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THE MINIMUM SUM OF ELEMENT ORDERS OF FINITE GROUPS

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Abstract. Let G be a finite group and $\psi(G) = \sum_{g\in G} o(g)$, where $o(g)$ denotes the order of $g \in G$. We show that the Conjecture 4.6.5 posed in [Group Theory and Computation, (2018) 59-90], is incorrect. In fact, we find a pair of finite groups G and S of the same order such that $\psi(G) < \psi(S)$, with G solvable and S simple.

1. Introduction

To determine algebraic properties of a finite group from its element orders sum is an interesting problem. If G is a finite group, then $\psi(G) = \sum_{g\in G} o(g)$ is the sum of element orders of G, where $o(g)$ denotes the order of $g \in G$. In [1], it's proved that the maximum value of ψ on the set of groups of order n will occur at the cyclic group C_n, namely, a cyclic group of order n can be characterized by the order n and the value ψ. Following this publication, many studies have been done on the function ψ, for example, see [2], [3], [6], [9], [15], [13], and [16], that let to find an exact upper bound for sums of element orders in non-cyclic finite groups, in [12].

Theorem 1.1. [1] *If G is noncyclic group and $|G| = n$, then $\psi(G) < \psi(C_n)$.*

Theorem 1.2. [12] *If G is noncyclic group and $|G| = n$, then $\psi(G) \leq \frac{7}{11}\psi(C_n)$.*

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It is natural to ask what one can say about the minimum of ψ on groups of the same order. In [3], authors turned their attention to the minimum value of G, and they showed that

Proposition 1.3. [3] *Among all nilpotent groups of order n, the minimum value of ψ is attained by groups with all Sylow subgroups of prime exponent.*

Theorem 1.4. [3] *Let G be a nilpotent group of order n and there are non-nilpotent groups of order n. Then, there exists a non-nilpotent group K of order n satisfying $\psi(K) < \psi(G)$.*

The minimal value of ψ on finite groups of the same order was also investigated in non-abelian simple groups. For simple groups of the small orders, the ψ value is the unique minimum of the values of ψ for groups of the corresponding orders. Hence, the authors in [3] posed the following Conjecture:

Conjecture 1.5. [3] *Let S be a simple group. If G is a non-simple group of order $|S|$, then $\psi(S) < \psi(G)$.*

A counterexample has been found to Conjecture 1.5 in [15]:

Proposition 1.6. [15] *There exists a non-simple group G of order $|L_2(64)|$ such that $\psi(G) < \psi(L_2(64))$.*

Since the counterexample to the Conjecture 1.5 was given by a non-solvable group, the authors in [11] stated the following new Conjecture:

Conjecture 1.7. [11] *If S is a simple group and G is a solvable group satisfying $|G| = |S|$, then $\psi(S) < \psi(G)$.*

Our aim in this note is to provide a counterexample to Conjecture 1.7. We prove the following theorem:

Theorem 1.8. *There exist two finite groups G and S of the same order, such that $\psi(G) < \psi(S)$, with G solvable and S simple.*

2. The groups

We describe our groups in Theorem 1.8 using the `SmallGroups` database in [5]. Since it does not appear to be possible to find examples in the database, our successful strategy has been to build up direct products of groups. A significant ingredient of our result is Lemma 2.1 of [3], which states that if G and H are finite groups, then $\psi(G \times H) \leq \psi(G)\psi(H)$, with equality if and only if $\gcd(|G|, |H|) = 1$. We need to find two solvable groups H and K and a simple group S with the following conditions:

i. $|H|.|K| = |S|$,
ii. $\psi(H \times K) < \psi(S)$.

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(1) To satisfy the second condition, we must choose those groups that have the value of $\psi(H \times K)$ as small as possible and the value of $\psi(S)$ is sufficiently large. Notice that the first condition always holds.

(2) It follows from Lemma 2.1 of [3] that increasing the value of $\gcd(|H|, |K|)$ increases the probability of decreasing the value of $\psi(H \times K)$. Therefore, we select a simple group that has in its order prime decomposition, small primes with large powers.

(3) The projective special linear groups $L_2(q)$ for $q = 2^m$, is a simple group of order $(q-1)q(q+1)$ and its ψ value is relatively large. The simple group $L_2(64)$ of order $63.64.65 = 2^6.3^2.5.7.13$ has small primes in its order prime decomposition and $42|L_2(64)| < \psi(L_2(64)$, because $L_2(64)$ has $\frac{1}{2}\varphi(65) = 24$ self-centralizing elements of order 65, and $\frac{1}{2}\varphi(63) = 18$ self-centralizing elements of order 63, see [10, Proposition 28.4]. Note that If g is a self-centralizing element of group G then $\langle g \rangle = C_G(g)$ and $o(g).|g^G| = |G|$. Therefore, we select the group $S = L_2(64)$.

(4) By 2, we consider groups H and K so that $|H| = 2^3.3.\alpha$, $|K| = 2^3.3.\beta$ and $\alpha.\beta = 5.7.13$. For the Frobenius group $F_{p,q} = C_p \rtimes C_q$, where p and q are two prime numbers and $q|(p-1)$, $\psi(F_{p,q}) < (q - 1 + \frac{p-1}{q})|F_{p,q}|$, and also $\mathrm{Aut}(C_{65}) \cong C_{12} \times C_4$. Hence we assume that $|H| = 2^3.3.5.13$ and therefore $|K| = 2^3.3.7$.

(5) Among all solvable groups of order 1560 and 168, we search for groups with the minimum ψ value, by Magma. Finaly, we let $H =$ `SmallGroup(1560,150)` and $K =$ `SmallGroup(168,43)`. We have that

$$H = C_{65} \rtimes (C_{12} \times C_2)$$

and

$$K = (C_2^3 \rtimes C_7) \rtimes C_3$$

are solvable groups. These groups satisfy $\psi(S) = 12106687$, $\psi(H) = 19297$ and $\psi(K) = 855$.

(6) Now, we let $G = H \times K$. We have that $|G| = |S|$ and

$$\psi(G) = \psi(H \times K) = 10954193,$$

and

$$\psi(G) < \psi(S).$$

Although we have found some other examples, for example direct product of $H =$ `SmallGroup(780,16)` or $H =$ `SmallGroup(780,17)` and $K =$ `SmallGroup(336,218)`, they do not differ significantly from the one that we are describing above.

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3. Discussion

Although we found a solvable group such that its ψ value is less than the ψ value of a simple group, there is a non-solvable group such that its ψ value is less than the ψ value of this group,

$$5482775 = \psi(3^2 \times S_z(8)) < 10954193 = \psi(H \times K) < \psi(L_2(64)) = 12106687.$$

So, we propose the following new conjecture:

Conjecture 3.1. *Let n be a positive integer such that there exists a non-abelian simple group of order n. Then there exist a non-solvable group G of order n with the property that $\psi(G) < \psi(H)$ for every solvable group H of order n.*

Algorithm 1 Search for two groups H and K

```
U:=[ ]; V:=[ ];
for H in SmallGroups(168, func <x| IsSolvable(x)>) do                 ▷ Step 5.
    u:=&+[Order(g): g in H ];
    U:=Append(U,u);
end for
for H in SmallGroups(168, func<x| IsSolvable(x)>) do
    if &+[Order(g): g in H ] eq Minimum(U) then
        IdentifyGroup(H);
        for K in SmallGroups(1560, func<x| IsSolvable(x) > ) do
            v:=&+[Order(g): g in K ];
            V:=Append(V,v);
        end for
        for K in SmallGroups(1560, func<x| IsSolvable(x)>) do
            if &+[Order(g): g in K ] eq Minimum(V) then
                IdentifyGroup(K);
                G:=DirectProduct(H,K);                                ▷ Step 6.
                IsSolvable(G);
                &+[Order(g): g in G ];
                &+[Order(g): g in PSL(2,64) ];
            end if
        end for
    end if
end for
```

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