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A NOTE ON LOCALLY SOLUBLE ALMOST SUBNORMAL SUBGROUPS IN DIVSION RINGS

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Abstract. Let D be a division ring with center F and assume that N is a locally soluble almost subnormal subgroup of the multiplicative group D^* of D. We prove that if N is algebraic over F, then N is central. This answers partially [11, Conjecture 1].

1. Introduction

Let D be a division ring with center F. The well-known result of Hua [15] stated that if the multiplicative group D^* of D is soluble, then $D = F$. In [19], Stuth generalized this result for subnormal subgroups of D^* and showed that every soluble subnormal subgroup N of D^* is central, i.e. $N \subseteq F$. In [20], Zalesskii investigated locally soluble property of normal subgroups of D^* and proved that every locally soluble normal subgroup of D^* is central. It was shown that every locally soluble subnormal subgroup of D^* is central if D is algebraic over F [10] or D is weakly locally finite [11]. For the subgroup structure of D^*, [14] is maybe a good survey. In this paper, we study locally soluble almost subnormal subgroups of D^*.

Recall that an element of D is *algebraic* over F if it is a root of some non-zero polynomial over F. If every element of a subset S of D is algebraic over F, then S is said to be *algebraic* over F. We say that a division ring D is *centrally finite* if D is a finite dimensional vector space over its center

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and *weakly locally finite* if every finitely generated division subring of D is centrally finite. From the results above, it is natural that the following conjecture was posed.

Conjecture 1.1. [11, Conjecture 1] *Let D be an arbitrary division ring and assume that N is a locally soluble subnormal subgroup of D^*. Then, N is central.*

The main aim of this note is to prove that Conjecture 1.1 holds in the case when N is algebraic over F. Moreover, we prove this result for almost subnormal subgroups of D^*. In fact, we show that if N is a locally soluble almost subnormal subgroup of D^* that is algebraic over F, then N is central (Theorem 2.8). Recall that, for a group G, according to Hartley [13], a subgroup N of G is said to be *almost subnormal* in G if there are subgroups

$$N = N_r \leq N_{r-1} \leq \cdots \leq N_1 = G$$

of G such that for each $1 < i \leq r$, either N_i is normal in N_{i-1} or N_i has finite index in N_{i-1}. By definition, every subnormal subgroup of a group is almost subnormal. It was noted in [3] that there is a vast number of division rings whose multiplicative groups contain almost subnormal subgroups that are not subnormal. Recently, the structure of almost subnormal subgroups in division rings has received considerable attention (e.g., see [3, 5, 8, 9, 18]).

By applying the main theorem, we obtain a result concerning maximal subgroups in division rings: assume that D is a non-commutative division ring that is algebraic over its center F and M is a maximal subgroup of a non-central almost subnormal subgorup of D^* such that $\mathrm{C}_D(M) \neq F$. We prove that every locally soluble almost subnormal subgroup of M is abelian (Corollary 2.9). From this fact, we can deduce that if M is a maximal subgroup of D^* such that $\mathrm{C}_D(M) \neq F$, then every locally soluble almost subnormal subgroup of M is abelian (Corollary 2.10).

The technique we use in this paper is from [6] concerning recent results on the existence of non-cyclic free subgroups in division rings. This technique was used in [4] to investigate Engel subnormal subgroups of skew linear groups and in [7] to study subnormal subgroups containing soluble subgroups in division rings.

Throughout this paper D is a division ring with center F. All notations and symbols in this note are standard. In particular, for any non-empty subset S of D, $\mathrm{C}_D(S)$ denotes the centralizer of S in D. Notations $F[S]$ and $F(S)$ denote the subring and the division subring of D generated by S over F respectively.

2. **Results**

We begin this section with the following lemma.

Lemma 2.1. *Let D be a division ring. For every non-central torsion element α of D^*, there exists a centrally finite division subring D_1 of D containing α as a non-central element.*

Proof. The lemma is implied directly from [2, Proposition 2.2]. □

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Lemma 2.1 has several applications. One of them is the following corollary on almost subnormal subgroups of D^* containing elments that are algebraic over finite fields.

Corollary 2.2. *Let D be a division ring with finite center F and N an almost subnormal subgroup of D^*. If $N \setminus F$ contains an algebraic element over F, then N is not locally soluble.*

Proof. Assume that $\alpha \in N \setminus F$ is algebraic over F. Since F is finite, $F(\alpha)$ is a finite field. Hence, α is torsion. By Lemma 2.1, there exists a centrally finite division subring D_1 of D such that $\alpha \in D_1 \setminus Z(D_1)$. Put $N_1 = N \cap D_1^*$. Then, N_1 is an almost subnormal subgroup of D_1^*. By [18, Theorem 4.2], N_1 contains a non-cyclic free subgroup. Thus, N is not locally soluble. □

Let D be a division ring with center F and $a \in D^*$. Assume that S is a non-empty subset of D. We say that S is *a-invariant* if $aSa^{-1} \subseteq S$. Put

$$S^{-1} = \{x^{-1} \in D \mid x \in S^* = S \backslash \{0\}\}.$$

Then, S is a-invariant iff S^{-1} is a-invariant. To see this, suppose that S is a-invariant. For any $s \in S^*$, we have $as^{-1}a^{-1} = (asa^{-1})^{-1} \in S^{-1}$. Therefore, $aS^{-1}a^{-1} \subseteq S^{-1}$. This means that S^{-1} is a-invariant. The reverse can be inferred from the above argument. For a subset N of D^*, we say that S is *N-invariant*, if $xS^*x^{-1} = S^*$ for any $x \in N$. In the following lemma, we prove that if S is a-invariant, then the division subring $F(S)$ of D generated by S over F is a-invariant.

Lemma 2.3. *Let D be a division ring with center F, S a subset of D and $a \in D^*$. If S is a-invariant, then $F(S)$ is a-invariant.*

Proof. We construct a family of subrings $\{L_n \mid n \in \mathbb{N}\}$ of D containing S as follows. Put $L_0 = F[S \cup S^{-1}]$ is a subring of D generated by $S \cup S^{-1}$ over F. Assume that L_n is defined. Put $L_{n+1} = F[L_n \cup L_n^{-1}]$, the subring of D generated by $L_n \cup L_n^{-1}$ over F. Then, we have the sequence of subrings of D containing S

$$L_0 \subseteq L_1 \subseteq \cdots \subseteq L_n \subseteq \cdots .$$

Put $L = \bigcup_{n\in\mathbb{N}} L_n$. We will prove that L is the division subring of D generated by S over F, i.e. $L = F(S)$. First, we prove that L is a division ring. Let $\alpha, \beta \in L$. We have $\alpha \in L_k, \beta \in L_t$ for some $k, t \in \mathbb{N}$. Then, $\alpha, \beta \in L_{k+t}$. We deduce that $\alpha + \beta, \alpha\beta \in L_{k+t} \subseteq L$. If $\alpha \neq 0$, then $\alpha^{-1} \in L_k^{-1}$. Hence, $\alpha^{-1} \in F[L_k \cup L_k^{-1}] = L_{k+1} \subseteq L$. Therefore, L is a division subring of D. Since $L \supseteq F$ and $L \supseteq S$, we have $L \supseteq F(S)$. Now, we prove that $L_n \subseteq F(S)$ by induction on n. We have $L_0 = F[S \cup S^{-1}] \subseteq F(S)$. Assume that $L_n \subseteq F(S)$. We prove that $L_{n+1} \subseteq F(S)$. Since $F(S)$ is a division ring, $L_n^{-1} \subseteq F(S)$. Hence, $L_{n+1} = F[L_n \cup L_n^{-1}] \subseteq F(S)$. Thus, $L_n \subseteq F(S)$ for all $n \in \mathbb{N}$ and $L = \bigcup_{n\in\mathbb{N}} L_n \subseteq F(S)$. We conclude that $L = F(S)$.

Next, we will prove that L_n is a-invariant for any $n \in \mathbb{N}$ by induction on n. We prove that L_0 is a-invariant. Let $\alpha \in L_0 = F[S \cup S^{-1}]$, then $\alpha = \sum_{i=1}^{\ell} \alpha_i s_{i_1} s_{i_2} \cdots s_{i_t}$, where $\alpha_i \in F, s_{i_1}, s_{i_2}, \ldots, s_{i_t} \in$

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$S\cup S^{-1}$, ℓ is a natural number and $i=1,2,\dots,\ell$. Then,

$$a\alpha a^{-1}=a(\sum_{i=1}^{\ell}\alpha_i s_{i_1}s_{i_2}\cdots s_{i_t})a^{-1}=\sum_{i=1}^{\ell}\alpha_i a s_{i_1}s_{i_2}\cdots s_{i_t}a^{-1}$$
$$=\sum_{i=1}^{\ell}\alpha_i a s_{i_1}a^{-1}as_{i_2}a^{-1}\cdots as_{i_t}a^{-1}.$$

Since S is a-invariant, S^{-1} is a-invariant. Therefore, $as_{i_j}a^{-1}\in S\cup S^{-1}$ for all $i=1,2,\dots,\ell$ and $i_j=i_1,i_2,\dots,i_t$ and

$$\sum_{i=1}^{\ell}\alpha_i a s_{i_1}a^{-1}as_{i_2}a^{-1}\cdots as_{i_t}a^{-1}\in F[S\cup S^{-1}].$$

Hence, $a\alpha a^{-1}\in F[S\cup S^{-1}]=L_0$. Thus, L_0 is a-invariant. Assume that L_n is a-invariant. We prove that L_{n+1} is a-invariant. Since $L_{n+1}=F[L_n\cup L_n^{-1}]$, by the similar argument as above, we conclude L_{n+1} is a-invariant. Thus, L_n is a-invariant for all natural number n.

Now, since $F(S)=L=\bigcup_{n\in\mathbb{N}}L_n$, we deduce that $F(S)$ is a-invariant. The proof is complete. □

For the convinience, we recall the following result.

Lemma 2.4. [3, Theorem 3.10] *Let D be a division ring with infinite center F and K be a division subring of D. Assume that N is a non-central almost subnormal subgroup of D^*. If K is N-invariant, then either $K\subseteq F$ or $K=D$.*

Using Lemma 2.4, we get the following corollary on abelian almost subnormal subgorup of D^*.

Corollary 2.5. *Let D be a division ring with infinite center and N is an almost subnormal subgroup of D^*. If N is abelian, then N is central.*

Proof. Suppose that N is non-central. Since $F(N)$ is N-invariant, by Lemma 2.4, we have $F(N)\subseteq F$ or $F(N)=D$. If $F(N)\subseteq F$, then $N\subseteq F$, a contradiction. If $F(N)=D$, then D is commutative. Hence, $D=F$ and $F(N)=F$. Therefore, $N\subseteq F$, a contradicion. Thus, N is central. □

The next two lemmas deal with the structure of subrings of division rings.

Lemma 2.6. *Let D be a division ring, K a subfield of D, and L a subring of D containing K. If L is a finite dimensional left vector space over K, then L is a centrally finite division ring.*

Proof. Let $\alpha\in L\backslash\{0\}$. Since $\dim_K L<\infty$, we have $\{\alpha^i\mid i\in\mathbb{N}\}$ are left dependent over K. Therefore, there exist $k_0,k_1,\dots,k_t\in K$ not all are zeros such that $k_0+k_1\alpha+k_2\alpha^2+\cdots+k_t\alpha^t=0$. Assume that t is the smallest integer having such a property. Then, $(k_1+k_2\alpha+\cdots+k_t\alpha^{t-1})\alpha=-k_0$. If $k_0=0$, then $(k_1+k_2\alpha+\cdots+k_t\alpha^{t-1})\alpha=0$. Hence $k_1+k_2\alpha+\cdots+k_t\alpha^{t-1}=0$. This contradicts the choice of t. Thus, $k_0\neq 0$ and

$$(-k_0)^{-1}(k_1+k_2\alpha+\cdots+k_t\alpha^{t-1})\alpha=1.$$

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We conclude that α is invertible and L is a division ring. Now, assume that E is a maximal subfield of L containing K. Then, $\dim_E L < \infty$. By [16, Theorem 15.8, p.242], L is centrally finite. □

Lemma 2.7. *Let D be a division ring with center F and K a subfield of D. Suppose that $a \in D^*$ such that K is a-invariant. Then $L = K + Ka + Ka^2 + \cdots$, the vector subspace of the left vector space D generated by $\{a^i \mid i \in \mathbb{N}\}$ over K, is a subring of D. Moreover, if a is algebraic over F, then L is a centrally finite division ring.*

Proof. It is clear that L is closed under addition. To show L is closed under multiplication, it is sufficient to prove that $(ka^i)(\ell a^j) \in L$, where i, j are natural numbers and $k, \ell \in K$. In fact, $(ka^i)(\ell a^j) = ka^i\ell a^j = k(a^i\ell a^{-i})a^{i+j}$. We have $a^i\ell a^{-i} = a \cdot a \cdots a \cdot \ell \cdot a^{-1} \cdots a^{-1} \cdot a^{-1}$. Since K is a-invariant, we have $a\ell a^{-1} \in K$ and $a \cdot (a \cdots (a \cdot \ell \cdot a^{-1}) \cdots a^{-1}) \cdot a^{-1} \in K$. Then, $k(a^i\ell a^{-i})a^{i+j} \in Ka^{i+j}$. Thus $(ka^i)(\ell a^j) \in L$ and we can conclude that L is a subring of D.

Now, assume that a is algebraic over F. Let m be the degree of the minimal polynomial of a. We will prove that $L = K + Ka + \cdots + Ka^{m-1}$. It is obvious that $K + Ka + \cdots + Ka^{m-1} \subseteq L$. To prove $L \subseteq K + Ka + \cdots + Ka^{m-1}$, it suffices to show that $a^{m+k} \in K + Ka + \cdots + Ka^{m-1}$ for any natural number k. Assume that $a^m + \alpha_{m-1}a^{m-1} + \cdots + \alpha_1 a + \alpha_0 = 0$, where $\alpha_i \in F, i = 0, 1, \ldots, m-1$. We consider two possible cases.

Case 1. $F \subseteq K$. Then

$$a^m = -\alpha_0 - \alpha_1 a - \cdots - \alpha_{m-1}a^{m-1} \in K + Ka + \cdots + Ka^{m-1},$$

and consequently $Ka^m \subseteq K + Ka + \cdots + Ka^{m-1}$. Hence,

$$a^{m+1} = a^m a \in (K + Ka + \cdots + Ka^{m-1})a = Ka + Ka^2 + \cdots + Ka^m,$$

which implies that $a^{m+1} \in K + Ka + \cdots + Ka^{m-1}$. By inductive method, we get $a^{m+k} \in K + Ka + \cdots + Ka^{m-1}$. Thus $L = K + Ka + \cdots + Ka^{m-1}$. From this fact, we deduce that $\dim_K L < \infty$. By Lemma 2.6, L is a centrally finite division ring.

Case 2. $F \not\subseteq K$. Put $E = F(K)$, the subfield of D generated by K over F. It is obvious that $aEa^{-1} \subseteq E$. By the similar argument as above,

$$Q = E + Ea + Ea^2 + \cdots$$

is a centrally finite division ring. By [12, Theorem 3], L is centrally finite. □

Now, we are ready to prove the main result of this note, which gives the affirmative answer to the Conjecture 1.1 in case of algebraic locally soluble almost subnormal subgroups and generalizes [10, Theorem 2.4].

Theorem 2.8. *Let D be a division ring with center F and assume that N is a locally soluble almost subnormal subgroup of D^*. If N is algebraic over F, then N is central.*

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Proof. Suppose that N is non-central. We consider two possible cases.

Case 1. F is finite. Since N is non-central, $N \setminus F \neq \emptyset$. Hence, there exists an element $\alpha \in N \setminus F$. By hypothesis, α is algebraic over F. By Corollary 2.2, N is not locally soluble, a contradiction. Thus, N is central.

Case 2. F is infinite. By Corollary 2.5, N is non-abelian and there exist $c, d \in N$ such that $cd \neq dc$. Put $H = \langle c, d \rangle$, then H is soluble and non-abelian. There exists a series of subgroups of H such that

$$H = H^{(0)} \triangleright H^{(1)} \triangleright H^{(2)} \triangleright \cdots \triangleright H^{(n-1)} \triangleright H^{(n)} = 1.$$

Since H is non-abelian, $n \geq 2$. Put $S = \{H^{(n-1)} \leq A \lneq H^{(n-2)} \mid A \text{ abelian}\}$. Since $H^{(n-1)} \neq 1$ and $H^{(n-1)}$ is abelian, $H^{(n-1)} \in S$. Thus, $S \neq \emptyset$. Suppose that $A_0 \leq A_1 \leq \cdots \leq A_k \leq \cdots$ is a chain in S. Put $B = \bigcup_{k \in \mathbb{N}} A_k$. If $B = H^{(n-2)}$, then there exists $k_0 \in \mathbb{N}$ such that $A_{k_0} = H^{(n-2)}$, a contradiction. Hence, $B \neq H^{(n-2)}$. For $x, y \in B$, we have $x \in A_i, y \in A_j$ for some $i, j \in \mathbb{N}$. Therefore, one has $x, y \in A_{i+j}$ and $xy = yx$. Thus, B is abelian and consequently $B \in S$. By Zorn's Lemma, there is a maximal element M in S. Since $M \geq (H^{(n-2)})' = H^{(n-1)}$, we have $M \triangleleft H^{(n-2)}$. Put $K = F(M)$, then K is a field because M is abelian. Since $M \neq H^{(n-2)}$, there exists $a \in H^{(n-2)} \setminus M$ such that $aMa^{-1} = M$. By Lemma 2.3, we have $aKa^{-1} \subseteq K$. Put $L = K + Ka + Ka^2 + \cdots + Ka^n + \cdots$. By Lemma 2.7, L is a centrally finite division ring. Now, consider $G = L \cap N$. We have G is almost subnormal in L^* and $G \supset \langle a, M \rangle \supset H^{(n-1)}$. Since M is maximal in S, we conclude that $\langle a, M \rangle$ is non-abelian and G is non-abelian. By [18, Theorem 4.2], G contains a non-cyclic free subgroup, which implies that N is not locally soluble. This is a contradiction and the proof is complete. □

The following corollary is an application of Theorem 2.8.

Corollary 2.9. *Let D be a non-commutative division ring that is algebraic over its center F and assume that N is a non-central almost subnormal subgroup of D^*. Assume that M is a maximal subgroup of N such that $\mathrm{C}_D(M) \neq F$. Then, every locally soluble almost subnormal subgroup of M is abelian.*

Proof. We consider two possible cases.

Case 1. F is finite. Then, D^* is algebraic over a finite field F. By [16, Theorem 13.11], D is commutative, a contradiction.

Case 2. F is infinite. Suppose that $\alpha \in \mathrm{C}_D(M) \setminus F$ is algebraic over F. Then, $[F(\alpha) : F] < \infty$. By the Double Centralizer Theorem, $\mathrm{C}_D(F(\alpha))$ is a division ring with center $F(\alpha)$. Since $\alpha \in \mathrm{C}_D(M)$, we have $M \leq \mathrm{C}_D(F(\alpha))^*$. Therefore, $M \leq N \cap \mathrm{C}_D(F(\alpha))^* \leq N$. By the maximality of M in N, we have $N \cap \mathrm{C}_D(F(\alpha))^* = N$ or $N \cap \mathrm{C}_D(F(\alpha))^* = M$. The first case can not occur. To prove this, we claim that $N \not\subseteq \mathrm{C}_D(F(\alpha))^*$. Suppose that $N \subseteq \mathrm{C}_D(F(\alpha))^*$. Then, $F(N) \subseteq \mathrm{C}_D(F(\alpha))$. Since $F(N)^*$ is N-invariant, by Lemma 2.4, we have $F(N) = D$ and consequently $\mathrm{C}_D(F(\alpha)) = D$. This contradicts the fact that α is not in F. Hence $N \not\subseteq \mathrm{C}_D(F(\alpha))^*$ and $N \cap \mathrm{C}_D(F(\alpha))^* = M$. By [5, Lemma 2.8], we have $N \cap \mathrm{C}_D(F(\alpha))^*$ is an almost subnormal subgroup of $\mathrm{C}_D(F(\alpha))^*$. Thus, M is an

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almost subnormal subgroup of $\mathrm{C}_D(F(\alpha))^*$. Now, assume that H is a locally soluble almost subnormal subgroup of M. Then H is a locally soluble almost subnormal subgroup of $\mathrm{C}_D(F(\alpha))^*$. By Theorem 2.8, $H \subseteq F(\alpha)$. Hence H is abelian. The proof is complete. □

We give another corollary of Theorem 2.8 which is implied directly from Corollary 2.9.

Corollary 2.10. *Let D be a non-commutative division ring that is algebraic over its center F and assume that M is a maximal subgroup of D^* such that $\mathrm{C}_D(M) \neq F$. Then, every locally soluble almost subnormal subgroup of M is abelian.*

We end this note with the following remark.

Remark 2.11. *The condition $\mathrm{C}_D(M) \neq F$ in Corollaries 2.9 and 2.10 is essential. For example, we consider the division ring of real quaternions*

$$\mathbb{H} = \mathbb{R} \oplus \mathbb{R}i \oplus \mathbb{R}j \oplus \mathbb{R}k.$$

It was proved in [1, Theorem 1] *and* [17, Example, p. 725] *that $\mathbb{C}^* \cup \mathbb{C}^* j$ is a non-abelian soluble maximal subgroup of $\mathbb{H}$ but $\mathrm{C}_{\mathbb{H}}(\mathbb{C}^* \cup \mathbb{C}^* j) = \mathbb{R}$.*

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