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WEAKLY TOTALLY PERMUTABLE PRODUCTS AND FITTING CLASSES

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ABSTRACT. It is known that if $G = AB$ is a product of its totally permutable subgroups A and B, then $G \in \mathfrak{F}$ if and only if $A \in \mathfrak{F}$ and $B \in \mathfrak{F}$ when $\mathfrak{F}$ is a Fischer class containing the class $\mathfrak{U}$ of supersoluble groups. We show that this holds when $G = AB$ is a weakly totally permutable product for a particular Fischer class, $\mathfrak{F} \diamond \mathfrak{N}$, where $\mathfrak{F}$ is a Fitting class containing the class $\mathfrak{U}$ and $\mathfrak{N}$ a class of nilpotent groups. We also extend some results concerning the $\mathfrak{U}$-hypercentre of a totally permutable product to a weakly totally permutable product.

1. Introduction

All groups considered in this article are finite.

Let A and B be subgroups of G. Recall that A and B are totally (mutually) permutable if every subgroup of A permutes with every subgroup of B(respectively if A permutes with every subgroup of B and B permutes with every subgroup of A). Products of totally (respectively mutually) permutable subgroups have been studied extensively. See [1] for more details on these types of products of finite groups.

A and B are weakly totally permutable if for every subgroup, U of A such that $U \leqslant A \cap B$ or $A \cap B \leqslant U$, permutes with every subgroup of B and if for every subgroup V of B such that $V \leqslant A \cap B$ or $A \cap B \leqslant V$, permutes with every subgroup of A. Products of weakly totally permutable subgroups

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were introduced in [7] and it was shown that the concept of products of weakly totally permutable subgroups is strictly more general than that of products of totally permutable subgroups and strictly less general than that of products of mutually permutable subgroups. We call them totally permutable products, weakly totally permutable products and respectively mutually permutable products for short.

In this article, we attempt to answer the question : which results on totally permutable products that do not hold on mutually permutable products can be extended to weakly totally permutable when either the product or the factors are contained in certain Fitting classes?

Recall that a non-empty class $\mathfrak{F}$ of finite groups is a Fitting class if and if the following two conditions hold:(a) if $G \in \mathfrak{F}$ and $N \trianglelefteq G$, then $N \in \mathfrak{F}$ and (b) if M and N are normal subgroups of G with $M, N \in \mathfrak{F}$, then $MN \in \mathfrak{F}$. Let $\mathfrak{F}$ be a Fitting class. The $\mathfrak{F}$-radical of a group G, denoted by $G_{\mathfrak{F}}$, is the unique maximal normal $\mathfrak{F}$-subgroup of G. The existence is guaranteed by the second property of a Fitting class. We now recall the definition of the Fitting product: If $\mathfrak{F}$ is a Fitting class and $\mathfrak{C}$ a class of finite groups, then

$$\mathfrak{F} \diamond \mathfrak{C} := \{G \mid G/G_{\mathfrak{F}} \in \mathfrak{C}\}.$$

If additionally $\mathfrak{C}$ is a Fitting class, it turns out that $\mathfrak{F} \diamond \mathfrak{C}$ is Fitting class [5, IX, Theorem 1.12(a)].

A Fischer class $\mathfrak{F}$ is a Fitting class such that if $N \trianglelefteq G \in \mathfrak{F}$ and H/N is a nilpotent subgroup of G/N, then $H \in \mathfrak{F}$. In [5, IX, Examples (3.7)(c)(2)], it was shown that $\mathfrak{F} \diamond \mathfrak{N}$ is a Fischer class when $\mathfrak{F}$ is a Fitting class and $\mathfrak{N}$ is the class of all finite nilpotent groups. In particular, $\mathfrak{N} \diamond \mathfrak{N} = \mathfrak{N}^2$ is a Fischer class containing $\mathfrak{U}$, the class of all finite supersoluble groups.

For totally permutable products it was shown that the following result holds:

Theorem A. [6, Theorem 2] *Let $\mathfrak{F}$ be a Fischer class containing the class $\mathfrak{U}$ of supersoluble groups. Let a group $G = AB$ be a weakly totally permutable product of subgroups A and B. Then $G \in \mathfrak{F}$ if and only if $A \in \mathfrak{F}$ and $B \in \mathfrak{F}$.*

In the same paper, the authors proved the result below:

Theorem B. [6, Theorem 5] *Let $\mathfrak{F}$ be a Fitting class containing the class $\mathfrak{N}$ of all nilpotent groups. Consider the Fitting product $\mathfrak{N} \diamond \mathfrak{F}$. Let the group $G = AB$ be a totally permutable product of subgroups A and B. Then $G \in \mathfrak{N} \diamond \mathfrak{F}$ if and only if $A \in \mathfrak{N} \diamond \mathfrak{F}$ and $B \in \mathfrak{N} \diamond \mathfrak{F}$.*

On the other hand for mutually permutable products Beidleman and Heineken [3] proved the following result:

Theorem C. [3, Theorem 1] *Let $\mathfrak{F}$ be a Fitting class. Let a group $G = AB$ be a mutually permutable product of subgroups A and B. If $G \in \mathfrak{F}$, then $A \in \mathfrak{F} \diamond \mathfrak{A}$ and $B \in \mathfrak{F} \diamond \mathfrak{A}$, where $\mathfrak{A}$ is the class of abelian groups.*

The dual statement of Theorem C is true as the following result states:

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Theorem D. [4, Corollary] *Let $\mathfrak{F}$ be a Fitting class. Let a group $G = AB$ be a mutually permutable product of subgroups A and B. If $A \in \mathfrak{F}$ and $B \in \mathfrak{F}$, then $G \in \mathfrak{F} \diamond \mathfrak{A}$.*

In this article, we extend Theorem A to a particular Fischer class, $\mathfrak{F} \diamond \mathfrak{N}$, where $\mathfrak{F}$ is a Fitting class containing $\mathfrak{U}$:

Theorem 1.1. *Let $\mathfrak{F}$ be a Fitting class containing the class $\mathfrak{U}$ of supersoluble groups. Consider $\mathfrak{F} \diamond \mathfrak{N}$, where $\mathfrak{N}$ is the class of all nilpotent groups. Let $G = AB$ be a weakly totally permutable product of subgroups A and B. Then $G \in \mathfrak{F} \diamond \mathfrak{N}$ if and only if $A \in \mathfrak{F} \diamond \mathfrak{N}$ and $B \in \mathfrak{F} \diamond \mathfrak{N}$.*

We also extend Theorem B to a Fitting product, $\mathfrak{N}^2 \diamond \mathfrak{F}$, where $\mathfrak{F}$ is a Fitting class containing $\mathfrak{N}$:

Theorem 1.2. *Let $\mathfrak{F}$ be a Fitting class containing the class $\mathfrak{N}$ of all nilpotent groups. Consider the Fitting product $\mathfrak{N}^2 \diamond \mathfrak{F}$. Let the group $G = AB$ be a weakly totally permutable product of subgroups A and B. Then $A \in \mathfrak{N}^2 \diamond \mathfrak{F}$ and $B \in \mathfrak{N}^2 \diamond \mathfrak{F}$ if and only if $G \in \mathfrak{N}^2 \diamond \mathfrak{F}$.*

We conclude this note by generalising some results concerning the $\mathfrak{U}$-hypercentre of a totally permutable product to a weakly totally permutable product. In particular, we show that the following holds:

Proposition 1.3. *Let a group $G = AB$ be the weakly totally permutable product of subgroups A and B. Then $[A, B] \leq Z_{\mathfrak{U}}(G)$. In particular, $A \cap B \leq Z_{\mathfrak{U}}(G)$, where $Z_{\mathfrak{U}}(G)$ denotes the $\mathfrak{U}$-hypercentre of G.*

Corollary 1.4. *Let a group $G = AB$ be the weakly totally permutable product of subgroups A and B. Then $Z_{\mathfrak{U}}(A) = Z_{\mathfrak{U}}(G) \cap A$, $Z_{\mathfrak{U}}(B) = Z_{\mathfrak{U}}(G) \cap B$ and $Z_{\mathfrak{U}}(G) = Z_{\mathfrak{U}}(A)Z_{\mathfrak{U}}(B)$. In particular, $A \cap B \leq Z_{\mathfrak{U}}(A) \cap Z_{\mathfrak{U}}(B)$.*

2. Preliminary Results

Lemma 2.1. [7, Lemma 9] *Let a group $G = AB$ be the weakly totally permutable product of subgroups A and B. Then*

$$[A, B^{\mathfrak{U}}] = [A^{\mathfrak{U}}, B] = 1.$$

Theorem 2.2. [7, Theorems 1-3] *Let $\mathfrak{F}$ be a formation containing $\mathfrak{U}$. Let a group $G = AB$ be the weakly totally permutable product of subgroups A and B.*

(i) *If $\mathfrak{F}$ is a saturated formation, then $G^{\mathfrak{F}} = A^{\mathfrak{F}} B^{\mathfrak{F}}$;*
(ii) *If A_1 and B_1 are $\mathfrak{F}_1$-projectors of A and B respectively, then A_1B_1 is an $\mathfrak{F}$-projector of G;*
(iii) *If A and B belong to $\mathfrak{F}$, then G also belongs to $\mathfrak{F}$.*

It is not known if given a weakly totally permutable product implies that $G \in \mathfrak{F}$ if and only if $A \in \mathfrak{F}$ and $B \in \mathfrak{F}$, when $\mathfrak{F}$ is a Fitting class containing $\mathfrak{U}$. However we look the properties of a minimal counterexample.

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Lemma 2.3. *Let $G = AB$ be a weakly totally permutable product of subgroups A and B. Let $\mathfrak{F}$ be a Fitting class containing the class $\mathfrak{U}$. Suppose that one of the following cases holds:*

Case 1 *: either $A, B \in \mathfrak{F}$, but $G \notin \mathfrak{F}$ with $|G| + |A| + |B|$ minimal, or*

Case 2 *: $G \in \mathfrak{F}$ but not both of A and B in $\mathfrak{F}$ with $|G| + |A| + |B|$ minimal.*

Then, after interchanging the roles of A and B if necessary, we have that B is a subnormal supersoluble subgroup of G, $\langle B^G \rangle$ is supersoluble and $G = A^{\mathfrak{N}} A_p \langle B^G \rangle$.

Proof. The group G has the following properties:

(i) We may assume that B is supersoluble and that A is not supersoluble. Moreover $[A^{\mathfrak{U}}, B] = 1$.

By Lemma 2.1, $[A^{\mathfrak{U}}, B] = [A, B^{\mathfrak{U}}] = 1$. Now A and B cannot be both supersoluble, because in Case 1 G would be supersoluble by [7, Theorem 3] and in Case 2 that would contradict the choice of (G, A, B).

Suppose that neither A nor B is supersoluble. Then $A^{\mathfrak{U}} \neq 1$ and $B^{\mathfrak{U}} \neq 1$. Note that $A^{\mathfrak{U}}$ cannot be central in A and $B^{\mathfrak{U}}$ cannot be central in B otherwise either A or B would be supersoluble contradicting our supposition. Then $B \leqslant \mathbf{C}_G(A^{\mathfrak{U}}) < G$ and $A \leqslant \mathbf{C}_G(B^{\mathfrak{U}}) < G$. Hence $\mathbf{C}_G(A^{\mathfrak{U}}) = B(A \cap \mathbf{C}_G(A^{\mathfrak{U}}))$ is a weakly totally permutable product of subgroups B and $A \cap \mathbf{C}_G(A^{\mathfrak{U}}) \in \mathfrak{F}$. Assume that Case 1 holds. By the choice of (G, A, B) we have that $\mathbf{C}_G(A^{\mathfrak{U}}) \in \mathfrak{F}$. Analogously $\mathbf{C}_G(B^{\mathfrak{U}}) \in \mathfrak{F}$. Since $\mathbf{C}_G(A^{\mathfrak{U}})$ and $\mathbf{C}_G(B^{\mathfrak{U}})$ are normal subgroups of G we have that $G = \mathbf{C}_G(A^{\mathfrak{U}})\mathbf{C}_G(B^{\mathfrak{U}}) \in \mathrm{N}_0\mathfrak{F} = \mathfrak{F}$. Assume now that Case 2 holds. Then $\mathbf{C}_G(A^{\mathfrak{U}})$ is an $\mathfrak{F}$-group since it is a normal subgroup of G. By the choice of (G, A, B), we obtain that $B \in \mathfrak{F}$. Analogously $A \in \mathfrak{F}$ contradicting the choice of (G, A, B).

Hence we may assume that B is supersoluble and that A is not supersoluble.

(ii) B is a subnormal subgroup of G and $\langle B^G \rangle \in \mathfrak{U} \subseteq \mathfrak{F}$.

We first show that $\langle B^G \rangle \in \mathfrak{U} \subseteq \mathfrak{F}$. Note first that $[A^{\mathfrak{U}}, B] = 1$ implies that $[\langle B^G \rangle, A^{\mathfrak{U}}] = 1$. Now

$$(\langle B^G \rangle \cap A)/(\langle B^G \rangle \cap A^{\mathfrak{U}}) \cong (\langle B^G \rangle \cap A^{\mathfrak{U}})/A^{\mathfrak{U}} \leqslant A/A^{\mathfrak{U}} \in \mathfrak{U}$$

and

$$\langle B^G \rangle \cap A^{\mathfrak{U}} \leqslant Z(\langle B^G \rangle \cap A^{\mathfrak{U}}),$$

so $\langle B^G \rangle \cap A$ is supersoluble. Since $A \cap B \leqslant \langle B^G \rangle \cap A$, we have that $\langle B^G \rangle = B(\langle B^G \rangle \cap A)$ is a weakly totally permutable product of supersoluble subgroups and so $\langle B^G \rangle \in \mathfrak{U} \subseteq \mathfrak{F}$ by [7, Theorem 3].

We now argue that B is subnormal in G. By [4, Theorem 2], there exists subgroups L, M such that $A'(A \cap B) \leqslant L \leqslant A$ and $B'(A \cap B) \leqslant M \leqslant B$, L and M are subnormal in G and $G' \leqslant LM$. If $L = A$ and $M = B$, then in Case 1 we have $G = AB \in \mathrm{N}_0\mathfrak{F} = \mathfrak{F}$ and in Case 2 we have that both A and B belong to $\mathfrak{F}$ since A and B are subnormal in G and $\mathfrak{F}$ is a Fitting class.

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Suppose that $L < A$ and $M < B$. Assume that Case 1 holds. Then BL and AM are normal $\mathfrak{F}$-subgroups of G by the choice (G, A, B). Hence $G = (BL)(AM) \in \mathrm{N}_0\mathfrak{F} = \mathfrak{F}$ which is a contradiction. Now suppose that Case 2 holds. Then since BL and AM are normal subgroups of G we have that $BL, AM \in \mathfrak{F}$. By the choice of (G, A, B), we have $A \in \mathfrak{F}$ which is a contradiction. Suppose $L = A$ and $M < B$. If we are in Case 1, then $G = A\langle B^G\rangle \in \mathrm{N}_0\mathfrak{F} = \mathfrak{F}$, a contradiction, and if we are in Case 2, then $A \in \mathfrak{F}$ since A is subnormal and $\mathfrak{F}$ is a Fitting class, which is also a contradiction. We may assume that $M = B$ and $L < A$ and so B is a subnormal supersoluble subgroup of G.

(iii) There exist a prime number p such that $G = A^{\mathfrak{N}}A_p\langle B^G\rangle$, where A_p is a Sylow p-subgroup of A.

Assume that $A^{\mathfrak{N}}A_q\langle B^G\rangle$ is a proper subgroup of G for all primes q, where A_q denotes a Sylow q-subgroup of A. Then $A^{\mathfrak{N}}A_q\langle B^G\rangle = A^{\mathfrak{N}}A_q\langle B^G\rangle \cap AB = A^{\mathfrak{N}}A_q(\langle B^G\rangle\ AB) = A^{\mathfrak{N}}A_q(\langle B^G\rangle \cap AB)$. Note also that $A^{\mathfrak{N}}A_q\langle B^G\rangle$ is a normal subgroup of G. The subgroups B and $X_q = A^{\mathfrak{N}}A_q(\langle B^G\rangle \cap A)$ are weakly totally permutable subgroups since $A \cap B \leqslant A^{\mathfrak{N}}A_q(\langle B^G\rangle \cap A)$. Assume that Case 1 holds. Then $X_q \in \mathfrak{F}$ since it is a normal subgroup of A. By minimality of (G, A, B), $X_qB \in \mathfrak{F}$. Hence $G = \prod_{q\in\mathbb{P}} X_qB \in \mathrm{N}_0\mathfrak{F} = \mathfrak{F}$, a contradiction. Assume we are in Case 2. Then since X_qB is a normal subgroup of G we have that $X_qB \in \mathfrak{F}$. By the choice of (G, A, B), $X_q \in \mathfrak{F}$. Hence $A = \prod_{q\in\mathbb{P}} X_q \in \mathrm{N}_0\mathfrak{F} = \mathfrak{F}$, a contradiction. □

3. Main Results

Theorem 3.1. *Let $\mathfrak{F}$ be a Fitting class containing the class $\mathfrak{U}$ of supersoluble groups. Consider $\mathfrak{F}\diamond\mathfrak{N}$, where $\mathfrak{N}$ is the class of all nilpotent groups. Let $G = AB$ be a weakly totally permutable product of subgroups A and B. Then $G \in \mathfrak{F}\diamond\mathfrak{N}$ if and only if $A \in \mathfrak{F}\diamond\mathfrak{N}$ and $B \in \mathfrak{F}\diamond\mathfrak{N}$.*

Proof. Suppose the theorem is not true and let G be a minimal counterexample. By Lemma 2.3 we have that B is a subnormal supersoluble subgroup of G. Hence $B \leqslant G_{\mathfrak{F}}$ and $G = AG_{\mathfrak{F}}$ for both implications. Assume that $A \in \mathfrak{F}\diamond\mathfrak{N}$, that is, $A/A_{\mathfrak{F}} \in \mathfrak{N}$. Then $A^{\mathfrak{N}} \leqslant A_{\mathfrak{F}}$ since $\mathfrak{N}$ is a formation. This implies that $A^{\mathfrak{N}}$ is a subnormal $\mathfrak{F}$-subgroup of G by [1, Corollary 4.1.25]. Thus $A^{\mathfrak{N}} \leqslant A \cap G_{\mathfrak{F}}$. Hence $A/(A\cap G_{\mathfrak{F}}) \cong AG_{\mathfrak{F}}/G_{\mathfrak{F}} = G/G_{\mathfrak{F}} \in \mathfrak{N}$, that is, $G \in \mathfrak{F}\diamond\mathfrak{N}$, a contradiction. Now assume that $G \in \mathfrak{F}\diamond\mathfrak{N}$. Then $G/G_{\mathfrak{F}} \cong A/(A\cap G_{\mathfrak{F}}) \in \mathfrak{N}$. Since $A^{\mathfrak{N}} \leqslant A'$ and A' is subnormal in G, by [1, Corollary 4.1.25], we have that $A^{\mathfrak{N}}$ is also a subnormal subgroup of G and so a subnormal subgroup of $G_{\mathfrak{F}}$, which implies that $A^{\mathfrak{N}} \in \mathfrak{F}$. Consequently, we have that $A^{\mathfrak{N}} \leqslant A \cap G_{\mathfrak{F}}$. Hence $A^{\mathfrak{N}} \leqslant A_{\mathfrak{F}}$ and $A/A_{\mathfrak{F}} \in \mathfrak{N}$, that is, $A \in \mathfrak{F}\diamond\mathfrak{N}$ which is a contradiction. Hence the result follows. □

Theorem 3.2. *Let $\mathfrak{F}$ be a Fitting class containing the class $\mathfrak{N}$ of all nilpotent groups. Consider the Fitting product $\mathfrak{N}^2\diamond\mathfrak{F}$. Let the group $G = AB$ be a weakly totally permutable product of subgroups A and B. Then $A \in \mathfrak{N}^2\diamond\mathfrak{F}$ and $B \in \mathfrak{N}^2\diamond\mathfrak{F}$ if and only if $G \in \mathfrak{N}^2\diamond\mathfrak{F}$.*

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Proof. Suppose the theorem is not true and let G be a minimal counterexample. Then by Lemma 2.3, since B is a subnormal supersoluble subgroup of G, we have that $B \leqslant G_{\mathfrak{N}^2}$ and $G = AG_{\mathfrak{N}^2}$ for both implications. Assume first that $A \in \mathfrak{N}^2 \diamond \mathfrak{F}$, that is, $A/A_{\mathfrak{N}^2} \in \mathfrak{F}$. By the quasi-$\mathrm{R}_0$-lemma [5, IX, Lemma 1.13], $A/(A^{\mathfrak{N}} \cap A_{\mathfrak{N}^2}) \in \mathfrak{F}$ since $A/A^{\mathfrak{N}} \in \mathfrak{N} \subseteq \mathfrak{F}$. But $A^{\mathfrak{N}} \cap A_{\mathfrak{N}^2} = A^{\mathfrak{N}} \cap G_{\mathfrak{N}^2}$ since $A^{\mathfrak{N}} \cap A_{\mathfrak{N}^2} \leqslant A^{\mathfrak{N}} \leqslant A'$ is a subnormal $\mathfrak{N}^2$-subgroup of G by [1, Corollary 4.1.25]. Consider $A/(A^{\mathfrak{N}}(G_{\mathfrak{N}^2} \cap A)) \in \mathfrak{N}$ and $A/A^{\mathfrak{N}} \in \mathfrak{N} \subseteq \mathfrak{F}$. It follows that $A/(G_{\mathfrak{N}^2} \cap A) \in \mathfrak{F}$ by the quasi-R_0 lemma [5, IX, Lemma 1.13] since $A/(A^{\mathfrak{N}} \cap G_{\mathfrak{N}^2}) \in \mathfrak{F}$. But $A/(G_{\mathfrak{N}^2} \cap A) \cong AG_{\mathfrak{N}^2}/G_{\mathfrak{N}^2} \in \mathfrak{F}$, that is, $G \in \mathfrak{N}^2 \diamond \mathfrak{F}$, a contradiction.

Assume now that $G \in \mathfrak{N}^2 \diamond \mathfrak{F}$. As in the previous argument $A^{\mathfrak{N}} \cap A_{\mathfrak{N}^2} = A^{\mathfrak{N}} \cap G_{\mathfrak{N}^2}$ is a subnormal $\mathfrak{N}^2$-subgroup of G. Since $A/(A \cap G_{\mathfrak{N}^2}) \cong G/G_{\mathfrak{N}^2} \in \mathfrak{F}$ and $A/A^{\mathfrak{N}} \in \mathfrak{N}$ we have $A/(A^{\mathfrak{N}} \cap A_{\mathfrak{N}^2}) \in \mathfrak{F}$ by the quasi-R_0 lemma [5, IX, Lemma 1.13]. Applying the quasi-R_0 lemma [5, IX, Lemma 1.13] again we have that $A/A_{\mathfrak{N}^2} \in \mathfrak{F}$, that is, $A \in \mathfrak{N}^2 \diamond \mathfrak{F}$. Hence the result follows. □

We extend [1, Corollary 4.2.11] to weakly totally permutable products:

Lemma 3.3. *Let a group $G = AB$ be the weakly totally permutable product of subgroups A and B. Then $[A, B]$ is a nilpotent normal subgroup of G.*

Proof. It is clear that $[A, B]$ is a normal subgroup of G. Let $A = A_1A^{\mathfrak{U}}$ and $B = B_1B^{\mathfrak{U}}$ where A_1, B_1 is an $\mathfrak{U}$-projector of A and B, respectively. Then since B centralises $A^{\mathfrak{U}}$ and A centralises $B^{\mathfrak{U}}$ by Lemma 2.1, it follows that

$$[A, B] = [A_1A^{\mathfrak{U}}, B_1B^{\mathfrak{U}}] = [A_1, B_1]$$

which is contained in $(A_1B_1)'$. It follows that A_1B_1 is supersoluble by Theorem 2.2(ii) and hence its derived subgroup is nilpotent. The result then follows. □

The following result generalises [6, Lemma 3] to weakly totally permutable products.

Proposition 3.4. *Let a group $G = AB$ be the weakly totally permutable product of subgroups A and B. Then $[A, B] \leq Z_{\mathfrak{U}}(G)$. In particular, $A \cap B \leq Z_{\mathfrak{U}}(G)$.*

Proof. By Theorem 2.2(i), it follows that $G^{\mathfrak{U}} = A^{\mathfrak{U}}B^{\mathfrak{U}}$. Let A_1 and B_1 be $\mathfrak{U}$-projectors of A and B. Then $[A, B] = [A^{\mathfrak{U}}A_1, B^{\mathfrak{U}}B_1] = [A_1, B_1] \leq A_1B_1$ by Lemma 2.1. Moreover, A_1B_1 is a $\mathfrak{U}$-projector of G by Theorem 2.2(ii). But $(A \cap B)[A, B] \leq \langle A^G \rangle \cap \langle B^G \rangle \leq C_G(A^{\mathfrak{U}}B^{\mathfrak{U}}) = C_G(G^{\mathfrak{U}})$ since $A^{\mathfrak{U}}$ and $B^{\mathfrak{U}}$ are normal subgroups of G by [2, Lemma 1] and the fact that $A^{\mathfrak{U}}$ and $B^{\mathfrak{U}}$ centralise B and A respectively by Lemma 2.1. Hence $(A \cap B)[A_1, B_1] \leq C_{A_1B_1}(G^{\mathfrak{U}}) = Z_{\mathfrak{U}}(G)$ by [5, IV, Theorem 6.14]. □

Corollary 3.5. *Let a group $G = AB$ be the weakly totally permutable product of subgroups A and B. Then*

$$G/Z_{\mathfrak{U}}(G) = (AZ_{\mathfrak{U}}(G)/Z_{\mathfrak{U}}(G)) \times (BZ_{\mathfrak{U}}(G)/Z_{\mathfrak{U}}(G)).$$

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Corollary 3.6. *Let a group $G = AB$ be the weakly totally permutable product of subgroups A and B. Then $Z_{\mathfrak{U}}(A) = Z_{\mathfrak{U}}(G) \cap A$, $Z_{\mathfrak{U}}(B) = Z_{\mathfrak{U}}(G) \cap B$ and $Z_{\mathfrak{U}}(G) = Z_{\mathfrak{U}}(A)Z_{\mathfrak{U}}(B)$. In particular, $A \cap B \leq Z_{\mathfrak{U}}(A) \cap Z_{\mathfrak{U}}(B)$.*

Proof. By Corollary 3.5, it follows that

$$G/Z_{\mathfrak{U}}(G) = (AZ_{\mathfrak{U}}(G)/Z_{\mathfrak{U}}(G)) \times (BZ_{\mathfrak{U}}(G)/Z_{\mathfrak{U}}(G)).$$

Since $Z_{\mathfrak{U}}(G) \cap A$ is a supersolubly embedded normal subgroup of A, it follows that $Z_{\mathfrak{U}}(G) \cap A \leq Z_{\mathfrak{U}}(A)$. Since $Z_{\mathfrak{U}}(G/Z_{\mathfrak{U}}(G)) = 1$, $Z(AZ_{\mathfrak{U}}(G)/Z_{\mathfrak{U}}(G)) = 1$ by Corollary 3.5. Therefore $A/(Z_{\mathfrak{U}}(G) \cap A)$ has a trivial $\mathfrak{U}$-hypercentre. Thus $Z_{\mathfrak{U}}(G) \cap A = Z_{\mathfrak{U}}(A)$. Analogously $Z_{\mathfrak{U}}(G) \cap B = Z_{\mathfrak{U}}(B)$
Let $T = Z_{\mathfrak{U}}(A)Z_{\mathfrak{U}}(B)$. Then T is a normal subgroup of G by [1, Lemma 4.1.21] and $T \leq Z_{\mathfrak{U}}(G)$. Hence $T \cap A$ is a subgroup of $Z_{\mathfrak{U}}(G)$. But by the definition of T, $Z_{\mathfrak{U}}(G) \cap A$ is a subgroup of $T \cap A$. So $Z_{\mathfrak{U}}(G) \cap A = T \cap A$. On the other hand, G/T is a totally permutable product of subgroups AT/T and BT/T by [1, Lemma 4.1.21] since $A \cap B \leq T$ by Proposition 3.4. Hence

$$\mid G/T \mid \leq \mid AT/T \mid\mid BT/T \mid = \mid A/(T \cap A) \mid\mid B/(T \cap B) \mid =$$
$$\mid A/(Z_{\mathfrak{U}}(G) \cap A) \mid\mid B/(Z_{\mathfrak{U}}(G) \cap B) \mid = \mid AZ_{\mathfrak{U}}(G)/Z_{\mathfrak{U}}(G) \mid\mid BZ_{\mathfrak{U}}(G)/Z_{\mathfrak{U}}(G) \mid.$$

This coincides with the order of $G/Z_{\mathfrak{U}}(G)$. Since $T \leq Z_{\mathfrak{U}}(G)$(which implies that $\mid G/Z_{\mathfrak{U}}(G) \mid \leq \mid G/T \mid$) and $\mid G/T \mid \leq \mid G/Z_{\mathfrak{U}}(G) \mid$, it follows that $T = Z_{\mathfrak{U}}(G)$. Hence the result follows. □

Corollary 3.5 and Corollary 3.6 generalises [1, Corollary 4.2.13] and [1, Corollary 4.2.14], respectively.

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