www.theoryofgroups.ir

International Journal of Group Theory
ISSN (print): 2251-7650, ISSN (on-line): 2251-7669
Vol. 2 No. 1 (2013), pp. 117-129.

www.ui.ac.ir

METAHAMILTONIAN GROUPS AND RELATED TOPICS

M. DE FALCO, F. DE GIOVANNI* AND C. MUSELLA

Communicated by Patrizia Longobardi

Abstract. A group G is called *metahamiltonian* if all its non-abelian subgroups are normal. The aim of this paper is to provide an updated survey of research concerning certain classes of generalized metahamiltonian groups, in various contexts, and to prove some new results. Some open problems are listed.

1. **Introduction**

It is well known that a group G has only normal subgroups (i.e. is a *Dedekind group*) if and only if G is either abelian or the direct product of a quaternion group of order 8 and a periodic abelian group with no elements of order 4. The structure of groups for which the set of non-normal subgroups is small in some sense has been studied by many authors in several different situations.

A group G is called *metahamiltonian* if all its non-abelian subgroups are normal. Metahamiltonian groups were introduced and investigated in a series of papers by G.M. Romalis and N.F. Sesekin (see [39],[40],[41]). Of course, any group whose proper subgroups are abelian is metahamiltonian; in particular, minimal non-abelian groups are metahamiltonian. Among these are the so-called Tarski groups (i.e. infinite simple groups whose proper non-trivial subgroups have prime order). In order to avoid them and other similar pathological examples, some weak solubility condition is necessary in the study of metahamiltonian groups. In fact, the best results on metahamiltonian groups have been proved within the universe of locally graded groups; here a group G is said to be *locally graded* if every finitely generated non-trivial subgroup of G contains a proper subgroup of finite index. It follows from the results of Romalis and Sesekin that locally graded metahamiltonian groups are soluble.

MSC(2010): Primary: 20F19; Secondary: 20F24.

Keywords: Metahamiltonian group, normalizer subgroup, lattice property.

Received: 30 December 2012, Accepted: 8 March 2013.

∗Corresponding author.

In the second section of this paper, the structure of metahamiltonian groups will be described, while Section 3 and Section 4 are devoted to the study of groups with many metahamiltonian subgroups and with many metahamiltonian images, respectively. Certain classes of generalized metahamiltonian groups are studied in Section 5, and groups with few normalizer subgroups are considered in Section 6. The final Section 7 deals with problems of metahamiltonian type in the subgroup lattice of a group.

Most of our notation is standard and can be found in [38].

2. **Metahamiltonian groups**

The class of metahamiltonian groups is obviously closed with respect to subgroups and homomorphic images. It is also clear that if H is any non-abelian subgroup of a metahamiltonian group G, then all subgroups of G containing H are normal in G, and so G/H is a Dedekind group. This remark suggests that the commutator subgroup of a (soluble) metahamiltonian group cannot be too large, and in fact Romalis and Sesekin proved the following relevant result.

Theorem 2.1. *Let G be a locally graded metahamiltonian group. Then the commutator subgroup G' of G is finite, and its order is a power of a prime number.*

As a consequence of this result, it follows that any locally graded metahamiltonian group is soluble and has boundedly finite conjugacy classes. For a proof of Theorem 2.1 we refer to [15]. A further restriction on the commutator subgroup of a metahamiltonian group was later obtained by N.F. Kuzennyi and N.N. Semko [24], who proved that any non-abelian subgroup of a soluble metahamiltonian group G must contain G'. This fact is actually obvious if G' has odd order, and it was first proved by A.A. Mahnev [33] for finite 2-groups. We will omit here the proof for arbitrary finite groups, but we present an easy argument which shows how to deduce the infinite case from the finite one.

Theorem 2.2. *Let G be an infinite locally graded metahamiltonian group. Then the commutator subgroup G' of G is contained in every non-abelian subgroup of G.*

Proof. As the commutator subgroup G' of G is a finite primary group by Theorem 2.1, it can be assumed without loss of generality that G is finitely generated, so that it is even polycyclic. In particular, every subgroup of G is the intersection of a collection of subgroups of finite index. Let X be any non-abelian subgroup of G, and let $\mathfrak{K}$ be the set of all normal subgroups K of G such that G/K is finite and XK/K is not abelian. Then $\mathfrak{K}$ is not empty, and

$$\bigcap_{K\in\mathfrak{K}} XK = X.$$

On the other hand, assuming that the result has already been proved in the finite case, we have that G' is contained in XK for any $K \in \mathfrak{K}$, and hence $G' \leq X$. □

A detailed description of soluble metahamiltonian groups has been obtained in a series of more recent papers (see for instance [25],[26],[27],[28],[29],[30],[31]). The next result summarizes a large part of these contributions (see [23], Proposition 1).

Theorem 2.3. *Let G be a locally graded metahamiltonian group whose commutator subgroup has order p^k (where p is a prime number).*

(a) *If G is nilpotent and $|G'| > p$, then the Sylow p-subgroups of G have bounded exponent.*

(b) *If G is not nilpotent, then $G = G' \times H$, where $H = \langle C, a\rangle$ for a central subgroup C of G and an element a such that $C \cap \langle a\rangle = \langle a^n\rangle$ $(n > 1)$, and one of the following conditions holds:*

(b1) *G' is a minimal normal subgroup of G and it coincides with the commutator subgroup of every non-abelian subgroup of G, $(n, p) = 1$, and a^i induces an irreducible automorphism on G' for all $i = 1, \dots, n-1$;*

(b2) *G' is a Sylow p-subgroup of G, and it is either a quaternion group of order 8 or a non-abelian group of order p^3 and exponent p.*

3. Groups with many metahamiltonian subgroups

Let $\mathfrak{X}$ be a class of groups. Recall that a group G is said to be *minimal non-*$\mathfrak{X}$ if it is not an $\mathfrak{X}$-group but all its proper subgroups belong to $\mathfrak{X}$. It is easy to show that every locally graded minimal non-abelian group is finite, and it has been shown that a corresponding statement holds for minimal non-metahamiltonian groups (see [7]). To prove this result we need the easy remark that metahamiltonian groups form a local class, i.e. a group is metahamiltonian if and only if all its finitely generated subgroups have the same property.

Theorem 3.1. *Let G be an infinite locally graded group whose proper subgroups are metahamiltonian. Then G is metahamiltonian.*

Proof. Assume for a contradiction that G is not metahamiltonian, so that G must be finitely generated since the class of metahamiltonian groups is local. Thus G contains a proper subgroup of finite index, and so it follows from Theorem 2.1 that G is polycyclic. Let X be a subgroup of G which neither is abelian nor normal. Then X is contained in a non-normal subgroup H of G of finite index, and H contains a normal subgroup N of G such that G/N is finite and H/N is not abelian. In particular, G/N is not metahamiltonian. As G is infinite, it is known that the Frattini factor group $G/\Phi(G)$ is likewise infinite (see [32]) and hence there exists a maximal subgroup M of G such that N is not contained in M. Then $G = MN$ and

$$G/N \simeq M/M \cap N$$

is a metahamiltonian group. This contradiction proves the statement. □

It is clear that if G is any group whose proper subgroups are either abelian or minimal non-abelian, then G is either metahamiltonian or minimal non-metahamiltonian. In particular, the alternating group A_5 is minimal non-metahamiltonian. The consideration of the unique stem cover of A_5 shows that there exist perfect non-simple minimal non-metahamiltonian groups. The structure of (finite) minimal non-metahamiltonian groups is still unknown, even in the soluble case.

It was also proved in [7] that within the universe of locally graded groups the minimal condition on non-metahamiltonian subgroups can occur only in the extreme cases.

Theorem 3.2. *Let G be a locally graded group satisfying the minimal condition on subgroups which are not metahamiltonian. Then G is either metahamiltonian or a Černikov group.*

The above theorem can be applied to the case of groups in which non-metahamitonian subgroups fall into finitely many conjugacy classes (see [7]). In fact, a group G with this latter property locally satisfies the maximal condition on subgroups, so that a result of D.I. Zaicev shows that if X is any subgroup of G such that $X^g \leq X$ for some element g of G, then $X^g = X$ (see for instance [1], Lemma 4.6.3), and hence G satisfies the minimal condition on non-metahamiltonian subgroups.

Theorem 3.3. *Let G be an infinite locally graded group with finitely many conjugacy classes of non-metahamiltonian subgroups. Then G is metahamiltonian.*

Theorem 3.2 will also be used in our next result, which proves that the class of locally graded groups whose infinite proper subgroups are metahamiltonian is exhausted by metahamiltonian groups and groups whose infinite proper subgroups are abelian, with the unique exception of central extensions of Prüfer groups by finite groups. Note here that groups whose infinite proper subgroups are abelian have been described in [3].

Theorem 3.4. *Let G be an infinite locally graded non-metahamiltonian group whose infinite proper subgroups are metahamiltonian. Then either all infinite proper subgroups of G are abelian or G is a finite extension of a central subgroup of type p^∞ (where p is a prime number).*

Proof. The group G clearly satisfies the minimal condition on non-metahamiltonian subgroups, and so Theorem 3.2 yields that G is a Černikov group. Moreover, as G is not metahamiltonian, it contains a finite subgroup E which is not metahamiltonian. Let J be the finite residual of G, and let K be any infinite G-invariant subgroup of J. Then EK is an infinite non-metahamiltonian subgroup of G, and so $EK = G$. It follows that J/K is finite, and hence $J = K$. Therefore J has no infinite proper G-invariant subgroups.

Suppose first that the group G/J is cyclic, and let X be any infinite proper subgroup of G. If $XJ = G$, the intersection $X \cap J$ is an infinite normal subgroup of G, so that $X \cap J = J$ and $J \leq X$, a contradiction. Thus XJ is properly contained in G, and so it is metahamiltonian. In particular, the commutator subgroup of XJ is finite, so that $J \leq Z(XJ)$ and hence XJ is abelian. Therefore all infinite proper subgroups of G are abelian in this case.

Assume now that G/J is not cyclic. Then for each element x of G the subgroup $\langle J, x\rangle$ is metahamiltonian, and so even abelian. Then J is contained in the centre of G, and in particular it is a group of type p^∞ for some prime p. □

A full description of metahamiltonian groups which are finite extensions of a central subgroup of type p^∞ can be found in the forthcoming paper [8].

It turns out that for several relevant group classes $\mathfrak{X}$ any soluble minimal non-$\mathfrak{X}$ group has finite rank (recall here that a group G is said to have *finite rank* r if every finitely generated subgroup of G can be generated by at most r elements, and r is the smallest positive integer with such property). In recent years, many authors have investigated the structure of groups whose proper subgroups of infinite rank belong to a fixed group class $\mathfrak{X}$. In particular, M.R. Dixon, M. Evans and H. Smith [17] have shown that if G is a locally (soluble-by-finite) group whose proper subgroups of infinite rank have finite commutator subgroup, then either G has finite rank or its commutator subgroup G' is finite. This result can be used to prove a corresponding statement for the class of metahamiltonian groups (see [14]).

Theorem 3.5. *Let G be a locally (soluble-by-finite) group of infinite rank whose proper subgroups of infinite rank are metahamiltonian. Then G is metahamiltonian.*

Proof. It follows from Theorem 2.1 that every proper subgroup of G either is finite-by-abelian or has finite rank, and hence the commutator subgroup G' of G must have finite rank (see [17]). Let X be any subgroup of G of finite rank. Then the product XG' has likewise finite rank, and so the abelian factor group G/XG' has infinite rank. It follows that XG' is contained in a proper subgroup of G of infinite rank, and hence X is metahamiltonian. Then G itself is a metahamiltonian group by Theorem 3.1. □

4. Groups with many metahamiltonian images

The argument used in the proof of Theorem 3.1 also proves that every polycyclic non-metahamiltonian group has a finite homomorphic image which is not metahamiltonian. Classical results show that a similar conclusion holds for other relevant group classes, like for instance that of nilpotent groups. Improving this result, it was proved by D.J.S. Robinson that a finitely generated soluble group is nilpotent if and only if all its finite homomorphic images are nilpotent (see [38] Part 2, Theorem 10.51). It turns out that also the property of being metahamiltonian can be detected from the behavior of finite homomorphic images within the larger universe of finitely generated soluble groups (see [7]).

Theorem 4.1. *Let G be a finitely generated hyper-(abelian or finite) group whose finite homomorphic images are metahamiltonian. Then G is metahamiltonian.*

The consideration of McLain's characteristically simple groups (see [38] Part 2, p.14) shows that a locally nilpotent group whose finite homomorphic images are metahamiltonian need not be metahamiltonian (and not even soluble). However, it is easy to prove that any residually finite group whose finite homomorphic images are metahamiltonian must be soluble (recall here that the *finite residual* of a group G is the intersection of all subgroups of finite index of G, and G is *residually finite* if its finite residual is trivial).

A group G is said to be *minimax* if it has a series of finite length whose factors satisfy either the minimal or the maximal condition on subgroups. If G is a soluble-by-finite minimax group, it is well known that the finite residual J of G is the direct product of finitely many Prüfer subgroups, the Fitting subgroup F/J of G/J is a nilpotent group with finitely many elements of finite order and G/F is finitely generated and abelian-by-finite (see [38] Part 2, Theorem 10.91); the set *spec*(G) of all prime numbers p such that G has a section of type p^∞ is an invariant, called the *spectrum* of G. Thus a group G is polycyclic-by-finite if and only if it is a soluble-by-finite minimax group with empty spectrum.

Another result of D.J.S. Robinson [37] proves that if G is a soluble minimax residually finite group whose finite homomorphic images are nilpotent, then G itself is nilpotent. A corresponding result for the metahamiltonian groups is false. In fact, let p be an odd prime number, and consider the semidirect product $H = \langle b \rangle \ltimes \langle a \rangle$, where a has order p^3 and b is an element of order p^2 such that $a^b = a^{1+p}$; if $\mathbb{Q}_p$ is the additive group of rational numbers whose denominators are powers of p, the direct product $G = \mathbb{Q}_p \times H$ is a metabelian minimax residually finite group whose finite homomorphic images are metahamiltonian, but G is not metahamiltonian. On the other hand, for minimax groups with metahamiltonian finite homomorphic images the following result has been proved (see [7]); it shows that the main obstacle in this case is the size of the spectrum.

Theorem 4.2. *Let G be a soluble minimax group whose finite homomorphic images are metahamiltonian. Then the commutator subgroup G' of G is periodic. Moreover, if spec$(G) \cap \pi(G') = \emptyset$, then G is metahamiltonian.*

In the situation described by the above theorem, if the group G is assumed to be also residually finite, it follows that the commutator subgroup G' is finite with prime-power order - as in the metahamiltonian case.

Let $\mathfrak{X}$ be a class of groups. Recall that a group G is *just non-*$\mathfrak{X}$ if it is not an $\mathfrak{X}$-group, but all its proper homomorphic images belong to $\mathfrak{X}$. The structure of soluble just non-$\mathfrak{X}$ groups has been described for several choices of the group class $\mathfrak{X}$, and the monograph [22] is a general reference on this subject. It would be interesting to obtain a classification of soluble just non-metahamiltonian groups.

5. Generalized metahamiltonian groups

Let ξ be a function associating to each group G a set $\xi(G)$ of non-abelian subgroups of G. The imposition of normality only to the members of $\xi(G)$ defines a class of generalized metahamiltonian groups, which coincides with that of metahamiltonian groups if $\xi(G)$ is the set of all non-abelian subgroups of G. The first of these classes was considered by S.N. Černikov [4], who restricted the attention to infinite subgroups and proved the following result.

Theorem 5.1. *Let G be a locally graded group in which every infinite non-abelian subgroup is normal. Then either G' is finite or G is a Černikov group.*

More recently, B. Bruno and R.E. Phillips [2] investigated the structure of groups in which all non-normal subgroups are locally nilpotent.

M.J. Evans and Y. Kim [19] have proved that if G is a locally (soluble-by-finite) group of infinite rank in which all subgroups of infinite rank are normal, then G is a Dedekind group. This result suggests applying the normality condition to the set of non-abelian subgroups of infinite rank, and in fact metahamiltonian groups can be characterized in terms of such subgroups (see [13]).

Theorem 5.2. *Let G be a locally (soluble-by-finite) group in which every non-abelian subgroup of infinite rank is normal. Then either G has finite rank or it is metahamiltonian.*

In the proof of Theorem 5.2, the intersection $M(G)$ of all normal subgroups of infinite rank of G plays a relevant role, and in fact the first main step is to show that such characteristic subgroup has finite rank. This step involves a result on the dimension of certain modules over the group algebra of an infinite cyclic group.

Other classes of generalized metahamiltonian groups, in the above sense, have been introduced imposing restrictions either on the commutator subgroup or on the conjugacy classes of non-normal subgroups. In particular, the authors have studied in [5] groups whose non-normal subgroups have finite commutator subgroup, while N.N. Semko and A.O. Yarovaya [43] considered groups in which non-normal subgroups have Černikov commutator subgroup. Finally, L.A. Kurdachenko, J. Otal, A. Russo and G. Vincenzi [21] investigated the structure of groups in which every non-normal subgroup has finite conjugacy classes.

Further generalizations of the class of metahamiltonian groups can be achieved by weakening the condition of normality. The first attempt of research in this direction is probably that of R.E. Phillips and J.S. Wilson [35], who studied, among other situations, groups in which all non-abelian subgroups are serial.

In a famous paper of 1955, B.H. Neumann [34] considered infinite groups in which every subgroup is close to be normal, with the obstruction of a finite section. In fact, he proved that a group G has finite conjugacy classes of subgroups (or equivalently that each subgroup is normal in a subgroup of finite index) if and only if the centre $Z(G)$ has finite index, and that groups in which every subgroup has finite index in its normal closure can be characterized as those with finite commutator subgroup. Much more recently, the authors and Y.P. Sysak [12] have investigated the behavior of groups in which Neumann's restrictions are imposed only to non-abelian subgroups, with special attention to the case of (locally) nilpotent groups. We quote here the following result, which describes groups with bounded normality conditions for non-abelian subgroups and generalizes the theorem of Romalis and Sesekin.

Theorem 5.3. *Let G be a locally graded group for which there exists a positive integer k such that for each non-abelian subgroup X of G either $|G : N_G(X)| \leq k$ or $|X^G : X| \leq k$. Then the commutator subgroup G' of G is finite.*

6. Groups with few normalizers

The above quoted theorem of Neumann stating that a group is finite over its centre if and only if all its conjugacy classes of subgroups are finite has been later improved by I.I. Eremin [18], who showed that in this statement it is enough to consider the conjugacy classes of abelian subgroups. Since a subgroup X of a group G has finitely many conjugates if and only if its normalizer $N_G(X)$ has finite index in G, these results suggest that the behaviour of normalizers of subgroups has a strong influence on the structure of a group. In fact, Y.D. Polovickiĭ [36] proved that a group is central-by-finite if and only if it has only finitely many normalizers of (abelian) subgroups (for a proof of this result see [15]).

Theorem 6.1. *Let G be a group with finitely many normalizers of abelian subgroups. Then the factor group $G/Z(G)$ is finite.*

In the study of groups with few normalizers of subgroups of a given type, and in particular in the proof of Polovickiĭ's theorem, a crucial role is played by the following lemma, which is also due to B.H. Neumann (see [38], Lemma 4.17).

Lemma 6.2. *Let the group $G = X_1 \cup \ldots \cup X_k$ be the union of finitely many subgroups $X_1, \ldots, X_k$. Then any X_i of infinite index can be omitted from this decomposition. In particular, at least one of the subgroups $X_1, \ldots, X_k$ has finite index in G.*

Of course, the requirement for a group to have only finitely many normalizers of non-abelian subgroups generalizes the property of being metahamiltonian, and hence the following result proved in [16] is another improvement of Theorem 2.1.

Theorem 6.3. *Let G be a locally graded group with finitely many normalizers of non-abelian subgroups. Then the commutator subgroup G' of G is finite.*

We leave here as an open question whether Theorem 5.2 can be generalized to the case of groups having finitely many normalizers of non-abelian subgroups of infinite rank. As a first step, we can prove here that in this case the commutator subgroup of the group is finite - like that of a metahamiltonian group.

Corollary 6.4. *Let G be a locally (soluble-by-finite) group of infinite rank. If G has only finitely many normalizers of non-abelian subgroups of infinite rank, then the commutator subgroup G' of G is finite.*

Proof. Let k be the number of proper normalizers of non-abelian subgroups of infinite rank of G. If $k = 0$, we have that all non-abelian subgroups of infinite rank of G are normal, so that G is metahamiltonian by Theorem 5.2, and hence G' is finite. Suppose now that $k > 0$, and let

$$N_G(X_1), \dots, N_G(X_k)$$

be the proper normalizers of non-abelian subgroups of infinite rank of G. For each $i = 1, \dots, k$, the group $N_G(X_i)$ has infinite rank and less than k proper normalizers of non-abelian subgroups of infinite rank, so that by induction on k we may suppose that its commutator subgroup $N_G(X_i)'$ is finite. Moreover, the subgroup $N_G(X_i)$ has obviously finitely many conjugates in G, and so the same property holds for $N_G(X_i)'$. Thus it follows from Dietzmann's Lemma that the normal closure

$$N = \langle N_G(X_1)', \dots, N_G(X_k)' \rangle^G$$

is a finite subgroup of G. Clearly, all non-abelian subgroups of infinite rank of the group G/N are normal, so that G/N is metahamiltonian by Theorem 5.2. In particular, the group G/N has finite commutator subgroup, and hence G' is finite. □

An argument similar to that used in the above proof shows that also Neumann's theorem can be extended to this situation. Observe that the corresponding generalization of the theorem of Eremin is another open question.

Theorem 6.5. *Let G be a locally (soluble-by-finite) group of infinite rank. If G has only finitely many normalizers of subgroups of infinite rank, then the factor group $G/Z(G)$ is finite.*

Proof. The commutator subgroup G' of G is finite by Theorem 6.4. Let k be the number of proper normalizers of subgroups of infinite rank of G. If $k = 0$, all subgroups of infinite rank of G are normal, and so G is a Dedekind group (see [19]). Suppose $k > 0$, and let

$$N_G(X_1), \dots, N_G(X_k)$$

be the proper normalizers of subgroups of infinite rank of G. For any $i \leq k$, the group $N_G(X_i)$ has infinite rank and less than k proper normalizers of subgroups of infinite rank, and hence we may suppose by induction on k that $N_G(X_i)$ is central-by-finite. If the index $|G : N_G(X_i)|$ is finite for some $i \leq k$, the group G is abelian-by-finite, and so $G/Z(G)$ is finite because G' is finite. Assume now for a contradiction that each normalizer $N_G(X_i)$ has infinite index in G. Then it follows from Lemma 6.2 that

$$N_G(X_1) \cup \ldots \cup N_G(X_k)$$

is a proper subset of G, and we may consider an element g in the set

$$G \setminus (N_G(X_1) \cup \ldots \cup N_G(X_k)).$$

As G' is finite, the centralizer $C_G(g)$ has finite index in G, and so it has infinite rank. Moreover, if X is any subgroup of infinite rank of $C_G(g)$, we have that X is normal in G, because $g \in N_G(X)$. Thus all subgroups of infinite rank of $C_G(g)$ are normal, and so $C_G(g)$ is a Dedekind group. In particular, G is abelian-by-finite and hence $G/Z(G)$ is finite. This contradiction proves the statement. □

7. **Lattice properties**

Recall that if $\mathfrak{L}$ is any lattice, an element a of $\mathfrak{L}$ is said to be *modular* if

$$(a \vee x) \wedge y = a \vee (x \wedge y)$$

for all elements x, y of $\mathfrak{L}$ such that $a \leq y$ and

$$(a \vee x) \wedge y = x \vee (a \wedge y)$$

if $x \leq y$. The lattice $\mathfrak{L}$ is called *modular* if all its elements are modular, i.e. if

$$(x \vee y) \wedge z = x \vee (y \wedge z)$$

for all elements x, y, z of $\mathfrak{L}$ such that $x \leq z$. Thus each projective image (i.e. every image under a lattice isomorphism) of an abelian group has modular subgroup lattice, and groups with this latter property can be considered as suitable lattice approximations of abelian groups. On the other hand, there exist also infinite simple groups with modular subgroup lattice, like for instance Tarski groups. The structure of groups with modular subgroup lattice has been completely described by K. Iwasawa and R. Schmidt. In particular, it turns out that locally graded groups with modular subgroup lattice are metabelian. We refer to [42] for a detailed account of results on groups with modular subgroup lattice, and in general for properties of the lattice of subgroups of a group.

In recent years, it has been proved that many results concerning finiteness conditions for infinite groups have analogues in the theory of subgroup lattices. For instance, this is the case for the theorems of Neumann quoted above and for the celebrated theorem of Schur on the finiteness of the commutator subgroup of any group which is finite over the centre (see for instance [6],[11],[20]). It must be observed that in the treatment of all these problems from a lattice point of view, a central role is played by a fundamental theorem of G. Zacher [44], which proves that the finiteness of the index of a subgroup is invariant under projectivities. The following lattice translation of the theorem of Romalis and Sesekin has been obtained in [10].

Theorem 7.1. *Let G be a periodic locally graded group in which every subgroup either has modular subgroup lattice or is a modular element of the subgroup lattice of G. Then G contains a finite normal subgroup N such that the factor group G/N has modular subgroup lattice. Moreover, the subgroup G'' is finite of prime-power order.*

In many problems concerning the lattice of subgroups of a group, a crucial role is played here by the so-called permutable subgroups: a subgroup X of a group G is said to be *permutable* if $XY = YX$ for all subgroups Y of G, and a group G is called *quasihamiltonian* if all its subgroups are permutable. It turns out that a subgroup is permutable if and only if it is modular and ascendant. It follows that quasihamiltonian groups are precisely the locally nilpotent groups with modular subgroup lattice, and in particular they are metabelian. Our last result, which has been proved in [9], is another analog of the theorem of Romalis and Sesekin.

Theorem 7.2. *Let G be a locally graded group in which every non-abelian subgroup is permutable. Then G contains a finite normal subgroup N such that the factor group G/N is quasihamiltonian. Moreover, G is soluble with derived length at most* 4.

References

[1] B. Amberg, S. Franciosi and F. de Giovanni, *Products of groups*, Oxford Mathematical Monographs, The Clarendon Press, Oxford University Press, New York, 1992.

[2] B. Bruno and R. E. Phillips, Groups with restricted non-normal subgroups, *Math. Z.*, **176** no. 2 (1981) 199–221.

[3] M. R. Celentani and U. Dardano, Groups whose infinite proper subgroups are T-groups, *Note Mat.*, **13** no. 2 (1993) 295–307.

[4] S. N. Černikov, Infinite nonabelian groups in which all infinite nonabelian subgroups are invariant, *Ukrain. Math. J.*, **23** (1971) 498–517.

[5] M. De Falco, F. de Giovanni and C. Musella, Groups whose non-normal subgroups have small commutator subgroup, *Algebra Discrete Math.*, (2007) no. 3 46–58.

[6] M. De Falco, F. de Giovanni and C. Musella, The Schur property for subgroup lattices of groups, *Arch. Math. (Basel)*, **91** no. 2 (2008) 97–105.

[7] M. De Falco, F. de Giovanni and C. Musella, Groups whose finite homomorphic images are metahamiltonian, *Comm. Algebra*, **37** no. 7 (2009) 2468-2476.

[8] M. De Falco, F. de Giovanni and C. Musella, On a class of metahamiltonian groups, *Ricerche Mat.*, to appear.

[9] M. De Falco, F. de Giovanni, C. Musella and R. Schmidt, Groups in which every non-abelian subgroup is permutable, *Rend. Circ. Mat. Palermo (2)*, **52** no. 1 (2003) 70–76.

[10] M. De Falco, F. de Giovanni, C. Musella and R. Schmidt, Groups with metamodular subgroup lattice, *Colloq. Math.*, **95** no. 2 (2003) 231–240.

[11] M. De Falco, F. de Giovanni, C. Musella and Y. P. Sysak, Periodic groups with nearly modular subgroup lattice, *Illinois J. Math.*, **47** no. 1-2 (2003) 189–205.

[12] M. De Falco, F. de Giovanni, C. Musella and Y. P. Sysak, Groups with normality conditions for non-abelian subgroups, *J. Algebra*, **315** no. 2 (2007) 665–682.

[13] M. De Falco, F. de Giovanni, C. Musella and Y. P. Sysak, On metahamiltonian groups of infinite rank, to appear.

[14] M. De Falco, F. de Giovanni, C. Musella and N. Trabelsi, Groups with restrictions on subgroups of infinite rank, *Rev. Mat. Iberoam.*, to appear.

[15] F. De Mari and F. de Giovanni, Groups with few normalizer subgroups, *Irish Math. Soc. Bull.*, **56** (2005) 103–113.

[16] F. De Mari and F. de Giovanni, Groups with finitely many normalizers of non-abelian subgroups, *Ric. Mat.*, **55** no. 2 (2006) 311–317.

[17] M. R. Dixon, M. J. Evans and H. Smith, Groups with all proper subgroups (finite rank)-by-nilpotent, *Arch. Math. (Basel)*, **72** (1999) 321–327.

[18] I. I. Eremin, Groups with finite classes of conjugate abelian subgroups, *Mat. Sb. (N.S.)*, **47** (1959) 45–54.

[19] M. J. Evans and Y. Kim, On groups in which every subgroup of infinite rank is subnormal of bounded defect, *Comm. Algebra*, **32** (2004) 2547–2557.

[20] F. de Giovanni, C. Musella and Y. P. Sysak, Groups with almost modular subgroup lattice, *J. Algebra*, **243** no. 2 (2001) 738–764.

[21] L. A. Kurdachenko, J. Otal, A. Russo and G. Vincenzi, Groups whose non-normal subgroups have finite conjugacy classes, *Math. Proc. R. Ir. Acad.*, **104A** no. 2 (2004) 177–189.

[22] L. A. Kurdachenko, J. Otal and I. Subbotin, *Groups with prescribed quotient groups and associate module theory*, World Scientific, Singapore, 2002.

[23] N. F. Kuzennyi, S. S. Levishchenko and N. N. Semko, Groups with invariant infinite non-abelian subgroups, *Ukrain. Math. J.*, **40** no. 3 (1988) 267–273.

[24] N. F. Kuzennyi and N. N. Semko, Structure of solvable nonnilpotent metahamiltonian groups, *Math. Notes*, **34** no. 2 (1983) 572–577.

[25] N. F. Kuzennyi and N. N. Semko, On the structure of infinite nilpotent periodic metahamiltonian groups, in *Structure of groups and their subgroup characterization*, Kiev, (1984) 101–111.

[26] N. F. Kuzennyi and N. N. Semko, Structure of solvable metahamiltonian groups, *Dokl. Akad. Nauk Ukrain. SSR Ser. A*, no. 2 (1985) 6–9.

[27] N. F. Kuzennyi and N. N. Semko, On the structure of nonperiodic metahamiltonian groups, *Izv. Vuzov Matematika*, **11** (1986) 32–38.

[28] N. F. Kuzennyi and N. N. Semko, Structure of periodic metabelian metahamiltonian groups with a nonelementary commutator subgroup, *Ukrain. Math. J.*, **39** no. 2 (1987) 149–153.

[29] N. F. Kuzennyi and N. N. Semko, The structure of periodic metabelian metahamiltonian groups with an elementary commutator subgroup of rank two, *Ukrain. Math. J.*, **40** no. 6 (1988) 627–633.

[30] N. F. Kuzennyi and N. N. Semko, Structure of periodic nonabelian metahamiltonian groups with an elementary commutator subgroup of rank three, *Ukrain. Math. J.*, **41** no. 2 (1989) 153–158.

[31] N. F. Kuzennyi and N. N. Semko, Metahamiltonian groups with elementary commutator subgroup of rank two, *Ukrain. Math. J.*, **42** no. 2 (1990) 149–154.

[32] J. C. Lennox, Finite Frattini factors in finitely generated soluble groups, *Proc. Amer. Math. Soc.*, **41** (1973) 356–360.

[33] A. A. Mahnev, On finite metahamiltonian groups, *Ural. Gos. Univ. Mat. Zap.*, **10** no. 1 (1976) 60–75.

[34] B. H. Neumann, Groups with finite classes of conjugate subgroups, *Math. Z.*, **63** (1955) 76–96.

[35] R. E. Phillips and J. S. Wilson, On certain minimal conditions for infinite groups, *J. Algebra*, **51** no. 1 (1978) 41–68.

[36] Y. D. Polovickiĭ, Groups with finite classes of conjugate infinite abelian subgroups, *Soviet Math. (Iz. VUZ)*, **24** (1980) 52–59.

[37] D. J. S. Robinson, Residual properties of some classes of infinite soluble groups, *Proc. London Math. Soc. (3)*, **18** (1968) 495–520.

[38] D. J. S. Robinson, *Finiteness conditions and generalized soluble groups*, Springer, Berlin, 1972.

[39] G. M. Romalis and N. F. Sesekin, Metahamiltonian groups, *Ural. Gos. Univ. Mat. Zap.*, **5** (1966) 101–106.

[40] G. M. Romalis and N. F. Sesekin, Metahamiltonian groups II, *Ural. Gos. Univ. Mat. Zap.*, **6** (1968) 52–58.

[41] G. M. Romalis and N. F. Sesekin, Metahamiltonian groups III, *Ural. Gos. Univ. Mat. Zap.*, **7** (1969/70) 195–199.

[42] R. Schmidt, *Subgroup lattices of groups*, de Gruyter, Berlin, 1994.

[43] N. N. Semko and O. A. Yarovaya, On some generalization of metahamiltonian groups, *Algebra Discrete Math.*, no. 2 (2009) 70–78.

[44] G. Zacher, Una caratterizzazione reticolare della finitezza dell'indice di un sottogruppo in un gruppo, *Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur. (8)*, **69** no. 6 (1980) 317–323.

Maria De Falco

Dipartimento di Matematica e Applicazioni, Università di Napoli Federico II, via Cintia, Napoli, Italy

Email: mdefalco@unina.it

Francesco de Giovanni

Dipartimento di Matematica e Applicazioni, Università di Napoli Federico II, via Cintia, Napoli, Italy

Email: degiovan@unina.it

Carmela Musella

Dipartimento di Matematica e Applicazioni, Università di Napoli Federico II, via Cintia, Napoli, Italy

Email: cmusella@unina.it