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OPEN NORMAL SUBGROUPS IN NORMALLY CONSTRAINED PRO-p GROUPS

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Abstract. In this paper we analyse properties satisfied by certain open normal subgroups in normally constrained pro-p groups and in a spread version of normally constrained pro-p groups. In the case of powerful normally constrained pro-p groups, we exhibit some kind of inheritance properties in certain open normal subgroups.

Notation. Our notation is standard, see [4, 9], where the reader can also find the basic notions about pro-p groups. In particular, if G is a pro-p group and $i \geq 1$, we denote by $\gamma_i(G)$ the ith term of the lower central series of G, and by $Z_i(G)$ the ith term of the upper central series of G. We denote by $P_i(G)$ the ith term of the exponent p lower central series of G (lower p-series for short), recursively defined by $P_1(G) = G$ and $P_{i+1}(G) = P_i(G)^p[P_i(G), G]$ for $i \in \mathbb{N}$. We define recursively also the *Frattini series* as the one obtained by iterating the Frattini subgroup operation, starting with the whole group. We will denote by $N \trianglelefteq_o G$ and by $N \trianglelefteq_c G$ respectively an open and a closed normal subgroup of G.

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1. Introduction

Pro-p groups whose open normal subgroups satisfy conditions of comparability with respect to inclusion raised some interest in the last decades. In this paper, we give a new characterization of normally constrained pro-p groups introduced in [5]: a topologically finitely generated pro-p group G is said to be normally constrained (NC for short) if every open normal subgroup of G is trapped between two consecutive terms of the lower central series of G. These groups are also known as pro-p groups of obliquity 0 or as 1-sandwich pro-p groups (see [1, 8]). Furthermore in the non-abelian case the requirement that G be topologically finitely generated is redundant (see [5, Section 2]).

The aim of this paper is twofold: firstly, to explore properties satisfied by open normal subgroups of a normally constrained pro-p group, and secondly, focusing on powerful normally constrained pro-p groups to exhibit some kind of inheritance properties in certain open normal subgroups. The concept of a powerful pro-p group was introduced in the 1980's by Lubotzky and Mann, and this concept has been used for instance to reinterpret the group-theoretic aspects of Lazard's work. In the present paper we study connections between certain subgroups in the class of NC groups and some general results related to the property of being of finite width in the class of pro-p groups are exhibited.

2. Preliminaries

Let us start this section recalling and giving some definitions. We recall that a waist W in a pro-p group G ([6]) is a subgroup which, respect to inclusion, is comparable with any open normal subgroup of G. Following [7], if V, U are proper non-trivial closed normal subgroups of G such that $V \leq U$ and $|U : V|$ is finite, then the pair (U, V) is said to be a waist pair in G (WP for short), if for every open normal subgroup N of G, either $N \leq U$ or $N \geq V$. In this case by [7, Proposition 3.2], the subgroup U and consequently V are necessarily open normal subgroups. We also recall that a pro-p group G is a just infinite group if G has no non-trivial closed normal subgroups of infinite index.

We recall two defininitions from [7].

Definition 2.1. *If H is an open normal subgroup of G, then we define:*

$$H{\downarrow} = H \cap \bigcap_{\substack{N \trianglelefteq_o G \\ N \not\leq H}} N \qquad \text{and} \qquad H{\uparrow} = H \cdot \prod_{\substack{N \trianglelefteq_o G \\ N \not\geq H}} N$$

Remark 2.2. *The following three conditions are equivalent:*

(1) $N \trianglelefteq_o G$ *is a waist*
(2) $N = N{\uparrow}$
(3) $N = N{\downarrow}$

Remark 2.3. *If for a given $H \triangleleft_o G$ the index $|H : H{\downarrow}|$ is finite (which is the case when G is a just infinite group by* [1, Theorem 36]*), and $H{\downarrow} \neq 1$ then $(H, H{\downarrow})$ is a WP, since for any open normal*

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subgroup K not contained in H, clearly $K \geq H\downarrow$. Moreover, if (H,V) is a WP, then $V \leq H\downarrow$. In a similar way, it is easy to prove that if $H\uparrow \neq G$, then $(H\uparrow, H)$ is a WP, and that $U \geq H\uparrow$ for any other waist pair (U,H).

Moreover, the way of building waist pairs exhibited in the last remark is a kind of idempotent process, (see [7, Lemmas 4.3 and 4.4]).

Lemma 2.4. *If H, K are open normal subgroups of G such that $H \leq K$, then:*

(1) $H\downarrow\uparrow \leq H$
(2) $H\uparrow\downarrow \geq H$
(3) $H\uparrow \leq K\uparrow$
(4) $H\downarrow \leq K\downarrow$

As a consequence,

(5) $H\downarrow\uparrow\downarrow = H\downarrow$
(6) $H\uparrow\downarrow\uparrow = H\uparrow$

To complete this section, let us introduce two new operators defined on the lattice of normal subgroups of a pro-p group.

Definition 2.5. *The upper central closure of a normal subgroup H is defined as $H^{\mathrm{uc}} = C_G(G/[H,G])$, and H is said to be upper central closed when $H = H^{\mathrm{uc}}$. Similarly, the lower central closure of a normal subgroup H of G is defined as $H_{\mathrm{lc}} = [C_G(G/H), G]$, and H is said to be lower central closed when $H = H_{\mathrm{lc}}$.*

It is clear that $H \leq H^{\mathrm{uc}}$ and $H_{\mathrm{lc}} \leq H$.

Lemma 2.6. *The upper and the lower central closure operators are idempotent, in the sense that $H^{\mathrm{uc}} = (H^{\mathrm{uc}})^{\mathrm{uc}}$ and $H_{\mathrm{lc}} = (H_{\mathrm{lc}})_{\mathrm{lc}}$.*

Proof. By the definition of H^{uc} it follows that $[H^{\mathrm{uc}}, G] \leq [H,G]$, and since $H \leq H^{\mathrm{uc}}$ we have that $[H,G] \leq [H^{\mathrm{uc}}, G]$. Thus $[H^{\mathrm{uc}}, G] = [H,G]$, and consequently $(H^{\mathrm{uc}})^{\mathrm{uc}} = C_G(G/[H^{\mathrm{uc}}, G]) = C_G(G/[H,G]) = H^{\mathrm{uc}}$. An analogous argument works for the lower central closure. □

Remark 2.7. *Notice that $H = H_{\mathrm{lc}}$ if and only if $H = [K,G]$ for a suitable $K \trianglelefteq G$, and that $H = H^{\mathrm{uc}}$ if and only if $H = C_G(G/K)$ for some $K \trianglelefteq G$. Thus, in a pro-p group G for $i \geq 1$ the terms of the upper central series are upper central closed, i.e., $Z_i(G)^{\mathrm{uc}} = Z_i(G)$, and for $i > 1$ the terms of the lower central series are lower central closed, i.e., $\gamma_i(G)_{\mathrm{lc}} = \gamma_i(G)$.*

3. Normally constrained pro-p groups and generalizations

In this section, we shall use repeately, without further mention three properties satisfied by any normally constrained pro-p group G:

(1) the "covering property", first introduced in [2], i.e., $[x, G]\gamma_{i+2}(G) = \gamma_{i+1}(G)$ for all $i \geq 1$ and $x \in \gamma_i(G) - \gamma_{i+1}(G)$,
(2) the upper central series and the lower central series coincide if G is nilpotent, and more generally $C_G(G/\gamma_{i+1}(G)) = \gamma_i(G)$ for $i \geq 1$ when G is not nilpotent,
(3) the sections of the lower and upper central series are elementary abelian.

We need two more definitions (see [3, Definitions 6, 1 and 7]).

Definition 3.1. *Let G be a pro-p group.*

(1) *G is said to be of weak finite width if $|\gamma_n(G)/\gamma_{n+1}(G)|$ is finite for all $n \geq 1$.*
(2) *G is said to be of finite width if $sup_{n\geq 1}|\gamma_n(G)/\gamma_{n+1}(G)|$ is finite.*

Definition 3.2. *Let G be a pro-p-group of finite width. Put*

$$\mu_n(G) = \gamma_{n+1}(G) \cap \bigcap\{N \triangleleft G : N \not\leq \gamma_{n+1}(G)\}.$$

The obliquity of G is defined to be

$$o(G) = \sup_n \log_p |\gamma_{n+1}(G) : \mu_n(G)|.$$

The group G is said to be of finite obliquity if $o(G) < \infty$.

Examples of pro-p groups of finite obliquity are

(1) infinite, insoluble p-adic analytic pro-p groups (see [8, III(d)]),
(2) the Nottingham pro-p group,
(3) periodic pro-p groups (see [1]).

It is well known that a pro-p group of finite obliquity is just infinite. It is also well known that a pro-p group of finite width is sandwich if and only if it has finite obliquity (see [3, Proposition 4]).

We have:

Proposition 3.3. *Let G be a normally constrained pro-p group with p odd. If $N \trianglelefteq_o G$ then $N{\uparrow} = N^{uc}$ unless G is finite and $N = 1$, and (dually) $N{\downarrow} = N_{lc}$ unless $N = G$.*

Proof. By the definition of normally constrained pro-p group, every term of the lower central series of G is a waist. Thus, by Remark 2.2, $\gamma_i(G){\uparrow} = \gamma_i(G)$ and $\gamma_i(G){\downarrow} = \gamma_i(G)$. Moreover, by definition, $\gamma_i(G)_{lc} = \gamma_i(G)$ and $\gamma_i(G) \leq \gamma_i(G)^{uc}$. Indeed $\gamma_i(G) = \gamma_i(G)^{uc}$, as the upper central series and the lower central series of the NC pro-p group $G/\gamma_{i+1}(G)$ coincide. We finally have that $\gamma_i(G)^{uc} = \gamma_i(G) = \gamma_i(G){\uparrow}$ and $\gamma_i(G) = \gamma_i(G)_{lc} = \gamma_i(G){\downarrow}$. Thus, the statement of the proposition is true if N is a term of the lower central series.

Assume now that $N \trianglelefteq_o G$ is not a term of the lower central series of G. Let $i \in \mathbb{N}$ be such that $\gamma_i(G) < N < \gamma_{i-1}(G)$. By the covering property it follows that $[N, G] = \gamma_i(G)$. Hence $N^{uc} =$

$C_G(G/[N,G]) = C_G(G/\gamma_i(G)) = \gamma_{i-1}(G)$ as G is a NC group. Similarly $N_{\mathrm{lc}} = [C_G(G/N), G] = [\gamma_{i-1}(G), G] = \gamma_i(G)$. By Lemma 2.4 (3) $N\!\uparrow \leq \gamma_{i-1}(G)\big\uparrow = \gamma_{i-1}(G)$. Since in a non-abelian NC pro-p group the sections of the lower central series are elementary abelian, there exists a normal subgroup M of G having index p in $\gamma_{i-1}(G)$ such that $N \not\leq M$. Then $N\!\uparrow \geq NM = \gamma_{i-1}(G)$, so that $N\!\uparrow = \gamma_{i-1}(G)$. Arguing dually it is possible to see that $N\!\downarrow = \gamma_i(G)$. Hence, $N\!\uparrow = N^{uc} = \gamma_{i-1}(G)$ and $N\!\downarrow = N_{\mathrm{lc}} = \gamma_i(G)$, for every $N \trianglelefteq_o G$. □

Remark 3.4. *If G is a pro-p group with p odd such that $N\!\downarrow = N_{\mathrm{lc}}$ for any $N \triangleleft_o G$ then trivially $\gamma_i(G)\big\downarrow = \gamma_i(G)_{\mathrm{lc}} = \gamma_i(G)$ for any $i > 1$, which, by Remark 2.2, implies that $\gamma_i(G)$ is a waist for any $i > 1$, and consequently that G is a NC pro-p group.*

By the statements of Proposition 3.3 and Remark 3.4, we deduce the following result which gives a characterization of NC pro-p groups not depending on the central series.

Theorem 3.5. *Let p be an odd prime. A pro-p group G is an NC group if and only if $N\!\downarrow = N_{\mathrm{lc}}$ for any $N \triangleleft_o G$.*

A natural generalization of the class of NC pro-p groups is given by:

Definition 3.6. *Let G be a pro-p group and r be a positive integer.*

1. *We say that G is an r-normally constrained pro-p group if for any $N \trianglelefteq_o G$ and any $i \geq 1$, either $N \leq \gamma_i(G)$ or $N \geq \gamma_{i+r-1}(G)$.*
2. *We say that G is an r-normally-p constrained pro-p group if for any $N \trianglelefteq_o G$ and any $i \geq 1$, either $N \leq P_i(G)$ or $N \geq P_{i+r-1}(G)$.*

Remark 3.7. *The item (i) in Definition 3.6 is clearly equivalent to saying that for any $N \trianglelefteq_o G$ there exists $i \in \mathbb{N}$ such that $\gamma_{i+r}(G) \leq N \leq \gamma_i(G)$; in other words G being an r-sandwich. Similarly item (ii) in Definition 3.6 is clearly equivalent to saying that for any $N \trianglelefteq_o G$ there exists $i \in \mathbb{N}$ such that $P_{i+r}(G) \leq N \leq P_i(G)$.*

Remark 3.8. *For a non-procyclic pro-p group G items (i) and (ii) of the previous definition are equivalent when $r = 1$, by* [7, Proposition 2.2].

Proposition 3.9. *Let G be a pro-p group. Then G is an r-normally constrained group if and only if for any $j \geq 1$, $\gamma_j(G)\big\downarrow \geq \gamma_{j+r-1}(G)$ and $\gamma_{j+r-1}(G)\big\uparrow \leq \gamma_j(G)$.*

Proof. Notice that by a consequence of Theorem 28 in [1] we have that the index between $\gamma_j(G)$ and $\gamma_{j+r-1}(G)$ is finite, and therefore, since G is an r-normally constrained pro-p group, it follows that $(\gamma_j(G), \gamma_{j+r-1}(G))$ is a waist pair of G for any $j \geq 1$. This yields $\gamma_j(G)\big\downarrow \geq \gamma_{j+r-1}(G)$ by Remark 2.3. Dually one can prove that $\gamma_{j+r-1}(G)\big\uparrow \leq \gamma_j(G)$. The converse is straightforward. □

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Proposition 3.10. *Let G be an r-normally constrained pro-p group. If $N \trianglelefteq_o G$ is any open normal subgroup then $N{\downarrow} \geq [N,_{2r-1} G]$.*

Proof. Since G is an r-normally constrained pro-p group, for some index i, $\gamma_{i+r}(G) \leq N \leq \gamma_i(G)$. By Proposition 3.9 G is r-normally constrained if and only if $\gamma_j(G){\downarrow} \geq \gamma_{j+r-1}(G)$ for all $j \geq 1$. It follows that

$$N{\downarrow} \geq \gamma_{i+r}(G){\downarrow} \geq \gamma_{i+2r-1}(G) = [\gamma_i(G),_{2r-1} G] \geq [N,_{2r-1} G],$$

as claimed. □

If N is as in the previous proof we have shown that $\gamma_i(G) \geq N \geq N{\downarrow} \geq \gamma_{i+2r-1}(G)$ so that we have:

Corollary 3.11. *Let G be an r-normally constrained pro-p group of finite width w. If $N \triangleleft_o G$, then $|N : N{\downarrow}| \leq p^{w(2r-1)}$.*

For $r = 1$ the previous corollary is the well known fact that in an NC pro-p group G the size of the Frattini factor bounds the size of all the lower central factors.

4. Powerful normally constrained pro-p groups

In this section, we exhibit some curious results of a kind of inheritance in powerful normally constrained pro-p groups. We recall that a pro-p group G is *powerful* if the commutator subgroup is contained in G^p for p odd (i.e., when G/G^p is abelian), and contained in G^4 for $p = 2$. Furthermore an open subgroup $N \leq_o G$ is *powerfullly embedded* in the pro-p group G (written N p.e. G for short) if $[N, G]$ is contained in N^p for p odd and is contained in N^4 if $p = 2$. In particular, if N is p.e. G, then N is normal powerful subgroup of G.

The property of being powerful is readily inherited by factor groups and by direct products, but straightforward examples show that often it is not inherited by subgroups. Moreover, a pro-p group G is *hereditarily powerful* if every open subgroup of G is powerful. For instance, Lubotzky and Mann in [10] gave a characterisation of hereditarily powerful finite p-groups, and, as a consequence, also a characterisation of finitely generated hereditarily powerful pro-p groups.

We recall that a pro-p group G is *uniformly powerful*, or uniform for short, if

(1) G is finitely generated
(2) G is powerful
(3) $|P_i(G) : P_{i+1}(G)| = |G : P_2(G)|$ for all $i \in \mathbb{N}$.

A useful characterisation of uniform pro-p groups is the following (see [4, Theorem 4.5]) : a pro-p group if uniform if and only if it is finitely generated, torsion-free and powerful.

Proposition 4.1. *Let G be a powerful normally constrained pro-p group with p odd. Then all terms of the lower central series of G are p.e. G, and for any $N \triangleleft G$ and $k \geq 1$ we have that $[N^{p^k}, G] \leq [N, G]^{p^k}$*

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and equality holds if N is p.e. G. Moreover, there exists a positive integer $\overline{k}$ such that $U := \gamma_{\overline{k}}(G)$ is uniform in G, and for any $N \lhd G$ contained in U, N is p.e. G if and only if N^p is p.e. G.

Proof. It is known that in a NC pro-p group the lower central series and the lower p-series coincide. Moreover the p-th power map induces a surjective homomorphism between consecutive lower central sections as G is powerful; (i.e., the lower central series consists of waists that are all p-powers, i.e., $\gamma_i(G)^p = \gamma_{i+1}(G)$ for all $i \geq 1$). Thus the terms of the lower central series are p.e. G, and in particular, they are powerful.

Let $N \trianglelefteq_o G$ be an open normal subgroup. We know that there exists a positive integer i such that $\gamma_i(G)^p = \gamma_{i+1}(G) < N \leq \gamma_i(G)$. By Proposition 3.3, $N_{lc} = \gamma_i(G)^p = \gamma_{i+1}(G) = [N, G]$. Thus $\gamma_{i+2}(G) = \gamma_{i+1}(G)^p = [N, G]^p$. Since $N^p \leq \gamma_{i+1}(G)$, then $[N^p, G] \leq \gamma_{i+2}(G)$. Hence $[N^p, G] \leq [N, G]^p$. If N is p.e. G then $\gamma_{i+1}(G) = [N, G] \leq N^p$ so that $[N, G] = N^p$ and $[N^p, G] = [[N, G], G] = [\gamma_{i+1}(G), G] = \gamma_{i+2}(G) = \gamma_{i+1}(G)^p = [N, G]^p$. The statement of the first part of the proposition follows then by induction.

Let us prove now the second part of the claim. Since G is a finitely generated powerful pro-p group, it is well known that there exists a term of the lower p-series which is open and uniform. Thus, since the lower p-series and the lower central series coincide, there exists a positive integer $\overline{k}$ such that $U := \gamma_{\overline{k}}(G)$ is uniform, and obviously powerful. Let N be an open normal subgroup contained in U. By [4, Proposition 2.3] N p.e. G implies N^p p.e. G. The converse is true as well, since U is uniform. □

Corollary 4.2. *Let G be a torsion free powerful normally constrained pro-p group and $N \lhd_o G$. Then N is p.e. G if and only if N^p is p.e. G.*

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