtain that $N(e) = A(e) := M$. Since $\mathbf{a} \subseteq \mathbf{g}' \subseteq \mathbf{n}_{\mathrm{rad}}$, the ideal $\mathbf{a}$ is nilpotent. As $\dim(\mathbf{a}) = 2$ the simply connected Lie group A of $\mathbf{a}$ is $\mathbb{R}^2$ and it acts sharply transitively on $A(e)$. Hence one has $A \cap Inn(L) = \{1\}$. As $\mathbf{a} \subset \mathbf{b} \subseteq \mathbf{g}'$ one has $B(e) = M$. Since $\dim(\mathbf{b}) \geq 3$ and $\dim(B(e)) = 2$ there is a subgroup of B of dimension $\dim(\mathbf{b}) - 2$, which fixes the identity element e of L. This proves the assertion. □

Proposition 3.4. *There does not exist any 3-dimensional connected topological loop* L *such that the Lie algebra of the group* $Mult(L)$ *is an indecomposable solvable 6-dimensional Lie algebra having one of the following nilradicals: (a)* $[e_2, e_4] = e_3$*,* $[e_2, e_5] = e_1$*,* $[e_4, e_5] = e_2$*; (b)* $[e_2, e_4] = e_1$*,* $[e_3, e_5] = e_1$*; (c)* $[e_3, e_4] = e_1$*,* $[e_2, e_5] = e_1$*,* $[e_3, e_5] = e_2$*; (d)* $[e_3, e_4] = e_1$*,* $[e_2, e_5] = e_1$*,* $[e_3, e_5] = e_2$*,* $[e_4, e_5] = e_3$*.*

Proof. We may assume that L is simply connected and hence it is homeomorphic to $\mathbb{R}^3$ (cf. Lemma 2.2). The 6-dimensional solvable indecomposable Lie algebras having nilradical as in cases (a) to (d) of the assertion are the Lie algebras $\mathbf{g}_{6,i}$, $i = 76, \dots, 99$, in [12], pp. 40-41. The Lie algebras $\mathbf{g}_{6,i}$, $i \in \{76, \dots, 99\} \setminus \{80, 81\}$, have the 1-dimensional ideal $\mathbf{i} = \langle e_1 \rangle$. There does not exist any ideal $\mathbf{s}$ of $\mathbf{g}_{6,i}$ such that $\mathbf{i} \le \mathbf{s}$ and the factor Lie algebras $\mathbf{g}_{6,i}/\mathbf{s}$ are isomorphic to an elementary filiform Lie algebra $\mathbf{f}_n$, $n \in \{4,5\}$. If $\mathbf{g}_{6,i}$, $i \in \{76, \dots, 99\} \setminus \{80, 81\}$, would be the Lie algebra of the multiplication group of L, then by Proposition 3.3 a) the orbit $I(e)$ is isomorphic to $\mathbb{R}$ and the factor loop $L/I(e)$ is isomorphic to $\mathbb{R}^2$. In this case the nilradical would contain a 4-dimensional abelian ideal of $\mathbf{g}_{6,i}$. None of the Lie algebras $\mathbf{g}_{6,i}$, $i = 76, \dots, 99$, have a 4-dimensional abelian ideal in their nilradical. Hence these Lie algebras are excluded.

The Lie algebras $\mathbf{g}'^{***}_{6,i}$, $i \in \{80, 81\}$, have trivial centre and the unique minimal ideal $\mathbf{s} = \langle e_1, e_3 \rangle$. Let S be the simply connected Lie group of $\mathbf{s}$. By Lemma 3.2 a) and c) the orbit $S(e)$ is a normal subgroup of L isomorphic to $\mathbb{R}$ such that the factor loop $L/S(e)$ is isomorphic to a loop $L_{\mathcal{F}}$. Since the factor Lie algebras $\mathbf{g}'^{***}_{6,i}/\mathbf{s}$, $i = 80, 81$, are not isomorphic to the elementary filiform Lie algebra $\mathbf{f}_4$ we obtain a contradiction. Hence these Lie algebras cannot be the Lie algebra of the group $Mult(L)$ of L. This yields the assertion. □

Proposition 3.5. *The solvable indecomposable* 6*-dimensional Lie algebras having a* 5*-dimensional abelian nilradical are not the Lie algebra of the multiplication group of a* 3*-dimensional connected topological loop* L*.*

Proof. We may assume that L is homeomorphic to $\mathbb{R}^3$ (cf. Lemma 2.2). The 6-dimensional solvable indecomposable Lie algebras having 5-dimensional abelian nilradical are given in [12], p. 37. All these Lie algebras $\mathbf{g}_{6,i}$, $i = 1, \dots, 12$ have the 1-dimensional ideal $\mathbf{i} = \langle e_1 \rangle$. With the exception of the Lie algebra $\mathbf{g}^{a=0}_{6,4}$ there does not exist any ideal $\mathbf{s}$ of $\mathbf{g}_{6,i}$ containing $\mathbf{i}$ such that the factor Lie algebras $\mathbf{g}_{6,i}/\mathbf{s}$ are isomorphic to an elementary filiform Lie algebra $\mathbf{f}_n$, $n = 4, 5$. Let I be the simply connected Lie group of the ideal $\mathbf{i}$. If $\mathbf{g}_{6,i}$, $i = 1, \dots, 12$, would be the Lie algebra of the group $Mult(L)$ of L, then the orbit $I(e)$ is isomorphic to $\mathbb{R}$ and the factor loop $L/I(e)$ is isomorphic to $\mathbb{R}^2$ (cf. Proposition 3.3 a). By Lemma 3.2 d) (i) $I(e)$ coincides with the 1-dimensional centre $Z(L)$ of L. By Proposition 3.3 a) the Lie algebra $\mathbf{inn(L)}$ of the group $Inn(L)$ lies in the 5-dimensional abelian nilradical of $\mathbf{g}_{6,i}$ which contains the ideal $\mathbf{p} = \mathbf{z} \oplus \mathbf{inn(L)} \cong \mathbb{R}^4$ of $\mathbf{g}_{6,i}$, where $\mathbf{z}$ is the 1-dimensional centre of $\mathbf{g}$. Then the normalizer $N_{\mathbf{g}_{6,i}}(\mathbf{inn(L)})$, $i = 1, \dots, 12$, is the nilradical of $\mathbf{g}_{6,i}$ which contradicts Lemma 2.3.

If the Lie algebra $\mathbf{g}^{a=0}_{6,4}$ would be the Lie algebra of the multiplication group of a 3-dimensional loop L, then from the above discussion it follows that the factor loop $L/Z(L)$ is isomorphic to a loop $L_{\mathcal{F}}$. In fact, for the ideal $\mathbf{s} = \langle e_1, e_5 \rangle$ the factor Lie algebra $\mathbf{g}^{a=0}_{6,4}/\mathbf{s}$ is isomorphic to the elementary filiform Lie algebra $\mathbf{f}_4$. Since the orbit $S(e)$, where $S = \exp(\mathbf{s})$, has dimension 1 we obtain that $\dim(\mathbf{s} \cap \mathbf{inn(L)}) = 1$. For the simply connected Lie group $I_2 = \{\exp(te_5); t \in \mathbb{R}\}$ of the ideal $\mathbf{i}_2 = \langle e_5 \rangle$ we obtain that the orbit $I_2(e)$ is a normal subgroup of L isomorphic to $\mathbb{R}$. Hence one has $\mathbf{i}_2 \cap \mathbf{inn(L)} = 0$. The abelian ideals $\mathbf{a} = \langle e_1, e_2 \rangle$, $\mathbf{b} = \langle e_1, e_2, e_3 \rangle$, $\mathbf{g}'^{a=0}_{6,4} = \langle e_1, e_2, e_3, e_5 \rangle$ of $\mathbf{g}^{a=0}_{6,4}$ satisfy the conditions of Proposition 3.3 b). Let A, B and N be the simply connected Lie group of $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{g}'_{6,4}$. Since $\langle e_1 \rangle = \mathbf{z} \subset \mathbf{a}$ the orbit $A(e)$ contains the centre $Z(L)$ of L. If $\dim(A(e)) = 1$, then $A(e) = Z(L)$. As the factor Lie algebra $\mathbf{g}_{6,4}^{a=0}/\mathbf{a}$ is not isomorphic to the elementary filiform Lie algebra $\mathbf{f}_4$ the factor loop $L/Z(L)$ is not isomorphic to a loop $L_{\mathcal{F}}$.

According to Proposition 3.3 b) the orbit $A(e)$ is a 2-dimensional connected normal subloop M of L containing $Z(L)$ and the orbits $B(e)$ and $N(e)$ coincide with M. Therefore the Lie algebra $\mathbf{inn}(\mathbf{L})$ contains the subalgebra $\langle e_3 + a_1e_1 + a_2e_2, e_5 + b_1e_1\rangle$, $a_i, b_1 \in \mathbb{R}$, $i = 1, 2$, $b_1 \neq 0$. The ideal $\mathbf{v}$ in Proposition 3.3 b) has one of the following forms: $\mathbf{v}_{1,k} = \langle e_1, e_2, e_3, e_5, e_4 + ke_6\rangle$, $k \in \mathbb{R}$, $\mathbf{v}_2 = \langle e_1, e_2, e_3, e_5, e_6\rangle$. Therefore the Lie algebra $\mathbf{inn}(\mathbf{L})$ has as generator either $e_4 + ke_6 + c_1e_1 + c_2e_2$ or $e_6 + c_1e_1 + c_2e_2$, $k, c_1, c_2 \in \mathbb{R}$. Only the subspace $\langle e_3 + a_1e_1 + a_2e_2, e_4 + c_1e_1 + c_2e_2, e_5 + b_1e_1\rangle \subset \mathbf{n}_{\text{rad}}$ is a 3-dimensional Lie algebra. Hence it would be the Lie algebra $\mathbf{inn}(\mathbf{L})$. The normalizer $N_{\mathbf{g}_{6,4}^{a=0}}(\mathbf{inn}(\mathbf{L}))$ equals to $\mathbf{n}_{\text{rad}}$ which contains $\mathbf{z} \oplus \mathbf{inn}(\mathbf{L})$ as a proper ideal. This contradiction to Lemma 2.3 yields the assertion. □

Theorem 3.6. *Let L be a connected topological loop of dimension 3 such that the Lie algebra $\mathbf{g}$ of its multiplication group is a 6-dimensional indecomposable solvable Lie algebra having one of the following nilradicals: (a) the direct sum $H \oplus \mathbb{R}^2$, where H is the 3-dimensional Heisenberg Lie algebra,*
(b) the 5-dimensional elementary filiform Lie algebra $\mathbf{f}_5$.
Then L is centrally nilpotent of class 2 and the nilradical of $\mathbf{g}$ is $H \oplus \mathbb{R}^2$. The following Lie algebra pairs can occur as the Lie algebra $\mathbf{g}$ of the group $Mult(L)$ and the subalgebra $\mathbf{k}$ of the subgroup $Inn(L)$:
$\mathbf{g}_1 := \mathbf{g}_{6,21}^{a=0}$: $[e_2, e_3] = e_1$, $[e_2, e_6] = e_3$, $[e_4, e_6] = e_4$, $[e_5, e_6] = be_5$, $0 < |b| \le 1$, $\mathbf{k}_1 = \langle e_3, e_4+e_1, e_5+e_1\rangle$,
$\mathbf{g}_2 := \mathbf{g}_{6,22}^{a=0}$: $[e_2, e_3] = e_1 = [e_5, e_6]$, $[e_2, e_6] = e_3$, $[e_4, e_6] = e_4$, $\mathbf{k}_2 = \langle e_3, e_4 + e_1, e_5\rangle$,
$\mathbf{g}_3 := \mathbf{g}_{6,24}$: $[e_2, e_3] = e_1 = [e_4, e_6]$, $[e_2, e_6] = e_3$, $[e_3, e_6] = e_4$, $[e_5, e_6] = e_5$, $\mathbf{k}_3 = \langle e_3, e_4, e_5 + e_1\rangle$,
$\mathbf{g}_4 := \mathbf{g}_{6,30}$: $[e_2, e_3] = e_1$, $[e_2, e_6] = e_3$, $[e_4, e_6] = e_4 + e_5$, $[e_5, e_6] = e_5$, $\mathbf{k}_4 = \langle e_3, e_4 + a_2e_1, e_5 + e_1\rangle$, $a_2 \in \mathbb{R}$,
$\mathbf{g}_5 := \mathbf{g}_{6,36}^{a=0}$: $[e_2, e_3] = e_1$, $[e_2, e_6] = e_3$, $[e_4, e_6] = be_4 + e_5$, $[e_5, e_6] = be_5 - e_4$, $b \ge 0$, $\mathbf{k}_5 = \langle e_3, e_4, e_5 + e_1\rangle$ *or* $\mathbf{k}_{5,a_3} = \langle e_3, e_4 + e_1, e_5 + a_3e_1\rangle$, $a_3 \in \mathbb{R}$,
$\mathbf{g}_6 := \mathbf{g}_{6,14}^{a=b=0}$: $[e_2, e_3] = e_1 = [e_5, e_6]$, $[e_4, e_6] = e_4$, $\mathbf{k}_{6,1} = \langle e_2, e_4 + e_1, e_5\rangle$, $\mathbf{k}_{6,2} = \langle e_3, e_4 + e_1, e_5\rangle$,
$\mathbf{g}_7 := \mathbf{g}_{6,17}^{\delta=1,a=\varepsilon=0}$: $[e_2, e_3] = e_1 = [e_4, e_6]$, $[e_3, e_6] = e_4$, $[e_5, e_6] = e_5$, $\mathbf{k}_{7,1} = \langle e_2, e_4, e_5 + e_1\rangle$, $\mathbf{k}_{7,2} = \langle e_3, e_4, e_5 + e_1\rangle$,
$\mathbf{g}_8 := \mathbf{g}_{6,15}^{a=0}$: $[e_2, e_3] = e_1 = [e_1, e_6]$, $[e_2, e_6] = e_2 + e_4$, $[e_3, e_6] = e_5$, $[e_4, e_6] = e_4$, $\mathbf{k}_8 = \langle e_1 + e_5, e_2 + a_2e_5, e_4 + a_3e_5\rangle$, $a_3 \in \mathbb{R} \setminus \{0\}$, $a_2 \in \mathbb{R}$,
$\mathbf{g}_9 := \mathbf{g}_{6,16}$: $[e_2, e_3] = e_1 = [e_1, e_6]$, $[e_2, e_6] = e_2 + e_4$, $[e_3, e_6] = e_5$, $[e_4, e_6] = e_1 + e_4$, $\mathbf{k}_{9,1} = \langle e_1 + e_5, e_2 + a_2e_5, e_4 + a_3e_5\rangle$, $a_2, a_3 \in \mathbb{R}$,
$\mathbf{g}_{10} := \mathbf{g}_{6,17}^{\delta=\varepsilon=0,a\neq 0}$: $[e_2, e_3] = e_1$, $[e_1, e_6] = ae_1$, $[e_2, e_6] = ae_2$, $[e_3, e_6] = e_4$, $[e_5, e_6] = e_5$, $\mathbf{k}_{10} = \langle e_1 + e_4, e_2 + a_2e_4, e_5 + e_4\rangle$, $a_2 \in \mathbb{R}$,
$\mathbf{g}_{11} := \mathbf{g}_{6,17}^{\delta=0,a=\varepsilon=1}$: $[e_2, e_3] = e_1 = [e_1, e_6]$, $[e_2, e_6] = e_2$, $[e_3, e_6] = e_4$, $[e_5, e_6] = e_1 + e_5$, $\mathbf{k}_{11} = \langle e_1 + e_4, e_2 + a_2e_4, e_5 + a_3e_4\rangle$, $a_2, a_3 \in \mathbb{R}$,
$\mathbf{g}_{12} := \mathbf{g}_{6,25}^{a=b=0}$: $[e_2, e_3] = e_1 = [e_1, e_6]$, $[e_2, e_6] = e_2$, $[e_4, e_6] = e_5$, $\mathbf{k}_{12} = \langle e_1 + e_5, e_2 + \varepsilon e_5, e_4\rangle$, $\varepsilon = 0, 1$,

$\mathbf{g}_{13} := \mathbf{g}_{6,27}^{a=1,b=\delta=0}$: $[e_2,e_3]=e_1=[e_1,e_6]$, $[e_2,e_6]=e_2$, $[e_3,e_6]=e_4$, $[e_4,e_6]=e_5$,
$\mathbf{k}_{13}=\langle e_1+e_5, e_2+a_2e_5, e_4\rangle$, $a_2\in\mathbb{R}$.

Proof. By Lemma 2.2 we may assume that L is homeomorphic to $\mathbb{R}^3$. The 6-dimensional solvable Lie algebras having nilradical as in the assertion are in [12], p. 38 and p. 40. The Lie algebra $\mathbf{g}_{6,27}^{a=1,b=\delta=0}$ has the centre $\mathbf{i}=\langle e_5\rangle$. For all other Lie algebras $\mathbf{g}_{6,i}$, $i=13,\dots,38$ and $71,\dots,75$ we consider the ideal $\mathbf{i}=\langle e_1\rangle$. With exception of the Lie algebras $\mathbf{g}_{6,23}^{\delta=0}$, $\mathbf{g}_{6,24}$ there does not exist any ideal $\mathbf{s}$ of $\mathbf{g}_{6,i}$ such that $\mathbf{i}\subseteq\mathbf{s}$ and the factor Lie algebras $\mathbf{g}_{6,i}/\mathbf{s}$ are isomorphic to an elementary filiform Lie algebra $\mathbf{f}_n$, $n=4$, 5. Let I be the simply connected Lie group of the ideal $\mathbf{i}$. If the Lie algebras $\mathbf{g}_{6,i}$, $i=13,\dots,38$ and $71,\dots,75$, are the Lie algebra of the group $Mult(L)$ of L, then the orbit $I(e)$ is a normal subloop of L isomorphic to $\mathbb{R}$, the factor loop $L/I(e)$ is isomorphic to $\mathbb{R}^2$ and $I(e)=Z(L)$ (cf. Proposition 3.3 a) (i) and Lemma 3.2 d) (i)). Hence the simply connected loop L is a central extension of the group $\mathbb{R}$ by the group $\mathbb{R}^2$. This means it is centrally nilpotent of class 2. By Proposition 3.3 a) (i) the Lie algebra $\mathbf{g}_{6,i}$ has a 4-dimensional abelian ideal $\mathbf{p}=\mathbf{z}\oplus\mathbf{k}$, where $\mathbf{z}$ is the 1-dimensional centre of $\mathbf{g}_{6,i}$ and $\mathbf{k}$ is the Lie algebra of the group $Inn(L)$ and $\mathbf{g}'_{6,i}\subset\mathbf{p}$. Then the nilradical of $\mathbf{g}_{6,i}$ is the direct sum of the 3-dimensional Heisenberg Lie algebra and $\mathbb{R}^2$. According to Lemma 2.3 the subalgebra $\mathbf{k}$ does not contain any non-zero ideal of $\mathbf{g}_{6,i}$ and the normalizer $N_{\mathbf{g}_{6,i}}(\mathbf{k})$ of $\mathbf{k}$ in $\mathbf{g}_{6,i}$ is $\mathbf{p}$. Then for the triples $(\mathbf{g}_{6,i},\mathbf{p},\mathbf{k})$ we obtain:
(a) The Lie algebras $\mathbf{g}_{6,i}^{a=0}$, $i=21,22,36$, and $\mathbf{g}_{6,j}$, $j=24,30$, have the centre $\mathbf{z}=\langle e_1\rangle$ and $\mathbf{p}$ is $\langle e_1,e_3,e_4,e_5\rangle$. The subalgebra $\mathbf{k}$ has the form: $\mathbf{k}_{a_1,a_2,a_3}=\langle e_3+a_1e_1, e_4+a_2e_1, e_5+a_3e_1\rangle$, $a_i\in\mathbb{R}$, $i=1,2,3$, such that in the case $\mathbf{g}_{6,21}^{a=0}$: $a_2\neq 0$, $a_3\neq 0$ since $\langle e_4\rangle$ and $\langle e_5\rangle$ are ideals of $\mathbf{g}_{6,21}^{a=0}$,
in the case $\mathbf{g}_{6,22}^{a=0}$: $a_2\neq 0$ because $\langle e_4\rangle$ is an ideal of $\mathbf{g}_{6,22}^{a=0}$,
in the cases $\mathbf{g}_{6,i}$, $i=24,30$: $a_3\neq 0$ since $\langle e_5\rangle$ is an ideal of $\mathbf{g}_{6,i}$,
in the case $\mathbf{g}_{6,36}^{a=0}$: $a_2\neq 0$ or $a_3\neq 0$ because $\langle e_4,e_5\rangle$ is an ideal of $\mathbf{g}_{6,36}^{a=0}$. Using the automorphism $\alpha(e_i)=e_i$, $i=1,2$, $\alpha(e_3)=e_3-a_1e_1$, $\alpha(e_4)=a_2e_4$, $\alpha(e_5)=a_3e_5$, $\alpha(e_6)=e_6-a_1e_3$ for $\mathbf{g}_{6,21}^{a=0}$, respectively $\alpha(e_5)=e_5-a_3e_1$ for $\mathbf{g}_{6,22}^{a=0}$, respectively $\alpha(e_4)=e_4-a_2e_1$, $\alpha(e_6)=e_6+a_2e_2-a_1e_3$ for $\mathbf{g}_{6,24}$, respectively $\alpha(e_j)=a_3e_j$, $j=4,5$, for $\mathbf{g}_{6,30}$, the Lie algebra $\mathbf{k}_{a_1,a_2,a_3}$ reduces to $\mathbf{k}=\langle e_3, e_4+e_1, e_5+e_1\rangle$, respectively $\mathbf{k}=\langle e_3,e_4+e_1,e_5\rangle$, respectively $\mathbf{k}=\langle e_3,e_4,e_5+e_1\rangle$, respectively $\mathbf{k}_{a_2}=\langle e_3,e_4+a_2e_1,e_5+e_1\rangle$, $a_2\in\mathbb{R}$. Applying the automorphism $\alpha(e_i)=e_i$, $i=1,2$, $\alpha(e_3)=e_3-a_1e_1$, $\alpha(e_j)=a_2e_j$, $j=4,5$, $\alpha(e_6)=e_6-a_1e_3$ for the Lie algebra $\mathbf{g}_{6,36}^{a=0}$ if $a_2\neq 0$, respectively $\alpha(e_j)=a_3e_j$, $j=4,5$, if $a_2=0$ and $a_3\neq 0$ we can reduce $\mathbf{k}_{a_1,a_2,a_3}$ to $\mathbf{k}_{a_3}=\langle e_3,e_4+e_1,e_5+a_3e_1\rangle$, $a_3\in\mathbb{R}$, respectively $\mathbf{k}_{a_1,0,a_3}$ to $\mathbf{k}=\langle e_3,e_4,e_5+e_1\rangle$.
(b) The Lie algebras $\mathbf{g}_{6,14}^{a=b=0}$ and $\mathbf{g}_{6,17}^{\delta=1,a=\varepsilon=0}$ have the centre $\mathbf{z}=\langle e_1\rangle$ and the ideal $\mathbf{p}$ has one of the forms: $\mathbf{p}_{1,k}=\langle e_1,e_2+ke_3,e_4,e_5\rangle$, $k\in\mathbb{R}$, and $\mathbf{p}_2=\langle e_1,e_3,e_4,e_5\rangle$. With respect to the ideals $\mathbf{p}_{1,k}$, $\mathbf{p}_2$ we obtain the subalgebras $\mathbf{k}_{1,k}=\langle e_2+ke_3+a_1e_1, e_4+a_2e_1, e_5+a_3e_1\rangle$, $k\in\mathbb{R}$, $\mathbf{k}_2=\langle e_3+a_1e_1,e_4+a_2e_1,e_5+a_3e_1\rangle$, $a_i\in\mathbb{R}$, $i=1,2,3$, such that for $\mathbf{g}_{6,14}^{a=b=0}$ one has $a_2\neq 0$ since $\langle e_4\rangle$ is an ideal of $\mathbf{g}_{6,14}^{a=b=0}$ and for $\mathbf{g}_{6,17}^{\delta=1,a=\varepsilon=0}$ we get $a_3\neq 0$ because $\langle e_5\rangle$ is an ideal of $\mathbf{g}_{6,17}^{\delta=1,a=\varepsilon=0}$. The automorphism $\alpha(e_i)=e_i$, $i=1,6$, $\alpha(e_4)=a_2e_4$, $\alpha(e_5)=e_5-a_3e_1$, $\alpha(e_2)=e_2-ke_3-a_1e_1$, $\alpha(e_3)=e_3$, respectively $\alpha(e_2)=e_2$, $\alpha(e_3)=e_3-a_1e_1$ of $\mathbf{g}_{6,14}^{a=b=0}$ maps the subalgebra $\mathbf{k}_{1,k}$ onto $\mathbf{k}=\langle e_2,e_4+e_1,e_5\rangle$, respectively $\mathbf{k}_2$ onto $\mathbf{k}=\langle e_3,e_4+e_1,e_5\rangle$. The automorphism $\alpha(e_1)=e_1$,

$\alpha(e_4) = e_4 - a_2 e_1$, $\alpha(e_5) = a_3 e_5$, $\alpha(e_6) = e_6 + a_2 e_2$, $\alpha(e_2) = e_2 - a_1 e_1 - k e_3$, $\alpha(e_3) = e_3$, respectively $\alpha(e_3) = e_3 - a_1 e_1$, $\alpha(e_2) = e_2$ of $\mathbf{g}_{6,17}^{\delta=1,a=\varepsilon=0}$ maps the subalgebra $\mathbf{k}_{1,k}$ onto $\mathbf{k} = \langle e_2, e_4, e_5 + e_1 \rangle$, respectively $\mathbf{k}_2$ onto $\mathbf{k} = \langle e_3, e_4, e_5 + e_1 \rangle$.
(c) The Lie algebras $\mathbf{g}_{6,17}^{\delta=\varepsilon=0,a\neq 0}$ and $\mathbf{g}_{6,17}^{\delta=0,a=\varepsilon=1}$ have the centre $\mathbf{z} = \langle e_4 \rangle$ and the ideal $\mathbf{p}$ equals to $\langle e_1, e_2, e_4, e_5 \rangle$. Hence the subalgebra $\mathbf{k}$ has the form $\mathbf{k}_{a_1,a_2,a_3} = \langle e_1 + a_1 e_4, e_2 + a_2 e_4, e_5 + a_3 e_4 \rangle$, $a_i \in \mathbb{R}$, $i = 1, 2, 3$ such that for $\mathbf{g}_{6,17}^{\delta=\varepsilon=0,a\neq 0}$ we have $a_1 \neq 0$, $a_3 \neq 0$ since it has the ideals $\langle e_1 \rangle$, $\langle e_5 \rangle$ and for $\mathbf{g}_{6,17}^{\delta=0,a=\varepsilon=1}$ one obtains $a_1 \neq 0$ because it has the ideal $\langle e_1 \rangle$. With the automorphism $\alpha(e_i) = a_1 e_i$, $i = 1, 2$, $\alpha(e_j) = e_j$, $j = 3, 4, 6$, $\alpha(e_5) = a_3 e_5$ of $\mathbf{g}_{6,17}^{\delta=\varepsilon=0,a\neq 0}$, respectively $\alpha(e_5) = a_1 e_5$ for $\mathbf{g}_{6,17}^{\delta=0,a=\varepsilon=1}$ we can change the subalgebra $\mathbf{k}_{a_1,a_2,a_3}$ onto $\mathbf{k}_{a_2} = \langle e_1 + e_4, e_2 + a_2 e_4, e_5 + e_4 \rangle$, $a_2 \in \mathbb{R}$, respectively $\mathbf{k}_{a_2,a_3} = \langle e_1 + e_4, e_2 + a_2 e_4, e_5 + a_3 e_4 \rangle$, $a_2, a_3 \in \mathbb{R}$.
(d) The Lie algebras $\mathbf{g}_{6,15}^{a=0}$, $\mathbf{g}_{6,16}$, $\mathbf{g}_{6,25}^{a=b=0}$, $\mathbf{g}_{6,27}^{a=1,b=\delta=0}$ have the centre $\mathbf{z} = \langle e_5 \rangle$ and their ideal $\mathbf{p}$ is $\langle e_1, e_2, e_4, e_5 \rangle$. Therefore the subalgebra $\mathbf{k}$ has the form $\mathbf{k}_{a_1,a_2,a_3} = \langle e_1 + a_1 e_5, e_2 + a_2 e_5, e_4 + a_3 e_5 \rangle$, $a_i \in \mathbb{R}$, $i = 1, 2, 3$. For $\mathbf{g}_{6,15}^{a=0}$ one has $a_1 \neq 0$, $a_3 \neq 0$ since $\langle e_1 \rangle$, $\langle e_4 \rangle$ are ideals of $\mathbf{g}_{6,15}^{a=0}$. For $\mathbf{g}_{6,k}$, $k = 16, 25, 27$, we have $a_1 \neq 0$ because $\langle e_1 \rangle$ is an ideal of $\mathbf{g}_{6,k}$. For $\mathbf{g}_{6,15}^{a=0}$ and $\mathbf{g}_{6,16}$ using the automorphism $\alpha(e_i) = a_1 e_i$, $i = 1, 2, 4$, $\alpha(e_j) = e_j$, $j = 3, 5, 6$, the subalgebra $\mathbf{k}_{a_1,a_2,a_3}$ reduces to $\mathbf{k}_{a_2} = \langle e_1 + e_5, e_2 + a_2 e_5, e_4 + a_3 e_5 \rangle$. For $\mathbf{g}_{6,25}^{a=b=0}$ applying the automorphism $\alpha(e_i) = e_i$, $i = 5, 6$, $\alpha(e_1) = a_1 e_1$, $\alpha(e_4) = e_4 - a_3 e_5$, $\alpha(e_2) = a_2 e_2$, $\alpha(e_3) = \frac{a_1}{a_2} e_3$ if $a_2 \neq 0$, respectively $\alpha(e_2) = e_2$, $\alpha(e_3) = a_1 e_3$ if $a_2 = 0$, we can change the subalgebra $\mathbf{k}_{a_1,a_2,a_3}$ to $\mathbf{k} = \langle e_1 + e_5, e_2 + e_5, e_4 \rangle$, respectively $\mathbf{k}_{a_1,0,a_3}$ to $\mathbf{k} = \langle e_1 + e_5, e_2, e_4 \rangle$. The automorphism $\alpha(e_i) = e_i$, $i = 5, 6$, $\alpha(e_j) = a_1 e_j$, $j = 1, 2$, $\alpha(e_3) = e_3 - a_3 e_4$, $\alpha(e_4) = e_4 - a_3 e_5$ of $\mathbf{g}_{6,27}^{a=1,b=\delta=0}$ maps $\mathbf{k}_{a_1,a_2,a_3}$ onto $\mathbf{k}_{a_2} = \langle e_1 + e_5, e_2 + a_2 e_5, e_4 \rangle$.
The Lie algebra $\mathbf{g}_{6,23}^{\delta=0}$ has 2-dimensional centre. Hence it is excluded by Lemma 3.2 b).
The Lie algebra $\mathbf{g}_{6,24}$ has the centre $\mathbf{z} = \langle e_1 \rangle$ and the ideals $\mathbf{i}_2 = \langle e_5 \rangle$, $\mathbf{s} = \langle e_1, e_5 \rangle$, $\mathbf{a} = \langle e_1, e_4 \rangle$, $\mathbf{b} = \langle e_1, e_3, e_4 \rangle$, $\mathbf{g}'_{6,24} = \langle e_1, e_3, e_4, e_5 \rangle$. Let Z, I_2, S, A, B, N be the simply connected Lie groups of $\mathbf{z}$, $\mathbf{i}_2$, $\mathbf{s}$, $\mathbf{a}$, $\mathbf{b}$, $\mathbf{g}'_{6,24}$ in this order. The factor Lie algebras $\mathbf{g}_{6,24}/\mathbf{s}$ is isomorphic to the elementary filiform Lie algebra $\mathbf{f}_4$. If $\mathbf{g}_{6,24}$ is the Lie algebra of the group $Mult(L)$ of L, then from the above discussion it follows that the factor loop $L/Z(e) = L/S(e)$ is isomorphic to a loop $L_{\mathcal{F}}$. Since $Z(e) = \mathbb{R} = S(e)$ one has $\dim(\mathbf{s} \cap \mathbf{inn(L)}) = 1$. The orbit $I_2(e)$ is a normal subgroup of L isomorphic to $\mathbb{R}$ (cf. Proposition 3.3 a). As $\mathbf{i}_2 \subset \mathbf{s}$ we have $I_2(e) = S(e)$ and $\mathbf{i}_2 \cap \mathbf{inn(L)} = 0$. For the ideals $\mathbf{a}$, $\mathbf{b}$, $\mathbf{g}'_{6,24}$ the conditions of Proposition 3.3 b) are satisfied. Since $\mathbf{z} \subset \mathbf{a}$ the orbit $A(e)$ contains the centre $Z(e)$ of L. If $\dim(A(e)) = 1$, then $A(e) = Z(e)$. As the factor Lie algebra $\mathbf{g}_{6,24}/\mathbf{a}$ is not isomorphic to the elementary filiform Lie algebra $\mathbf{f}_4$ the factor loop $L/Z(e)$ cannot be isomorphic to a loop $L_{\mathcal{F}}$.

According to Proposition 3.3 b) we obtain that $A(e) = B(e) = N(e) = M$, where M is a 2-dimensional connected normal subloop of L such that $\mathbf{a} \cap \mathbf{inn(L)} = 0$, $\mathbf{b} \cap \mathbf{inn(L)}$ has dimension 1 whereas $\mathbf{g}'_{6,24} \cap \mathbf{inn(L)}$ has dimension 2 and $Z(e) \subset M$. For the ideal $\mathbf{v}$ in Proposition 3.3 b) we obtain one of the following forms: $\mathbf{v}_{1,k} = \langle e_1, e_3, e_4, e_5, e_2 + k e_6 \rangle$, $k \in \mathbb{R}$, $\mathbf{v}_2 = \langle e_1, e_3, e_4, e_5, e_6 \rangle$. Hence the Lie algebra $\mathbf{inn(L)}$ has either the generators $b_1 = e_3 + a_1 e_1 + a_2 e_4$, $b_2 = e_5 + b_1 e_1$, $b_{3,k} = e_2 + k e_6 + c_1 e_1 + c_2 e_4$ or b_1, b_2, $b_3 = e_6 + c_1 e_1 + c_2 e_4$, $a_i, b_1, k, c_i \in \mathbb{R}$, $i = 1, 2$, $b_1 \neq 0$. None of the vector spaces $\langle b_1, b_2, b_{3,k} \rangle$, $\langle b_1, b_2, b_3 \rangle$ are 3-dimensional Lie algebras. Hence we get a contradiction and the assertion is proved. □

Theorem 3.7. *Let L be a connected topological loop of dimension 3 such that the Lie algebra $\mathbf{g}$ of the group $Mult(L)$ is a 6-dimensional solvable indecomposable Lie algebra having nilradical isomorphic either to the direct sum of the 4-dimensional elementary filiform Lie algebra $\mathbf{f}_4$ and $\mathbb{R}$ or to the Lie algebra defined by $[e_3, e_5] = e_1$, $[e_4, e_5] = e_2$. Then the loop L is centrally nilpotent of class 2 and the following pairs can occur as Lie algebra $\mathbf{g}$ of the group $Mult(L)$ and the subalgebra $\mathbf{k}$ of the group $Inn(L)$:*
$\mathbf{g}_1 := \mathbf{g}_{6,49} : [e_1, e_5] = e_2 = [e_2, e_6]$, $[e_4, e_5] = e_1 = [e_1, e_6]$, $[e_4, e_6] = \varepsilon e_2 + e_4$, $[e_5, e_6] = e_3$, $\varepsilon = 0, \pm 1$, $\mathbf{k}_1 = \langle e_1 + a_1 e_3, e_2 + e_3, e_4 + a_3 e_3\rangle$, $a_1, a_3 \in \mathbb{R}$,
$\mathbf{g}_2 := \mathbf{g}_{6,51} : [e_1, e_5] = e_2$, $[e_4, e_5] = e_1$, $[e_3, e_6] = e_3$, $[e_4, e_6] = \varepsilon e_2$, $\varepsilon = \pm 1$, $\mathbf{g}_3 := \mathbf{g}_{6,52} : [e_1, e_5] = e_2$, $[e_4, e_5] = e_1$, $[e_3, e_6] = e_3$, $[e_4, e_6] = \varepsilon e_2$, $[e_5, e_6] = e_4$, $\varepsilon = 0, \pm 1$, $\mathbf{k}_2 = \mathbf{k}_3 = \langle e_1 + a_1 e_2, e_3 + e_2, e_4\rangle$, $a_1 \in \mathbb{R}$,
$\mathbf{g}_4 := \mathbf{g}^{a=b=0}_{6,54} : [e_3, e_5] = e_1 = [e_1, e_6]$, $[e_4, e_5] = e_2$, $[e_3, e_6] = e_3$, $\mathbf{g}_5 := \mathbf{g}^{a=0}_{6,57} : [e_3, e_5] = e_1 = [e_1, e_6]$, $[e_4, e_5] = e_2$, $[e_3, e_6] = e_3$, $[e_5, e_6] = e_4$, $\mathbf{g}_6 := \mathbf{g}_{6,59} : [e_3, e_5] = e_1 = [e_1, e_6]$, $[e_4, e_5] = e_2 = [e_4, e_6]$, $[e_3, e_6] = e_3$, $[e_5, e_6] = \delta e_4$, $\delta = 0, 1$, $\mathbf{g}_7 := \mathbf{g}^{a=0}_{6,63} : [e_3, e_5] = e_1 = [e_1, e_6]$, $[e_3, e_6] = e_3$, $[e_4, e_5] = e_2 = [e_4, e_6]$, $\mathbf{k}_4 = \mathbf{k}_5 = \mathbf{k}_6 = \mathbf{k}_7 = \langle e_1 + e_2, e_3 + a_2 e_2, e_4\rangle$, $a_2 \in \mathbb{R}$.

Proof. We may assume that L is homeomorphic to $\mathbb{R}^3$ (cf. Lemma 2.2). The 6-dimensional solvable indecomposable Lie algebras having nilradical as in the assertion are the Lie algebras $\mathbf{g}_{6,i}$, $i = 39, \dots, 70$, in [12], p. 39.

The Lie algebra $\mathbf{g}_{6,70}$ has trivial centre and the unique minimal ideal $\mathbf{i} = \langle e_1, e_2\rangle$. Let I be the simply connected Lie group of $\mathbf{i}$. By Lemma 3.2 a) and c) the orbit $I(e)$ is a 1-dimensional normal subloop of L such that the factor loop $L/I(e)$ is isomorphic to a loop $L_{\mathcal{F}}$. The factor Lie algebra $\mathbf{g}_{6,70}/\mathbf{i}$ is not isomorphic to the elementary filiform Lie algebra $\mathbf{f}_4$. Hence the Lie algebra $\mathbf{g}_{70}$ is not the group $Mult(L)$ of L.

All other Lie algebras have the ideal $\mathbf{i} = \langle e_2\rangle$. With the exception of the Lie algebra $\mathbf{g}_{6,52}$ there does not exist any ideal $\mathbf{s}$ of $\mathbf{g}_{6,i}$, $i = 39, \dots, 69$, such that $\mathbf{i} \le \mathbf{s}$ and the factor Lie algebras $\mathbf{g}_{6,i}/\mathbf{s}$ are isomorphic to an elementary filiform Lie algebra $\mathbf{f}_n$, $n \in \{4, 5\}$. If $\mathbf{g}_{6,i}$, $i = 39, \dots, 69$, would be the Lie algebra of the group $Mult(L)$ of L, then the simply connected loop L has a 1-dimensional centre $Z(L) = I(e) \cong \mathbb{R}$, where $I = \exp \mathbf{i}$, and the factor loop $L/I(e)$ is isomorphic to $\mathbb{R}^2$ (cf. Lemma 3.2 a) and d) (i). Hence L is centrally nilpotent of class 2. According to Proposition 3.3 a) (i) we seek for Lie algebras $\mathbf{g}_{6,i}$ such that the nilradical of $\mathbf{g}_{6,i}$ contains an ideal $\mathbf{p} = \mathbf{z} \oplus \mathbf{inn(L)} \cong \mathbb{R}^4$ of $\mathbf{g}_{6,i}$ and the commutator ideal $\mathbf{g}'_{6,i}$ of $\mathbf{g}_{6,i}$ lies in $\mathbf{p}$. Here $\mathbf{z}$ is the 1-dimensional centre of $\mathbf{g}_{6,i}$ and $\mathbf{inn(L)}$ is the Lie algebra of the group $Inn(L)$. By Lemma 2.3 the Lie algebra $\mathbf{k}$ does not contain any non-zero ideal of $\mathbf{g}_{6,i}$ and the normalizer $N_{\mathbf{g}_{6,i}}(\mathbf{k})$ of $\mathbf{k}$ in $\mathbf{g}_{6,i}$ is $\mathbf{p}$. The following pairs $(\mathbf{g}_{6,i}, \mathbf{k})$ have the above properties:
(a) The Lie algebra $\mathbf{g}_{6,49}$ has the centre $\mathbf{z} = \langle e_3\rangle$ and the ideal $\mathbf{p}$ is $\langle e_1, e_2, e_3, e_4\rangle$. Hence for the subalgebra $\mathbf{k}$ we obtain $\mathbf{k}_{a_1,a_2,a_3} = \langle e_1 + a_1 e_3, e_2 + a_2 e_3, e_4 + a_3 e_3\rangle$, $a_2 \neq 0$, because $\langle e_2\rangle$ is an ideal of $\mathbf{g}_{6,49}$ and $a_1, a_3 \in \mathbb{R}$. The automorphism $\alpha(e_i) = a_2 e_i$, $i = 1, 2, 4$, $\alpha(e_j) = e_j$, $j = 3, 5, 6$, maps the subalgebra $\mathbf{k}_{a_1,a_2,a_3}$ onto $\mathbf{k}_{a_1,a_3} = \langle e_1 + a_1 e_3, e_2 + e_3, e_4 + a_3 e_3\rangle$.

(b) The Lie algebras $\mathbf{g}_{6,k}$, $k = 51, 52$, $\mathbf{g}_{6,54}^{a=b=0}$, $\mathbf{g}_{6,57}^{a=0}$, $\mathbf{g}_{6,59}$, $\mathbf{g}_{6,63}^{a=0}$ have the centre $\mathbf{z} = \langle e_2 \rangle$ and the ideal $\mathbf{p}$ equals to $\langle e_1, e_2, e_3, e_4 \rangle$. Hence the Lie algebra $\mathbf{k}$ has the form $\mathbf{k}_{a_1,a_2,a_3} = \langle e_1+a_1e_2, e_3+a_2e_2, e_4+a_3e_2 \rangle$, $a_i \in \mathbb{R}$, $i = 1, 2, 3$, such that $a_2 \neq 0$ for the Lie algebras $\mathbf{g}_{6,k}$, $k = 51, 52$, since $\langle e_3 \rangle$ is their ideal and $a_1 \neq 0$ for the Lie algebras $\mathbf{g}_{6,k}$, $k = 54, 57, 59, 63$, because $\langle e_1 \rangle$ is their ideal. Applying the automorphism $\alpha(e_i) = e_i$, $i = 1, 2, 5$, $\alpha(e_3) = a_2e_3$, $\alpha(e_4) = e_4 - a_3e_2$, $\alpha(e_6) = e_6$ for $\mathbf{g}_{6,51}$, respectively $\alpha(e_6) = e_6 + a_3e_1$ for $\mathbf{g}_{6,52}$ the subalgebra $\mathbf{k}_{a_1,a_2,a_3}$ reduces to $\mathbf{k}_{a_1} = \langle e_1 + a_1e_2, e_3 + e_2, e_4 \rangle$. The automorphism $\alpha(e_i) = e_i$, $i = 2, 5, 6$, $\alpha(e_j) = a_1e_j$, $j = 1, 3$, $\alpha(e_4) = e_4 - a_3e_2$ for $\mathbf{g}_{6,k}$, $k = 54, 63$, respectively $\alpha(e_6) = e_6 + a_3e_4$ for $\mathbf{g}_{6,l}$, $l = 57, 59$, maps $\mathbf{k}_{a_1,a_2,a_3}$ onto $\mathbf{k}_{a_2} = \langle e_1 + e_2, e_3 + a_2e_2, e_4 \rangle$.
The Lie algebra $\mathbf{g}_{6,52}$ has the centre $\mathbf{z} = \langle e_2 \rangle$ and the ideals $\mathbf{i}_2 = \langle e_3 \rangle$, $\mathbf{s} = \langle e_2, e_3 \rangle$, $\mathbf{a} = \langle e_1, e_2 \rangle$, $\mathbf{b}_1 = \langle e_1, e_2, e_3 \rangle$, $\mathbf{b}_2 = \langle e_1, e_2, e_4 \rangle$, $\mathbf{g}'_{6,52} = \langle e_1, e_2, e_3, e_4 \rangle$. Denote by Z, I_2, S, A, B_i, $i = 1, 2$, and N the simply connected Lie group of $\mathbf{z}$, $\mathbf{i}_2$, $\mathbf{s}$, $\mathbf{a}$, $\mathbf{b}_i$, $i = 1, 2$, and $\mathbf{g}'_{6,52}$. The factor Lie algebra $\mathbf{g}_{6,52}/\mathbf{s}$ is isomorphic to the elementary filiform Lie algebra $\mathbf{f}_4$. If $\mathbf{g}_{6,52}$ would be the Lie algebra of the group $Mult(L)$ of L, then the above discussion yields that the factor loop $L/Z(e) = L/S(e)$ is isomorphic to a loop $L_{\mathcal{F}}$. As $Z(e) = \mathbb{R} = S(e)$ we have $\dim(\mathbf{s} \cap \mathbf{inn(L)}) = 1$. Since the orbit $I_2(e)$ is a normal subgroup of L isomorphic to $\mathbb{R}$ and $\mathbf{i}_2 \subset \mathbf{s}$ we obtain $I_2(e) = S(e)$ and $\mathbf{i}_2 \cap \mathbf{inn(L)} = 0$. The ideals $\mathbf{a}$, $\mathbf{b}_i$, $i = 1, 2$, and $\mathbf{g}'_{6,52}$ have the properties as in Proposition 3.3 b). Since $\mathbf{z} \subset \mathbf{a}$ one has $Z(e) \subset A(e)$. If $\dim(A(e)) = 1$, then we get $A(e) = Z(e)$. Since the factor Lie algebra $\mathbf{g}_{6,52}/\mathbf{a}$ is not isomorphic to the elementary filiform Lie algebra $\mathbf{f}_4$ the factor loop $L/Z(e)$ is not isomorphic to a loop $L_{\mathcal{F}}$.

Hence the orbit $A(e)$ is a 2-dimensional connected normal subloop M of L and $B_1(e) = B_2(e) = N(e) = M$ (cf. Proposition 3.3 b) such that $Z(e) \subset A(e)$. The ideal $\mathbf{v}$ in Proposition 3.3 b) has one of the following forms: $\mathbf{v}_{1,k} = \langle e_1, e_2, e_3, e_4, e_5 + ke_6 \rangle$, $k \in \mathbb{R}$, $\mathbf{v}_2 = \langle e_1, e_2, e_3, e_4, e_6 \rangle$. Therefore the Lie algebra $\mathbf{inn(L)}$ has either the basis elements $b_1 = e_3 + a_1e_2$, $b_2 = e_4 + b_1e_1 + b_2e_2$, $b_{3,k} = e_5 + ke_6 + c_1e_1 + c_2e_2$ or b_1, b_2, $b_3 = e_6 + c_1e_1 + c_2e_2$, $a_1, b_i, k, c_i \in \mathbb{R}$, $i = 1, 2$, $a_1 \neq 0$. The vector spaces $\langle b_1, b_2, b_{3,k} \rangle$, $\langle b_1, b_2, b_3 \rangle$ are not 3-dimensional Lie algebras. This contradiction proves the assertion. □

If the loop L in Theorems 3.6, 3.7 is not simply connected, then in the universal covering loop $\widetilde{L}$ of L there exists a discrete central normal subgroup Z such that L is isomorphic to $\widetilde{L}/Z$. Since every element of Z associates and commutes with any element of $\widetilde{L}$ we have $\lambda_z\lambda_g(k) = \lambda_g\lambda_z(k)$ and $\lambda_z\rho_g(k) = \rho_g\lambda_z(k)$ for any $z \in Z$, $g, k \in \widetilde{L}$. Hence the set $\widetilde{Z} = \{\lambda_z; z \in Z\}$ is a discrete central subgroup of the group $Mult(\widetilde{L})$. The centre of $Mult(\widetilde{L})$ is isomorphic to $\mathbb{R}$ (cf. Theorems 3.6, 3.7) and hence $\widetilde{Z}$ is isomorphic to $\mathbb{Z}$. Hence the fundamental group of L is isomorphic to $\mathbb{Z}$. Together with Theorems 14 and 16 in [7] and Theorem in [5] we obtain:

Corollary 3.8. *Every connected not simply connected 3-dimensional topological loop L having an indecomposable solvable Lie group of dimension 6 as the group $Mult(L)$ of L is homeomorphic to $\mathbb{R}^2 \times \mathbb{R}/\mathbb{Z}$ and the centre of the group $Mult(L)$ is isomorphic to the group $SO_2(\mathbb{R})$.*

Acknowledgments

We thank to the referee for the valuable suggestions. This research is supported by the EFOP-3.6.1-16-2016-00022 project. The project is co-financed by the European Union and the European Social Fund.

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