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CERTAIN COMBINATORIAL TOPICS IN GROUP THEORY

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In memory of Narain Gupta

Abstract. This article is intended to be a survey on some combinatorial topics in group theory. The bibliography at the end is neither claimed to be exhaustive, nor is it necessarily connected with a reference in the text. I include it as I see it revolves around the concepts which are discussed in the text.

1. **Introduction**

Automorphisms of free groups.

Let $F = F_n = \langle x_1, x_2, \ldots, x_n \rangle$, $n \geq 2$, be a free group of rank n and let $\mathrm{Aut}(F)$ denote the group of all automorphisms of F. Here we begin with **Elementary Nielsen (transformations) automorphisms**

$$\big\{x_i \longrightarrow x_{i_\sigma}^{\pm 1},\ \sigma \text{ permutation of } \{1, 2, \ldots, n\}\big\};$$
$$\{x_i \longrightarrow x_i x_j^{\pm 1},\ x_k \longrightarrow x_k,\ k \neq i\}; \quad \{x_i \longrightarrow x_j^{\pm 1} x_i,\ x_k \longrightarrow x_k,\ k \neq i\};$$
$$\{x_i \longrightarrow x_j^{-1} x_i x_j,\ x_k \longrightarrow x_k,\ k \neq i\}; \quad \{x_i \longrightarrow x_j x_i x_j^{-1},\ x_k \longrightarrow x_k,\ k \neq i\}.$$

Every automorphism of the free group F is a power product of these elementary Nielsen transformations. Next, we state a significant result of Nielsen.

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Theorem 1.1 (**Nielsen,** [68]). $\mathrm{Aut}(F)$ *can be generated by the following four elementary automorphisms:*

$$\tau_1 = \{x_1 \longrightarrow x_1^{-1},\ x_i \longrightarrow x_i,\ i \neq 1\}; \quad \tau_2 = \{x_1 \longrightarrow x_1x_2,\ x_i \longrightarrow x_i,\ i \neq 1\};$$
$$\tau_3 = \{x_1 \longrightarrow x_2,\ x_2 \longrightarrow x_1,\ x_i \longrightarrow x_i,\ i \neq 1,2\};$$
$$\tau_4 = \{x_1 \longrightarrow x_2,\ x_2 \longrightarrow x_3, \ldots, x_n \longrightarrow x_1\}.$$

If $n \geq 4$, then $\mathrm{Aut}(F)$ can, in fact, be generated by a set of two automorphisms.

Theorem 1.2 (**B. H. Neumann,** [67]). *If* $n = 4, 6, 8, \ldots,$ *then* $\mathrm{Aut}(F) = \langle \tau_4, \varphi \rangle$, *where*

$$\varphi\colon x_1 \longrightarrow x_2^{-1}, x_2 \longrightarrow x_1, x_3 \longrightarrow x_3, \ldots,$$
$$x_{n-2} \longrightarrow x_{n-2}, x_{n-1} \longrightarrow x_n x_{n-1}^{-1}, \quad x_n \longrightarrow x_{n-1}^{-1};$$

if $n = 5, 7, 9, \ldots,$ *then* $\mathrm{Aut}(F) = \langle \varphi, \chi \rangle$, *where* $\chi = \tau_4\tau_{1i}\tau_{2i}\cdots\tau_{ni}$, *and* $\tau_{ki}\colon x_k \to x_k^{-1}, x_i \to x_i$, $i \neq k$.

Let $F' = [F, F]$ denote the commutator subgroup of F. Consider the natural homomorphism $\alpha\colon \mathrm{Aut}(F) \to \mathrm{Aut}(F/F')$, the kernel of the homomorphism consists of all those automorphisms of F which induce the identity automorphism on F module F'. These are so-called IA-automorphisms of F. We denote by IA-Aut(F) the subgroup of all IA-automorphisms of F. Elements of IA-Aut(F) may be defined as

$$\alpha = \{x_1 \longrightarrow x_1d_1, x_2 \longrightarrow x_2d_2, \ldots, x_n \longrightarrow x_nd_n\},$$

where $d_i \in F'$ are such that $\{x_1d_1, x_2d_2, \ldots, x_nd_n\}$ is a basis of F.

Next, we mention an interesting result of Magnus.

Theorem 1.3 (**W. Magnus,** [57]). IA-Aut(F_n), $n \geq 2$, *is finitely generated. And*

$$\text{IA-Aut}(F_n) = \mathrm{sgp}\big\{\alpha_{ijk}\colon x_i \longrightarrow x_i[x_j, x_k], x_t \longrightarrow x_t, t \neq i$$
$$\textit{for all } i,j,k \;\big|\; j = i \textit{ or } i \notin \{j,k\} \textit{ and } j < k\big\}.$$

(Here, we write the commutator $[x_j, x_k] = x_j^{-1}x_k^{-1}x_jx_k$.)

An inner automorphism of F is clearly an IA-automorphism. We denote by Inner-Aut(F) = Inn-Aut(F) the subgroup of $\mathrm{Aut}(F)$ which consists of all inner automorphisms of F. The centre of F is trivial, so Inn-Aut$(F) \cong F$ and the following inclusions of normal subgroups of $\mathrm{Aut}(F)$ are now clear:

$$F \cong \text{Inn-Aut}(F) \leq \text{IA-Aut}(F) \leq \mathrm{Aut}(F).$$

Nielsen [68] proved that if F is a free group of rank 2, then IA-Aut(F) = Inn-Aut(F), see Lyndon and Schupp [56] for a proof. Formanek and Procesi [26] proved that Inn-Aut(F_n), $n \geq 2$, is a unique normal free subgroup of rank n. Also, it is clear that $\mathrm{Aut}(F_n/F_n') \cong \mathrm{GL}(n, \mathbb{Z})$, $n \geq 2$, the group of $n \times n$ invertible matrices over the integers. Here $F_n' = [F_n, F_n]$ is the commutator subgroup of F_n.

We recall the question: What IA-endomorphisms of the free group $F_n = F = \langle x_1, x_2, \ldots, x_n \rangle$ $(\varphi\colon F_n \to F_n)$ are automorphisms?

Nielsen [69] and Mal'tsev [60] have proved that if an endomorphism φ of the free group F_2 fixes the commutator of a pair of generators, then φ is an automorphism; Durnev [23] showed that if φ fixes the commutator modulo F_2'' $(= [F_2', F_2'])$, then φ defines an automorphism of F_2; Narain Gupta and Shpilrain [48] proved that if an endomorphism φ of F_2 fixes the commutator modulo the derived series $\delta_k(F_2)$, $k \geq 3$, then φ is not an automorphism. (Here, the Nielsen commutator test fails.)

2. **Test words for automorphisms of free groups**

Definition. A word $w \in F_n$ is a test word if any endomorphism fixing w is necessarily an automorphism.

Here we begin with some known results: Zieschang [101] showed that if φ is an endomorphism of the free group $F_n = F = \langle x_1, x_2, \ldots, x_n \rangle$, $n \geq 2$, then φ is an automorphism if it fixes $x_1^k x_2^k \cdots x_n^k$ for some $k \geq 2$ or $[x_1, x_2][x_3, x_4] \cdots [x_{n-1}, x_n]$ if n is even; Rips [77] and Shpilrain [90] showed that if $\varphi([x_1, x_2, \ldots, x_n]) = [x_1, x_2, \ldots, x_n]$, $n \geq 2$, then φ is an automorphism; Dold [20] has found a series of test elements described in graph-theoretic methods; Edward Turner [97] proved that **test words in** F_n are those words that are not contained in any proper retracts of F_n. A retraction $\rho\colon G \to G$ is a homomorphism of a group G satisfying $\rho^2 = \rho$ and a retract is the image of a retraction, *i.e.*, $R = \rho(G)$ is a retract.

Example. Let $\varphi\colon F_2 \to F_2$ be an endomorphism defined by

$$\varphi(x_1) = x_1^2 x_2 x_1^{-1} x_2^{-1}, \quad \varphi(x_2) = 1.$$

Then φ fixes $x_1^2 x_2 x_1^{-1} x_2^{-1}$ but $x_1^2 x_2 x_1^{-1} x_2^{-1}$ lies in a proper retract (namely, the image of φ). Therefore, $x_1^2 x_2 x_1^{-1} x_2^{-1}$ is not a test word, whereas one can check that $x_1 x_2 x_1^{-1} x_2^{-1}$ is a test word.

Test sets and test rank: Let $G = \langle x_1, \ldots, x_n \rangle$ be an n-generator group. A set of elements $\{g_1, \ldots, g_m\}$, $m \leq n$, is a test set of a group G if whenever $\varphi(g_i) = g_i$, $i = 1, \ldots, m$ for some endomorphism φ of G, then φ must be an automorphism of G. The test rank of G is the minimal cardinality of a test set.

3. **Test ranks of free nilpotent groups**

Theorem 3.1 (**C. K. Gupta, Roman'kov, Timoshenko** [36]). *Let $N = N_{rc}$ be a free nilpotent group of rank $r \geq 2$, class $c \geq 2$. Then*

(1) $\mathrm{tr}(N) = 2$ *for r odd and $c = 2$;*
(2) $\mathrm{tr}(N) = 1$ *in all other cases.*
(3) *An element $g \in N_{2q,2}$ is a test element if and only if it can be written as*

$$g = [x_1, x_2]^{t_1} \cdots [x_{2q-1}, x_{2q}]^{t_q}$$

in some basis $x_1, x_2, \ldots, x_{2q}$, and $t_1, \ldots, t_q$ are non-zero integers.

Proof. First we show that **if a test element g exists in N_{r2}, then $g \in N'_{r2}$,** the commutator subgroup of N_{r2}. Suppose $g \notin N'_{r2}$, without loss of generality, assume $g = x_1^n u$, where

$$u = \prod_{i<j} [x_i, x_j]^{k_{ij}} \in N'_{r2}, \quad n \geq 1.$$

Let φ be an endomorphism of N_{r2} defined by

$$\begin{cases} x_1 \longrightarrow x_1 v, & v \in N'_{r2} \\ x_i \longrightarrow x_i^{n+1}, & i = 2, \ldots, r. \end{cases}$$

Applying φ to g,

$$\begin{aligned} \varphi(g) = \varphi(x_1^n u) &= (x_1 v)^n \prod_{1<j} [x_1, x_j]^{k_{ij}(n+1)} \prod_{1<i<j<n} [x_1, x_j]^{k_{ij}(n+1)^2} \\ &= x_1^n v^n u w^n \end{aligned}$$

for some $w \in N'_{r2}$.

$$\begin{aligned} \varphi(g) = \varphi(x_1^n u) &= x_1^n v^n u w^n \\ &= x_1^n (w^{-1})^n u w^n \quad \text{let } v = w^{-1} \\ &= x_1^n u, \end{aligned}$$

$u \in N'_{r2}$ and N_{r2} is nilpotent of class 2. Hence $\varphi(g) = g$. Obviously, φ is not an automorphism of N_{r2}. Thus, g is not a test element. So, we can assume that $g \in N' = N'_{rc}$. For instance, the set of elements $\{x_1, [x_2, x_3] \cdots [x_{r-1}, x_r]\}$ is a test set. Let

$$\alpha \colon \begin{cases} x_1 \longrightarrow y_1 \\ \cdots \\ x_r \longrightarrow y_r \end{cases}$$

be an endomorphism of N, $y_i = x_1^{\alpha_{i1}} \cdots x_r^{\alpha_{ir}} c_i$, $\alpha_{ij} \in \mathbb{Z}$, $c_i \in N'$. Let $A = (\alpha_{ij})$. An endomorphism α of N is an automorphism if α induces an automorphism on N/N' [66]. It turns out that the matrix $A = (\alpha_{ij})$ is invertible over $\mathbb{Z}$. Thus, α is an automorphism.

For instance, r odd, $c \geq 3$, N_{rc} contains a test element $g = \big[[x_2, x_3] \cdots [x_{r-1}, x_r], x_1\big]$. If $r = 2q$ even, $c \geq 3$, then $g = [x_1, x_2]^{t_1} \cdots [x_{2q-1}, x_{2q}]^{t_q}$ is a test element in N. For details, see our paper. □

4. Test rank of a direct product $F_r \times A_n$

[$F_r = \langle x_1, x_2, \ldots, x_r \rangle$ free group of rank r; $A_n = \langle a_1, a_2, \ldots, a_n \rangle$ free abelian group of rank n.]

It follows from the works of Zieschang, Rips, Rosenberger and Turner: A free group F_r has test rank 1; a free abelian group A_n has test rank n. Fine, Rosenberger, Spellman, Stille [25] have shown that for the given integers n and k, there exists a group of rank n with test rank k.

Theorem 4.1 (Fine, Rosenberger, Spellman, Stille [25]**).** *Test rank of $G = G_{rn} = F_r \times A_n$ equals $n+1$, for $r \geq 2$ and $n \geq 1$.*

Something is wrong with the statement of the theorem. We have now proved the following.

Theorem 4.2 (C. K. Gupta, Roman'kov, Timoshenko [36]**).** *The group $G = F_r \times A_n = G_{rn}$ has test rank n, where F_r: free group of rank $r \geq 2$, A_n: free abelian group of rank $n \geq 1$.*

Proof. Let $x_1, \ldots, x_r$ be a basis of F_r. Shpilrain [90] has shown that the commutator

$$w_1 = [\cdots [x_1, x_2], \ldots, x_r]$$

is a test element in F_r. He started with Nielsen's test element $[x_1, x_2]$ of F_2, and his proof used induction on r. Also, as proved by Fine, Rosenberger, Spellman, Stille [25], every commutator $[x_1^m, x_2]$, $m \geq 2$, is also a test element in F_2. We started with $[x_1^m, x_2]$ for $m = 2$, and using similar arguments as Shpilrain, we derive that the commutator

$$w_2 = [\cdots [x_1^2, x_2], \ldots, x_r]$$

is a test element in F_r. So, we have obtained w_1, w_2 two non-commuting test elements in F_r. Here, we have shown that the set of elements $g_1 = w_1 a_1$, $g_2 = w_2 a_2, \ldots, g_n = w_2 a_n$ is a test set for G [here $a_1, \ldots, a_n$ is a basis of A_n]. That concludes the proof. □

5. Invertibility criterion for endomorphisms, test ranks of metabelian products of abelian groups

Here we begin with a definition.

Definition. An endomorphism of a group G is an IA-endomorphism if it induces the identity map on G/G'.

Let $G = \prod * A_i$ be a free product of some non-trivial abelian groups $A_1, \ldots, A_n$. Let $A = G/G''$ be the metabelian product of groups $A_1, \ldots, A_n$. Denote $A = \prod_{i=1}^{n} *_{\mathfrak{A}^2} A_i$.

P. Ushakov studied IA-automorphisms of metabelian products of abelian groups. He considered an embedding of the IA-Aut A group in a group of matrices over some ring for metabelian products A of abelian torsion-free groups. He showed a stronger version of Bachmuth embedding. It follows from Ushakov's results: an IA-endomorphism φ is an IA-Aut A if and only if the determinant of the matrix $(D_i a_j)$ over the ring $\mathbb{Z}(A/A')$ can be written as

$$\det(D_i a_j) = \pm g(a_1 - 1) \cdots (a_n - 1),$$

where $g \in A/A'$, B_i a basis of A_i and $a_i \in B_i$.

Here, we give necessary and sufficient conditions for an endomorphism of a metabelian product of free abelian groups to be an automorphism.

Theorem 5.1 (**C. K. Gupta and E. I. Timoshenko** [44]). *Let $A = \prod_{i=1}^{n} *_{\mathfrak{A}^2} A_i$ be a metabelian product of torsion-free abelian groups of finite ranks; B_i: a basis of A_i; a_i some fixed element of B_i; φ an endomorphism of A and $\varphi(a_i) = y_i$. $A_1, \dots, A_m$ non-cyclic groups ($m \leq n$). Then $\varphi \in \mathrm{Aut}(A)$ if and only if*

(1) *the images of elements $\varphi(b)$, $b \in \bigcup_{i=1}^{n} B_i$ in the group A/A' generate this group;*
(2) *$\det(D_i y_j) = J_{n\times n}$ over $\mathbb{Z}(A/A')$ can be written as*

$$\det J = \pm g(y_i - 1)\cdots(y_m - 1)(a_{m+1} - 1)\cdots(a_n - 1)$$

for some $g \in A/A'$.

Next we begin with a definition.

Definition. Let G be a group and $g_1, \dots, g_n \in G$. The set of elements $\{g_1, \dots, g_m\}$ has the **property** $(*)$ if, whenever $\varphi(g_i) = g_i$, for $i = 1, \dots, m$, for some IA-endomorphism φ of G, then this φ is an automorphism of G.

Next, we state the following.

Theorem 5.2 (**C. K. Gupta and E. I. Timoshenko** [44]). *Let $A = \prod_{i=1}^{n} *_{\mathfrak{A}^2} A_i$ be a metabelian product of torsion-free abelian groups of finite ranks, $n \geq 2$. Then*

(1) *there is a system $\{g_1, \dots, g_{n-1}\}$ of elements of the group A with the property $(*)$;*
(2) *there exists no system of elements $\{g_1, \dots, g_{n-2}\}$ with the property $(*)$;*
(3) *a system of elements $\{g_1, \dots, g_{n-1}\}$ of the group A has the property $(*)$ if and only if all elements belong to A' and are independent over the ring $\mathbb{Z}(A/A')$.*

In the proof of (1), we demonstrate that the set of elements $[a_1, a_2], \dots, [a_1, a_n]$ has the property $(*)$, where B_i is a basis of the group A_i, and a_i some fixed element of B_i, $i = 1, \dots, n$. Let $\varphi \in \mathrm{IA\text{-}End}(A)$ and let $\varphi([a_1, a_i]) = [a_1, a_i]$ for $i = 2, \dots, n$. Then it can be shown that the IA-endomorphism φ induces the identity map on the commutator subgroup A'. From this we conclude that φ is actually an automorphism of A. For the details of (2) and (3), see our paper.

Next, we have obtained the following:

Theorem 5.3 (**C. K. Gupta and E. I. Timoshenko** [44]). *Let $A = \prod_{i=1}^{n} *_{\mathfrak{A}^2} A_i$ be a metabelian product of torsion-free abelian groups of finite ranks, $n \geq 2$. Then the test rank of the group A equals $n - 1$.*

6. Test ranks for some free polynilpotent groups

Theorem 6.1 (**C. K. Gupta and E. I. Timoshenko** [45]). *Let $F = \langle x_1, \dots, x_r \rangle$ be a free group. Let $\mathbb{Z}(F/R)$ be a domain, having trivial units only, $R \leq F'$. If $\{\widetilde{g_1}, \dots, \widetilde{g_m}\} \in F/R'$ is a test collection,* *then for every solution* $\overline{\lambda_1}, \ldots, \overline{\lambda_m} \in \mathbb{Z}(F/R)$ *of the system of equations*

$$\partial_1 \widetilde{g_1} \cdot \overline{\lambda_1} + \cdots + \partial_r \widetilde{g_1} \cdot \overline{\lambda_r} = 0$$

$$\cdots$$

$$\partial_1 \widetilde{g_m} \cdot \overline{\lambda_1} + \cdots + \partial_r \widetilde{g_m} \cdot \overline{\lambda_r} = 0$$

and for any $\tilde{c} \in R/R'$, *the following condition should hold:*

$$\partial_1 \tilde{c} \cdot \overline{\lambda_1} + \cdots + \partial_r \tilde{c} \cdot \overline{\lambda_r} = 0.$$

Proposition. *Let* F *be a free group with basis* $x_1, \ldots, x_r$, $\{1\} \neq R \trianglelefteq F$, $\tilde{c} \in R/R'$, $\mathbb{Z}(F/R)$ *be a domain,* $\overline{\lambda_i} \in \mathbb{Z}(F/R)$ *for* $i = 1, \ldots, r$. *Then the endomorphism*

$$\varphi \colon \begin{cases} \widetilde{x_1} \longrightarrow \tilde{c}^{\overline{\lambda_1}} \cdot \widetilde{x_1} \\ \cdots \\ \widetilde{x_r} \longrightarrow \tilde{c}^{\overline{\lambda_r}} \cdot \widetilde{x_r} \end{cases}$$

of F/R' *will be an automorphism if and only if*

$$\partial_1 \tilde{c} \cdot \overline{\lambda_1} + \cdots + \partial_r \tilde{c} \cdot \overline{\lambda_r} + 1$$

is an invertible element of the ring $\mathbb{Z}(F/R)$.

Theorem 6.2 (C. K. Gupta and E. I. Timoshenko [45]**).** *Let* $F = \langle x_1, \ldots, x_r \rangle$ *be a free group. Let* $\mathbb{Z}(F/R)$ *be a domain with trivial units only,* R *a non-trivial verbal subgroup of* F', *and* $\mathrm{tr}(F/R')$ *be the test rank. Then* $\mathrm{tr}(F/R')$ *equals* $r-1$, *or* r.

From these theorems, we obtain the following:

Corollary 1. *For* $r \geq 2$ *and every collection* $(c_1, \ldots, c_l)$ *of classes, a free polynilpotent group* $F_r(\mathbb{A}\mathbb{N}_{c_1} \cdots \mathbb{N}_{c_l})$ *has its test rank equal to* $r-1$, *or* r.

Corollary 2. *Let* $r \geq 2$, $c \geq 2$ *and* F *be a free group of rank* r. *Then the test rank for a group* $F/[\gamma_c(F), \gamma_c(F)]$ *equals to* $r-1$.

Here, we mention Shmel'kin's theorems.

Theorem 6.3. *Let* $F = \langle x_1, \ldots, x_r \rangle$ *be a free group of rank* $r \geq 2$. *Let* R *be a verbal subgroup of* F *and* F/R *a free polynilpotent group. Assume endomorphism* φ *of* F/R' *acts identically on the subgroup* $A = R/R'$. *Then* φ *is an inner automorphism of* F/R' *induced by some element of* $A = R/R'$.

Theorem 6.4. *Let* $F = \langle x_1, x_2 \rangle$ *be a free group of rank* 2. *Let* $c \geq 2$, $t \geq 1$, $R = \gamma_c(F)$ *and* $\widetilde{F} = F/R'$. *Assume* $\widetilde{y_1}, \widetilde{y_2} \in \widetilde{F}$ *and* $[\widetilde{x_1}, \widetilde{x_2}, \ldots, \widetilde{x_2}] = [\widetilde{y_1}, \widetilde{y_2}, \ldots, \widetilde{y_2}]$ *of weight* $c+t$ *are equal. Then the map* $\varphi \colon \{\widetilde{x_i} \to \widetilde{y_i}, i = 1, 2\}$ *is an automorphism of* $\widetilde{F}$.

7. Generating elements of groups F/R'

Theorem 7.1 (**C. K. Gupta and E. I. Timoshenko** [46])**.** *Let $g_1, \dots, g_m$ be elements of the group $F = A_1 * \cdots * A_n$ and $R \trianglelefteq F$; $R \cap A_i = \{1\}$ for $i = 1, \dots, n$; $A = F/R$, $T = t_1\mathbb{Z}A + \cdots + t_n\mathbb{Z}A$ is a right $\mathbb{Z}A$-module with basis $t_1, \dots, t_n$; $\tau_j = t_1 D_1 g_j + \cdots + t_n D_n g_j$; $L = t_1\triangle_1\mathbb{Z}A + \cdots + t_n\triangle_n\mathbb{Z}A$, where $\triangle_i$ is the augmentation ideal of $\mathbb{Z}A_i$. Then the elements $g_1, \dots, g_m$ generate the group F/R' if and only if the elements $\tau_1, \dots, \tau_m$ generate a right $\mathbb{Z}A$-module L.*

Corollary. *Let F_m be a free group of rank m with basis $x_1, \dots, x_m$; $g_1, \dots, g_n \in F_m$, $n \geq m$; R is a normal subgroup of F_m with $R \cap \mathrm{gp}(x_i) = \{1\}$ for $i = 1, \dots, m$; $\mathbb{Z}(F_m/R)$ is a domain. Then the elements $g_1, \dots, g_n$ generate the group F_m/R' if and only if for the matrix $J(g) = (\partial_i g_j)_{m\times n}$, there exists a matrix $B_{n\times m}$ over $\mathbb{Z}(F_m/R)$ for which $J(g) \cdot B = E_{m\times m}$ (identity matrix).*

The following is a stronger version of Birman's theorem.

Corollary. *Elements $g_1, \dots, g_n \in F_m$ ($n \geq m$) of a free group F_m of rank m generate F_m if and only if for the matrix $J(g) = (\partial_i g_j)_{m\times n}$ there exists a matrix $B_{n\times m}$ such that $J(g) \cdot B = E_{m\times m}$ (identity matrix).*

Next we obtain the following:

Theorem 7.2 (**C. K. Gupta and E. I. Timoshenko** [46])**.** *Let A_i ($i = 1, \dots, n$) be free abelian groups of ranks m_i; $m = m_1 + \cdots + m_n$; $r \leq m$, $F = A_1 * \cdots * A_n$; D is a Cartesian subgroup of F; $A = F/D$; $G = F/D'$; $\triangle_i$ is the augmentation ideal of the ring $\mathbb{Z}A_i$; $g_1, \dots, g_r \in G$; $T = t_1\mathbb{Z}A + \cdots + t_n\mathbb{Z}A$ is a free $\mathbb{Z}A$-module with basis $t_1, \dots, t_n$; $L = t_1\triangle_1\mathbb{Z}A + \cdots + t_n\triangle_n\mathbb{Z}A$; $\tau_j = t_1 D_1 g_j + \cdots + t_n D_n g_j \in L$, $j = 1, \dots, r$. Then there exist elements $g_{r+1}, \dots, g_m \in G$ such that $g_1, \dots, g_r, g_{r+1}, \dots, g_m$ generate the group G if and only if there exist elements $\tau_{r+1}, \dots, \tau_m \in L$ such that the elements $\tau_1, \dots, \tau_r, \tau_{r+1}, \dots, \tau_m$ generate the $\mathbb{Z}A$-module L.*

The above theorem collects earlier results of Timoshenko, Roman'kov, Narain Gupta–Noskov–C. K. Gupta.

Corollary. *Let S_n be a free metabelian group of rank n; $g_1, \dots, g_r \in S_n$ ($r \leq n$). Then the elements $g_1, \dots, g_r$ can be included in a basis of S_n if and only if the ideal generated by the $r \times r$ minors of the matrix $(\partial_i g_j)$ over the ring $\mathbb{Z}(S_n/S_n')$ coincides with the whole ring.*

8. Test ranks for certain solvable groups

Theorem 8.1 (**C. K. Gupta and E. I. Timoshenko** [43])**.** *The test rank of solvable products of m non-trivial free abelian groups $A_1, \dots, A_m$ of finite ranks is equal to $m - 1$ if we consider the variety $\mathfrak{A}^n$, $n \geq 2$, of solvable groups.*

This statement implies the following.

Corollary 8.2. *Let $G = \mathfrak{A}^2 \prod_{i=1}^{m} A_i$ be a metabelian product of torsion-free abelian groups of finite ranks, $m \geq 2$. Then the test rank of G equals $m - 1$.*

Corollary 8.3. *The test rank of a free solvable group $F_r(\mathfrak{A}^n)$, $r \geq 2$, $n \geq 2$, is $r-1$.*

Corollary 2 answers a question by Fine and Shpilrain (unsolved problems in [62, problem 14.88]): **Does a free solvable group of rank 2 and class $n \geq 3$, possess test elements?** Here, we answer their question positively.

9. Torsion in factors of polynilpotent series

A well-known result of Karrass, Magnus, Solitar [51] **states:** The group is torsion-free if and only if the relation is not a proper power of any word w in the group. Earlier, an analog with the Karrass, Magnus, Solitar theorem: Romanovskii [84] proved that the factors of the derived subgroup series of the group G are torsion-free if and only if r is not a proper power of any element of F modulo $F^{(k+1)}$.

Here, we consider a more general situation: let $F = \langle x_1, \ldots, x_n \rangle$ be a free group. We consider in F some polynilpotent series of subgroups

$$F = F_{11} \geq F_{12} \geq \cdots \geq F_{1,m_1+1} = F_{21} \geq \cdots \geq F_{2,m_2+1} = F_{31} \geq \cdots . \tag{9.1}$$

We define $F_{ij} = \gamma_j(F_{i1})$, $\gamma_j(F_{i1})$ is the j-th term of the lower central series of F_{i1}.

The group $F/F_{s,m_s+1}$ is a free group of the variety $\mathbf{N}_{m_1}\mathbf{N}_{m_2}\cdots\mathbf{N}_{m_s}$, where $\mathbf{N}_m$ is a variety of nilpotent groups of class $\leq m$. [The product variety $\mathbf{N}_{c_1}\mathbf{N}_{c_2}$ is the variety of all extensions of groups in $\mathbf{N}_{c_1}$ by groups in $\mathbf{N}_{c_2}$.]

Let $G = \langle x_1, \ldots, x_n \mid r \rangle$ be a group with a single defining relation. Denote by G_{ij} a canonical image of F_{ij} in G. We have a polynilpotent series in G:

$$\begin{aligned} G &= G_{11} \geq G_{12} \geq \cdots \geq G_{1,m_1+1} \\ &= G_{21} \geq \cdots \geq G_{2,m_2+1} = G_{31} \geq \cdots . \end{aligned} \tag{9.2}$$

We prove more generally the following.

Theorem 9.1 (**C. K. Gupta and N. S. Romanovskii** [39]). *Let $G = F/r^F$ be a group with a single defining relation, $r \in F_{km} \setminus F_{k,m+1}$ ($m \leq m_k$), F_{ij} the term of some polynilpotent series of the free group F. We show that the factors of the corresponding polynilpotent series of the group G are torsion-free if and only if r is not a proper power of any element of $F \mod F_{k,m+1}$.*

Denote $\gamma_i(G)$ as the i-th term of the lower central series of a group G, *i.e.*, $\gamma_1(G) = G$, $\gamma_{i+1}(G) = [\gamma_i(G), G]$.

We also give a description of the lower central series of a group $F/[R,R]$, when F/R is a nilpotent group with torsion-free lower central factors. Earlier, Narain Gupta, Frank Levin, and C. K. Gupta [31] gave a description of the lower central series of a group $F/[R,R]$, when F/R is a free nilpotent group. We make use of the Magnus Embedding in the proofs. We recall:

Let $F = \langle x_i \mid i \in I \rangle$ be a free group, $A = F/R$, a_i canonical image of x_i in A. Consider a right free $\mathbb{Z}A$-module T with a basis $\{t_i \mid i \in I\}$, a group of matrices $\left(\begin{smallmatrix} A & 0 \\ T & 1 \end{smallmatrix}\right)$ and a group homomorphism

$$\varphi\colon F \longrightarrow \begin{pmatrix} A & 0 \\ T & 1 \end{pmatrix} \quad \text{defined as} \quad x_i \longrightarrow \begin{pmatrix} a_i & 0 \\ t_i & 1 \end{pmatrix}.$$

Kernel of $\varphi = [R, R]$. Here φ defines an embedding of the group $F/[R, R]$ into the group of matrices $\left(\begin{smallmatrix} A & 0 \\ T & 1 \end{smallmatrix}\right)$ **known as the Magnus embedding**. $R\varphi$ consists of matrices of the type $\left(\begin{smallmatrix} 1 & 0 \\ t_1u_1+\cdots+t_nu_n & 1 \end{smallmatrix}\right)$ for which $(a_1 - 1)u_1 + \cdots + (a_n = 1)u_n = 0$. It follows from the construction, replace the ring $\mathbb{Z}A$ by $(\mathbb{Z}/m\mathbb{Z})A$, then $\ker \varphi = R^m[R, R]$.

10. Word problem for certain polynilpotent groups

We write $G = \langle x_1, \ldots, x_n \mid r_1, \ldots, r_m; \mathfrak{M} \rangle$, where a group G is presented in a variety $\mathfrak{M}$ by generating elements $x_1, \ldots, x_n$ and by defining relations $r_1, \ldots, r_m$ if it is a quotient group of a free group in the variety $\mathfrak{M}$ with basis $\{x_1, \ldots, x_n\}$ w.r.t. a normal subgroup generated by $r_1, \ldots, r_m$. In other words, all relations between $x_1, \ldots, x_n$ in G are consequences of the given defining relations and identities defining $\mathfrak{M}$.

For $k \geq 3$, there are examples of groups that are finitely presented in a variety $\mathbf{A}^k$ of soluble groups (here, $\mathbf{A}$ the variety of Abelian groups) that have an algorithmically undecidable word problem [72, 53]. $\mathbf{N}_k$: the variety of nilpotent groups of class at most k. Similar examples for varieties $\mathbf{N}_k\mathbf{A}$, with $k \geq 3$, are given by Epanchintsev and Kukin [24]. On the other hand, the word problem is decidable for groups that are finitely presented in varieties $\mathbf{N}_k$ (Mal'tsev, [60]), $\mathbb{A}^2$ (Romanovskii, [84]), and $\mathbb{N}_2\mathbf{A}$ (Kharlampovich, [52]).

A question of Kargapolov (in [62, Problem 3.16]**): Decide whether the word problem is decidable for groups with one defining relation in $\mathbb{A}^k$, $k \geq 3$, remaining open.**

We define an element r of a free soluble group F of derived length n is called primitive if $r \in F^{(i)} \setminus F^{(i+1)}$, in a series of derived subgroups, implies that r modulo $F^{(i+1)}$ is not a proper power of another element. Here we deal with a more general case—of a polynilpotent variety $\mathfrak{M} = \mathbf{N}_{ms} \cdots \mathbf{N}_{m1}$. We prove the following.

Theorem 10.1 (**C. K. Gupta and N. S. Romanovskii** [38]**).** *Polynilpotent groups with a single primitive defining relation have a decidable word problem.*

In the proof, essential use is made of earlier results [39] **from our paper:** where factors of a polynilpotent series in a group G are torsion-free.

11. Property of being equationally Noetherian

A group A is equationally Noetherian if, for any n, every system of equations in $x_1, \ldots, x_n$ with coefficients from A is equivalent to some finite subsystem of a given system. The condition of being equationally Noetherian is equivalent to being Noetherian for the Zariski topology defined on a set

A^n, where as the subbase of a system of closed sets we take algebraic sets, that is, sets of solutions to systems of equations over the group A. More details about this fact can be found in papers by G. Baumslag, Myasnikov, Remeslennikov [5, 6, 7].

Every group representable by matrices over a commutative Noetherian unitary ring is equationally Noetherian (G. Baumslag, Myasnikov, Remeslennikov [6]). They conclude: free groups are equationally Noetherian. Every abelian group is also equationally Noetherian. R. Bryant [9] proved that a finitely generated group, which is an extension of an abelian by a nilpotent group, has this property.

We give two simple examples of groups which are not equationally Noetherian. The first example is nilpotent group of class 2 (not finitely generated); the second example is centre-by-metabelian group and 2-generated. It is worth mentioning another result of Baumslag, Myasnikov, Roman'kov [7] which states as follows: a wreath product of any non-abelian group and any infinite group is not an equationally Noetherian group.

Let $\mathfrak{B}$ be a class of groups A which are soluble, equationally Noetherian, and have a central series as

$$A = A_1 \geq A_2 \geq \cdots \geq A_n \geq \cdots ,$$

for which $\bigcap A_n = 1$ and all factors A_n/A_{n+1} are torsion-free groups.

Here, we have proved the following.

Theorem 11.1 (**C. K. Gupta and N. S. Romanovskii** [37]). *Let D be a direct product of finitely many cyclic groups of infinite or prime orders, with $A \in \mathfrak{B}$. Then the wreath product $D \wr A$ is an equationally Noetherian group.*

Here, we have obtained the following two interesting corollaries.

Corollary 11.2 (**C. K. Gupta and N. S. Romanovskii** [37]). *Let a group $A \in \mathfrak{B}$ be representable as a factor group F/R, where F is a free group of finite rank. Then $F/[R, R] \in \mathfrak{B}$.*

Corollary 11.3. *Free soluble groups of arbitrary derived lengths and ranks are equationally Noetherian.*

Some remarks.

(a) The class of equationally Noetherian groups is closed w.r.t. subgroups and finite direct products.

(b) A group is equationally Noetherian if all of its countable subgroups have this property.

In [37], Romanovskii and I give two simple examples of groups which fail to be equationally Noetherian.

Example 1. Let a group A in a variety of nilpotent groups of class 2 be given by the following generators and relations:

$$\begin{aligned} A = \langle a_1, a_2, \ldots, b_1, b_2, \cdots \mid [b_1, a_1] = 1, [b_2, a_1] = [b_2, a_2] = 1, \ldots, \\ [b_n, a_1] = [b_n, a_2] = \cdots = [b_n, a_n] = 1, [b_{n+1}, a_1] = [b_{n+1}, a_2] = \cdots \\ = [b_{n+1}, a_n] = [b_{n+1}, a_{n+1}] = 1, \ldots \rangle. \end{aligned}$$

Consider a system of equations in one variable such as

$$[x, a_1] = 1, \quad [x, a_2] = 1, \quad [x, a_3] = 1, \quad \ldots;$$

this system is not equivalent to any one of its finite subsystems of the form

$$[x, a_1] = 1, \quad [x, a_2] = 1, \quad \ldots, \quad [x, a_n] = 1.$$

Since $x = b_n$ is a solution to the subsystem, but $[b_n, a_{n+1}] \neq 1$. Therefore, the group A is not equationally Noetherian.

Example 2. Consider a free centre-by-metabelian group of rank 2. Its derived subgroup is a free nilpotent group of class 2 of countable rank (C. K. Gupta [29]). We identify the basis of the derived subgroup with a set $\{a_1, a_2, \ldots, b_1, b_2, \ldots\}$, imposing on elements certain relations which the group A in the previous example enjoyed. We obtain a 2-generated centre-by-metabelian group C, whose derived subgroup is equal to A. Thus the group C is not equationally Noetherian, because its subgroup A has this property.

12. Automorphisms of free groups of countable infinite rank

$F = F_\infty = \langle x_1, x_2, \ldots \rangle$: **Methods of Nielsen and Whitehead do not reduce an infinite set of generators of F in a finite number of steps.** D. H. Wagner extended Nielsen transformations to apply to infinite subsets of a group.

Wagner's τ-transformations: elementary simultaneous transformations, where generators of $\tau(x)$ have bounded lengths.

Question (D. Solitar): **Is the group of all automorphisms of F that are bounded relative to the given basis $\{x_i \mid i \in I\}$ generated by generalized elementary simultaneous Nielsen transformations?**

The following natural generalizations of the elementary Nielsen transformations:

(i) automorphisms permuting the x_i;

(ii) automorphisms inverting any subset of the x_i's and leaving the remainder fixed;

(iii) automorphisms of the form: given any partition I_1 and I_2 (in I)

$$\{x_{i_1} \longrightarrow x_{i_1}x_{i_2} \quad i_1 \in I_1, i_2 \in I_2, \quad x_{i_2} \longrightarrow x_{i_2} \quad \text{for all } i_2 \in I_2\};$$

(iv)

$$\{x_{i_1} \longrightarrow x_{i_1}x_{i_2}^{\pm 1} \quad \text{or} \quad x_{i_1} \longrightarrow x_{i_2}^{\pm 1}x_{i_1}, \quad i_2 \in I_2\}.$$

We call automorphisms of these four types generalized Nielsen elementary transformations.

An automorphism $\varphi \in \mathrm{Aut}(F_\infty)$ is bounded if the lengths of the words $x_i\varphi$ and $x_i\varphi^{-1}$ are bounded: there is an n such that $|x_i\varphi|, |x_i\varphi^{-1}| \leq n$ for all $i \in I$. (See also R. Cohen [18], Burns & Pi [15].) The problem of Solitar still remains open.

Theorem 12.1 (C. K. Gupta and W. Holubowski [35]). *We described a generating set for the group* $\operatorname{Aut} F_\infty$*. We show: the group of all automorphisms (modulo the* IA*-automorphisms) is generated by strings and the lower triangular automorphisms. Some new subgroups of* $\operatorname{Aut} F_\infty$ *are presented.*

13. Automorphisms of rooted trees

Here we prove the following:

Theorem 13.1 (C. K. Gupta, N. D. Gupta, and A. S. Oliynyk [33]). *Let a finite number of finite groups be given. Let* n *be the largest order of their composition factors. We prove explicitly that the group of finite state automorphisms of rooted* n*-tree contains subgroups isomorphic to the free product of given groups.*

14. Universal theories for partially commutative groups

Here Γ is a finite graph without loops, whose vertex set $\{x_1, \ldots, x_n\}$ is denoted by X, and the set of edges (x_i, x_j) by E. For a graph Γ, a partially commutative group given by

$$G(c, \Gamma) = \langle x_1, \ldots, x_n \mid x_i x_j = x_j x_i \Longleftrightarrow (x_i, x_j) \in E; \quad \mathfrak{A}^2 \wedge \mathfrak{N}_c \rangle.$$

$\mathfrak{A}^2 \wedge \mathfrak{N}_c$: class of nilpotent metabelian groups.

We define certain transformations of a defining graph: we prove that these transformations do not change the universal theory of a partially commutative nilpotent metabelian group for each defining graph.

Theorem 14.1 (C. K. Gupta and E. I. Timoshenko [40]). *We give necessary and sufficient conditions for two partially commutative nilpotent metabelian groups defined by trees to be universally equivalent.*

Theorem 14.2 (C. K. Gupta and E. I. Timoshenko [41]). *Two partially commutative metabelian groups defined by cycles are universally equivalent if and only if the cycles are isomorphic.*

Definition. Two groups G and H are said to be universally equivalent if their universal theories coincide. That is, any $\exists$-formula is true on one of these groups if and only if it is true on the other.

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