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PARTIALLY S-EMBEDDED MINIMAL SUBGROUPS OF FINITE GROUPS

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Abstract. Suppose that H is a subgroup of G, then H is said to be s-permutable in G, if H permutes with every Sylow subgroup of G. If $HP = PH$ hold for every Sylow subgroup P of G with $(|P|, |H|) = 1)$, then H is called an s-semipermutable subgroup of G. In this paper, we say that H is partially S-embedded in G if G has a normal subgroup T such that HT is s-permutable in G and $H \cap T \leq H_{\bar{s}G}$, where $H_{\bar{s}G}$ is generated by all s-semipermutable subgroups of G contained in H. We investigate the influence of some partially S-embedded minimal subgroups on the nilpotency and supersolubility of a finite group G. A series of known results in the literature are unified and generalized.

1. Introduction

All groups considered in this paper are finite and G stands for a group. We use conventional notions and notation, as in Robinson [15]. Let $\mathcal{F}$ be a formation, $G^{\mathcal{F}}$ and $Z_{\mathcal{F}}(G)$ denote the $\mathcal{F}$-residual and $\mathcal{F}$-hypercenter of G, respectively.

From minimal subgroup's generalized normalities to characterize the structure of a finite group is an active topic in the group theory. For example, Buckley and Itô have got some well-known results about the supersolublity and nilpotency of a finite group, respectively. Since then, a series of papers have dealt with generalizations of the results of Itô and Buckley by using the theory of formations and some generalized normal subgroups (see, for example, [1], [2], [4], [9], [14]). Following Kegel in [13], a subgroup H of G is said to be s-permutable in G, if H permutes with every Sylow subgroup of G. After that, Wang in [17] introduced that: a subgroup H is said to be c-normal in G, if G has a normal subgroup T such that $G = HT$ and $H \cap T \leq H_G$, where H_G is the normal core of H in G. These

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concepts have been investigated extensively by many scholars, a lot of meaningful results have been obtained. Recently, Guo et al [8] integrated the above two concepts and introduced that: a subgroup H is said to be S-embedded in G, if there exists a normal subgroup N such that HN is s-permutable in G and $H \cap N \leq H_{sG}$, where H_{sG} is the largest s-permutable subgroup of G contained in H. By assuming that some subgroups of G satisfy the S-embedded property, many meaningful results have been derived (see [8], [10] etc.), a series of known results in the literature were generalized. As another generation of the s-permutable subgroup, following Chen in [3], a subgroup H of a group G is said to be s-semipermutable (or s-seminormal) in G if $PH = HP$ for all Sylow p-subgroups P of G with $(p, |H|) = 1$. Based on the above concepts, in this paper we introduce that:

Definition 1.1. *A subgroup H of a group G is said to be partially S-embedded in G, if G has a normal subgroup T such that HT is an s-permutable subgroup of G and $H \cap T \leq H_{\bar{s}G}$, where $H_{\bar{s}G}$ is generated by all those subgroups of H which are s-semipermutable in G.*

It is easy to see that $H_{\bar{s}G}$ is the largest s-semipermutable subgroup of G contained in H. Besides that, every subgroup of G, whether it is s-permutable, c-normal, s-semipermutable or S-embedded in G is partially S-embedded in G. In general, a partially S-embedded subgroup of G need not be S-embedded or s-semipermutable in G. For instance, let $G = S_5$ be the symmetric group of degree 5.

Example 1. Since $H = S_4$ permutes with every Sylow 5-subgroup of G, H is s-semipermutable and thus partially S-embedded in G. But neither H nor $H \cap A_5 = A_4$ is s-permutable in G, as they are not subnormal subgroups of G. Since the only normal subgroups of G are A_5 and G itself, $H = S_4$ is not S-embedded in G.

Example 2. Let $K = \langle(12)\rangle$ and $T = A_5$. Since $T \trianglelefteq G$ and $K \cap T = 1 \leq K_{\bar{s}G}$, K is partially S-embedded in G. But the fact $K\langle(12345)\rangle \neq \langle(12345)\rangle K$ implies that K is not s-semipermutable in G.

In this paper, we investigate the influence of some partially S-embedded minimal subgroups on the structure of a finite group, our results generalized some known ones.

2. Preliminaries

We list here some basic results which are useful in the sequel.

Lemma 2.1. *([13]) Suppose that H is an s-permutable subgroup of G, $H \leq G$ and $N \trianglelefteq G$.*

(1) *If $K \leq G$, then $H \cap K$ is s-permutable in K.*
(2) *HN and $H \cap N$ are s-permutable in G, HN/N is s-permutable in G/N.*
(3) *H is subnormal in G.*
(4) *If H is a p-group for some prime p, then $N_G(H) \geq O^p(G)$.*

Lemma 2.2. *([18]) Let G be a group and $H \leq K \leq G$.*

(1) *If H is s-semipermutable in G, then H is s-semipermutable in K.*
(2) *Suppose that N is normal in G, and H is a p-group. If H is s-semipermutable in G, then HN/N is s-semipermutable in G/N.*

Lemma 2.3. *([18, Lemma 3]) Let H be a subnormal p-subgroup of G. If H is s-semipermutable in G, then H is s-permutable in G.*

Lemma 2.4. *Suppose that $H \leq K \leq G$, N is a normal subgroup of G.*

(1) *If H is partially S-embedded in G, then H is partially S-embedded in K.*
(2) *Suppose that H is a partially S-embedded p-subgroup of G. If $N \leq H$ or $(p, |N|) = 1$, then HN/N is partially S-embedded in G/N.*
(3) *If $H \leq K \trianglelefteq G$ and H is partially S-embedded in G, then G has a normal subgroup N contained in K such that HN is s-permutable in G and $H \cap N \leq H_{\overline{s}G}$.*

Proof. Suppose that for some $T \trianglelefteq G$, HT is s-permutable in G and $H \cap T \leq H_{\overline{s}G}$.

(1) Clearly, $K \cap T$ is a normal subgroup of K. By Lemmas 2.1 and 2.2, we know that $H(K \cap T) = K \cap HT$ is s-permutable in K and $H \cap (K \cap T) = H \cap T \leq H_{\overline{s}G} \leq H_{\overline{s}K}$. Hence, H is partially S-embedded in K.

(2) It is easy to see that $TN/N \trianglelefteq G/N$ and $(HN/N)(TN/N) = HTN/N$ is s-permutable in G/N. If $N \leq H$, then $H/N \cap TN/N = (H \cap T)N/N \leq H_{\overline{s}G}N/N$. If N is a p'-group, then

$$|H \cap TN| = \frac{|H| \cdot |TN|_p}{|HTN|_p} = \frac{|H| \cdot |T|_p}{|HT|_p} = |H \cap T|.$$

This implies that $H \cap TN = H \cap T$, we also deduce that $(HN/N) \cap (TN/N) = (HN \cap TN)/N = (H \cap TN)N/N = (H \cap T)N/N \leq H_{\overline{s}G}N/N$. By Lemma 2.2, we know $H_{\overline{s}G}N/N$ is s-semipermutable in G/N. Hence, HN/N is partially S-embedded in G/N.

(3) Let $N = K \cap T$, then $HN = H(K \cap T) = K \cap HT$ is s-permutable in G and $H \cap N = H \cap K \cap T = H \cap T \leq H_{\overline{s}G}$. □

$T \leq G$ is said to be a supplement of H in G, if $HT = G$. If some supplement K of H belongs to a formation $\mathcal{F}$, then we say that K is an $\mathcal{F}$-supplement of H in G. We can easily deduce that:

Lemma 2.5. *Suppose that K is an $\mathcal{F}$-supplement of H in G.*

(1) *If $N \trianglelefteq G$, then HN/N is an $\mathcal{F}$-supplement of KN/N in G/N.*
(2) *If $H \leq M \leq G$ and $\mathcal{F}$ is a subgroup-closed formation, then $K \cap M$ is an $\mathcal{F}$-supplement of H in M.*

Lemma 2.6. *([9, Lemma 2.5]) Let G be a group and p a prime such that $p^{n+1} \nmid |G|$ for some integer $n \geq 1$. If $(|G|, (p-1)(p^2-1)\cdots(p^n-1)) = 1$, then G is p-nilpotent.*

Lemma 2.7. *([14, Lemma 2.8]) Suppose G is a group and P is a normal p-subgroup of G contained in $Z_\infty(G)$, then $C_G(P) \geq O^p(G)$.*

Lemma 2.8. *Let $\mathcal{F}$ be a saturated formation containing the class of all nilpotent groups $\mathcal{N}$, H a normal subgroup of G. If $G/H \in \mathcal{F}$ and $H \leq Z(G)$, then $G \in \mathcal{F}$.*

Proof. Let f and F be the canonical definitions of $\mathcal{N}$ and $\mathcal{F}$, respectively. Pick an chief factor M/N of G contained in H, then M/N is a p-group for some prime p. Since $M \leq H \leq Z(G)$, $M/N \leq Z(G/N)$.

Thus $G/C_G(M/N) = 1 \in f(p)$. Since $\mathcal{N} \subseteq \mathcal{F}$, $f(p) \subseteq F(p)$ by [6, IV, Proposition 3.11]. It follows that $G/C_G(M/N) \in F(p)$. The arbitrary choice of M/N implies that there exists a normal chain of G contained in H such that every G-chief factor is F-central. Since $G/H \in \mathcal{F}$, it follows that $G \in \mathcal{F}$. □

Lemma 2.9. *([12, X. 13]) Let $F^*(G)$ be the generalized Fitting subgroup of G.*

(1) *If M is a normal subgroup of G, then $F^*(M) \leq F^*(G)$.*
(2) *$F^*(G) \neq 1$, if $G \neq 1$; in fact, $F^*(G)/F(G) = Soc(F(G)C_G(F(G))/F(G))$.*
(3) *$F^*(F^*(G)) = F^*(G) \geq F(G)$; if $F^*(G)$ is soluble, then $F^*(G) = F(G)$.*
(4) *If $K \leq Z(G)$, then $F^*(G/K) = F^*(G)/K$.*

3. Main results

Theorem 3.1. *Let p be a prime and G a group with $(|G|, (p-1)(p^2-1)\cdots(p^n-1)) = 1$ for some integer $n \geq 1$. Then G is p-nilpotent if and only if every subgroup H of $P \in Syl_p(G)$ with order p^n or cyclic of order 4 (if P is a non-abelian 2-group, $n = 1$ and $|P| > 4$) not having a p-nilpotent supplement in G is partially S-embedded in G.*

Proof. The necessity is obvious, we need to prove only the sufficiency. Suppose that the result is false and let G be a counterexample of minimal order. Then we have:

(1) $p^{n+1}||G|$ and every proper subgroup of G is p-nilpotent.

The fact that $p^{n+1}||G|$ follows from Lemma 2.6. Let L be a proper subgroup of G, then $(|L|, (p-1)(p^2-1)\cdots(p^n-1)) = 1$. If $p^{n+1} \nmid |L|$, then Lemma 2.6 implies that L is p-nilpotent. Now we assume that $p^{n+1}||L|$ and let H be a subgroup of L with order p^n. By hypothesis, H either has a p-nilpotent supplement T in G or is partially S-embedded in G. In the former case, $L = L \cap HT = H(L \cap T)$ and $L \cap T$ is a p-nilpotent supplement of H in L. In the latter case, H is partially S-embedded in L by Lemma 2.4. Thus L satisfies the hypothesis of the theorem. The minimal choice of G implies that L is p-nilpotent. Then by [11, IV, Theorem 5.4] we have: $G = PQ$, where P is a normal Sylow p-subgroup and Q a non-normal cyclic Sylow q-subgroup of G for some prime $q \neq p$; $P/\Phi(P)$ is a minimal normal subgroup of $G/\Phi(P)$; $exp(P) = p$ if $p > 2$, while $exp(P)$ is at most 4 if $p = 2$.

(2) Every subgroup of P with order p^n or cyclic of order 4 (if P is a non-abelian 2-group, $n = 1$ and $|P| > 4$) not contained in $\Phi(P)$ is s-permutable in G.

Let H be a subgroup of P with order p^n or cyclic of order 4 (if P is a non-abelian 2-group, $n = 1$ and $|P| > 4$) and T a supplement of H in G. Then $HT = G$ and $P = P \cap HT = H(P \cap T)$. Since $P/\Phi(P)$ is a chief factor of G, $P/\Phi(P)$ is an elementary abelian p-group and hence $(P \cap T)\Phi(P)/\Phi(P)$ is normal in $G/\Phi(P)$. It follows that $P \cap T \leq \Phi(P)$ or $P \cap T = P$. If $P \cap T \leq \Phi(P)$, then $H = P$ is of order p^n, which contradicts with (1). If $P \cap T = P$, then $T = G$ is not p-nilpotent. Thus, every subgroup of P with order p^n is partially S-embedded in G.

Suppose that $H \nleq \Phi(P)$ is a subgroup of P with order p^n. Then by Lemma 2.4(3), there exists a normal subgroup T of G contained in P such that HT is s-permutable in G and $H \cap T \leq H_{\overline{s}G}$. If $T < P$, then $T \leq \Phi(P)$ as $P/\Phi(P)$ is a chief factor of G. Thus $H\Phi(P)/\Phi(P) = HT\Phi(P)/\Phi(P)$ is

s-permutable in $G/\Phi(P)$. By Lemma 2.1(4), we know $N_{G/\Phi(P)}(H\Phi(P)/\Phi(P)) \geq O^p(G/\Phi(P))$. Since $H\Phi(P)/\Phi(P) \trianglelefteq P/\Phi(P)$, $H\Phi(P)$ is normal in G. Therefore, we conclude that $H\Phi(P) = P$. This contradicts with (1). Thus we have $T = P$ and then $H = H_{\overline{s}G}$ is an s-semipermutable subgroup of G. Since $H \leq P \trianglelefteq \trianglelefteq G$, Lemma 2.3 implies that H is s-permutable in G.

(3) The final contradiction.

For any element $x \in P\backslash\Phi(P)$, we have $|x| = p$ or 4 by (1). If $n > 1$, then there exists a subgroup H of P with order p^n such that $x \in H \nsubseteq \Phi(P)$. Since H is s-permutable in G, HQ is a proper subgroup of G. Hence HQ is p-nilpotent by (1), which implies that $\langle x\rangle \leq H \leq N_G(Q)$. If $n = 1$, then by hypothesis and (2), $\langle x\rangle$ is s-permutable in G. Since P is not cyclic, $\langle x\rangle Q$ is a proper subgroup of G. Thus $\langle x\rangle Q$ is p-nilpotent and so $\langle x\rangle \leq N_G(Q)$. Therefore, in any case, we can both conclude that Q is normalized by P and so $Q \trianglelefteq G$. This final contradiction completes the proof of the theorem. □

In Theorem 3.1, when we take $n = 1$ we have:

Corollary 3.2. *Let P be a Sylow p-subgroup of G, where p is a prime divisor of $|G|$ with $(|G|, p-1) = 1$. Then G is p-nilpotent if and only if every minimal subgroup and cyclic subgroup with order 4 (if $p = 2$ and P is non-abelian) of P not having a p-nilpotent supplement in G is partially S-embedded in G.*

Next, we prove that:

Theorem 3.3. *Let $\mathcal{F}$ be a saturated formation containing the class of all supersolvable groups $\mathcal{U}$. Then $G \in \mathcal{F}$ if and only if there is a normal subgroup E of G such that $G/E \in \mathcal{F}$ and every cyclic subgroup of any noncyclic Sylow subgroup of E with prime order or order 4 (if the Sylow 2-subgroup of E is non-abelian) not having a supersoluble supplement in G is partially S-embedded in G.*

Proof. The necessity is obvious, we need to prove only the sufficiency. Suppose that the result is false and let G be a counterexample of minimal order. Then we have:

(1) The Sylow q-subgroup Q of E is normal in G, where $q = max\pi(E)$.

Let p be the smallest prime divisor of $|E|$ and $P \in Syl_p(E)$. If P is cyclic, then E is p-nilpotent by [11, IV, Theorem 2.8]. Now we assume that P is not cyclic. Since a supersoluble subgroup of E is p-nilpotent, any subgroup of E has no p-nilpotent supplement in E also has no supersoluble supplement in E. Lemma 2.4 and Corollary 3.2 implies that E is p-nilpotent. Let K be the normal p-complement of E, then K is of odd order and every minimal subgroup of any noncyclic Sylow subgroup of K not having a supersoluble supplement in K is partially S-embedded in K. Then we can also deduce that K is r-nilpotent for its minimal prime divisor r. This implies that E is a Sylow tower group of supersoluble type. Let q be the largest prime divisor and Q a Sylow q-subgroup of E, then Q *char* $E \trianglelefteq G$ which implies that $Q \trianglelefteq G$.

(2) The $\mathcal{F}$-residual $G^{\mathcal{F}}$ of G is contained in Q and it is not cyclic, $G^{\mathcal{F}}/\Phi(G^{\mathcal{F}})$ is a chief factor of G and $exp(G^{\mathcal{F}}) = q$ or 4.

By Lemmas 2.4 and 2.5, we can easily deduce that every cyclic subgroup of any non-cyclic Sylow subgroup of E/Q with prime order or order 4 (if the Sylow 2-subgroup of E/Q is non-abelian) not having a supersoluble supplement in G/Q is partially S-embedded in G/Q. Thus $G/Q \in \mathcal{F}$ by induction and $G^{\mathcal{F}} \leq Q$ is a q-group. By [16, Lemma 2.16], we know $G^{\mathcal{F}}$ is not cyclic. Since $\mathcal{F}$ is a saturated formation, $G^{\mathcal{F}} \nleq \Phi(G)$. Let M be a maximal subgroup of G such that $G^{\mathcal{F}} \nleq M$, then $G = MG^{\mathcal{F}} = MF(G)$. Since $M/M \cap E \cong ME/E = G/E \in \mathcal{F}$, a trivial argument shows that the hypothesis holds for M (respect to $M \cap E$). Thus $M \in \mathcal{F}$ by induction. By [7, Theorem 3.4.2], we know $G^{\mathcal{F}}/\Phi(G^{\mathcal{F}})$ is a chief factor of G, $G^{\mathcal{F}}$ has exponent q if $q > 2$ and exponent at most 4 if $q = 2$.

(3) Some minimal subgroup $X/\Phi(G^{\mathcal{F}})$ of $G^{\mathcal{F}}/\Phi(G^{\mathcal{F}})$ is not s-permutable in $G/\Phi(G^{\mathcal{F}})$.

If every minimal subgroup of $G^{\mathcal{F}}/\Phi(G^{\mathcal{F}})$ is s-permutable in $G/\Phi(G^{\mathcal{F}})$, then by [16, Lemma 2.11] we know some maximal subgroup of $G^{\mathcal{F}}/\Phi(G^{\mathcal{F}})$ is normal in $G/\Phi(G^{\mathcal{F}})$. Since $G^{\mathcal{F}}/\Phi(G^{\mathcal{F}})$ is a chief factor of G, $|G^{\mathcal{F}}/\Phi(G^{\mathcal{F}})| = q$ and so $G^{\mathcal{F}}$ is cyclic, contradicts with (2). Thus, there exists some minimal subgroup $X/\Phi(G^{\mathcal{F}})$ of $G^{\mathcal{F}}/\Phi(G^{\mathcal{F}})$ which is not s-permutable in $G/\Phi(G^{\mathcal{F}})$.

(4) The final contradiction.

Let $x \in X \backslash \Phi(G^{\mathcal{F}})$, then by (2) we know $\langle x \rangle$ is a cyclic group of prime order or order 4. Let T be any supplement of $\langle x \rangle$ in G, then $G = \langle x \rangle T$ and $G^{\mathcal{F}} = G^{\mathcal{F}} \cap \langle x \rangle T = \langle x \rangle (G^{\mathcal{F}} \cap T)$. Since $G^{\mathcal{F}}/\Phi(G^{\mathcal{F}})$ is abelian, $(G^{\mathcal{F}} \cap T)\Phi(G^{\mathcal{F}})/\Phi(G^{\mathcal{F}}) \trianglelefteq G/\Phi(G^{\mathcal{F}})$ and hence $(G^{\mathcal{F}} \cap T)\Phi(G^{\mathcal{F}}) \trianglelefteq G$. Thus $G^{\mathcal{F}} \cap T \leq \Phi(G^{\mathcal{F}})$ or $G^{\mathcal{F}} \cap T = G^{\mathcal{F}}$, as $G^{\mathcal{F}}/\Phi(G^{\mathcal{F}})$ is a chief factor of G. If $G^{\mathcal{F}} \cap T \leq \Phi(G^{\mathcal{F}})$ for some supplement T, then $G^{\mathcal{F}} = \langle x \rangle$ is cyclic, which contradicts with (2). Thus we have $G^{\mathcal{F}} \cap T = G^{\mathcal{F}}$, hence $T = G$ is the unique supplement of $\langle x \rangle$ in G. Since G is not supersoluble, by the hypothesis $\langle x \rangle$ is partially S-embedded in G. Then there exists a normal subgroup K of G contained in $G^{\mathcal{F}}$ such that $\langle x \rangle K$ is s-permutable in G and $\langle x \rangle \cap K \leq \langle x \rangle_{\overline{s}G}$. Since $G^{\mathcal{F}}/\Phi(G^{\mathcal{F}})$ is a chief factor of G, $K \leq \Phi(G^{\mathcal{F}})$ or $K = G^{\mathcal{F}}$. If $K \leq \Phi(G^{\mathcal{F}})$, then $X/\Phi(G^{\mathcal{F}}) = \langle x \rangle K\Phi(G^{\mathcal{F}})/\Phi(G^{\mathcal{F}})$ is s-permutable in $G/\Phi(G^{\mathcal{F}})$, this contradicts with (3). If $K = G^{\mathcal{F}}$, then $\langle x \rangle = \langle x \rangle \cap K = \langle x \rangle_{\overline{s}G}$ is an s-semipermutable subgroup of G. Since $\langle x \rangle \leq O_p(G)$ is subnormal in G, by Lemma 2.3 we know $\langle x \rangle$ is s-permutable in G. Thus we can conclude that $X/\Phi(G^{\mathcal{F}}) = \langle x \rangle \Phi(G^{\mathcal{F}})/\Phi(G^{\mathcal{F}})$ is s-permutable in $G/\Phi(G^{\mathcal{F}})$, which contradicts with (3). This contradiction completes the proof of the theorem. □

From Theorem 3.3, when we take $\mathcal{F} = \mathcal{U}$, we have the following result:

Corollary 3.4. *A group G is supersoluble if and only if there exists a normal subgroup E of G such that G/E is supersoluble, and every noncyclic Sylow subgroup of E with prime order or order 4 (if the Sylow 2-subgroup of E is non-abelian) not having a supersoluble supplement in G is partially S-embedded in G.*

Next, by assuming that some cyclic subgroups of prime order or order 4 having the partially S-embedded properties, we give out some new criteria about the nilpotency of a group G.

Theorem 3.5. *Let E be a normal subgroup of G such that G/E is nilpotent. If every minimal subgroup of E is contained in $Z_\infty(G)$ and every cyclic subgroup of E with order 4 is partially S-embedded in G or lies in $Z_\infty(G)$, then G is nilpotent.*

Proof. Suppose that the result is false and let G be a counterexample of minimal order. Then we have:

(1) Every proper subgroup of G is nilpotent.

Let K be an arbitrary proper subgroup of G. Since G/E is nilpotent, $K/K \cap E \cong KE/E$ is nilpotent. Let H be a minimal subgroup of $K \cap E$, then $H \leq Z_\infty(G) \cap K \leq Z_\infty(K)$. By hypotheses, any cyclic subgroup U of $K \cap E$ with order 4 is partially S-embedded in G or lies in $Z_\infty(G)$. By Lemma 2.4, U is partially S-embedded in K or lies in $Z_\infty(G) \cap K \leq Z_\infty(K)$. Thus $(K, K \cap E)$ satisfies the hypothesis of the theorem in any case. The minimal choice of G implies that K is nilpotent, thus G is a group which is not nilpotent but every proper subgroup of G is nilpotent. By [11, III, Theorem 5.2], we can deduce that $G = PQ$, where P is a normal Sylow p-subgroup and Q a non-normal cyclic Sylow q-subgroup of G; $P/\Phi(P)$ is a chief factor of G; $exp(P) = p$ or 4.

(2) $p = 2$, $exp(P) = 4$. Moreover, every cyclic subgroup of $P \leq E$ with order 4 is partially S-embedded in G.

Since both G/E and G/P are nilpotent, $G/P \cap E \lesssim G/P \times G/E$ is nilpotent. If $P \nleq E$, then $P \cap E < P$ and $(P \cap E)Q$ is a proper subgroup of G. By (1), we know $(P \cap E)Q$ is nilpotent. Thus, $(P \cap E)Q = (P \cap E) \times Q$ and Q *char* $(P \cap E)Q$. On the other hand, $G/P \cap E = P/P \cap E \times Q(P \cap E)/P \cap E$, it follows that $Q(P \cap E)/P \cap E \trianglelefteq G/P \cap E$ and $Q(P \cap E) \trianglelefteq G$. Therefore, $Q \trianglelefteq G$ and $G = P \times Q$, a contradiction. Thus we have $P \leq E$. Since $P \trianglelefteq G$, all elements of order p or 4 (if $p = 2$) of G are contained in P and so contained in E. If $p > 2$ or $p = 2$ and every cyclic subgroup of P with order 4 lies in $Z_\infty(G)$, then $P \leq Z_\infty(G)$ by the hypotheses. Therefore, Lemma 2.7 implies that $G = PQ = P \times Q$ is nilpotent, a contradiction. Hence, (2) holds.

(3) Every element $x \in P \backslash \Phi(P)$ is of order 4.

If there exists some $x \in P \backslash \Phi(P)$ such that $o(x) = 2$, denote $M = \langle x \rangle^G \leq P$. Then $M \leq Z_\infty(G)$ and $M\Phi(P)/\Phi(P) \trianglelefteq G/\Phi(P)$. Since $P/\Phi(P)$ is a chief factor of G and $M \nsubseteq \Phi(P)$, $P = M\Phi(P) = M \leq Z_\infty(G)$. Therefore, Lemma 2.7 implies that $G = PQ = P \times Q$ is nilpotent, a contradiction. Thus every element of P not contained in $\Phi(P)$ is of order 4. By (2), we know that it is partially S-embedded in G.

(4) Some minimal subgroup $X/\Phi(P)$ of $P/\Phi(P)$ is not s-permutable in $G/\Phi(P)$.

If every minimal subgroup of $P/\Phi(P)$ is s-permutable in $G/\Phi(P)$, then by [16, Lemma 2.11] we know some maximal subgroup of $P/\Phi(P)$ is normal in $G/\Phi(P)$. Since $P/\Phi(P)$ is a chief factor of G, $|P/\Phi(P)| = p$. Since $exp(P) = 4$, P is a cyclic group of order 4. Then [11, IV, Theorem 2.8] implies that G is 2-nilpotent and thus it is nilpotent, a contradiction. Thus some minimal subgroup $X/\Phi(P)$ of $P/\Phi(P)$ is not s-permutable in $G/\Phi(P)$.

(5) The final contradiction.

Let $x \in X \backslash \Phi(P)$, by (2) and (3) we know x is of order 4 and $\langle x \rangle$ is partially S-embedded in G. Thus there exists a normal subgroup K of G contained in P such that $\langle x \rangle K$ is s-permutable in G and $\langle x \rangle \cap K \leq \langle x \rangle_{\overline{s}G}$. Since $\langle x \rangle_{\overline{s}G} \leq P$ is subnormal in G, $\langle x \rangle_{\overline{s}G}$ is s-permutable in G. Since $P/\Phi(P)$ is a chief factor of G, $K \leq \Phi(P)$ or $K = P$. If $K \leq \Phi(P)$, then $X/\Phi(P) = \langle x \rangle K\Phi(P)/\Phi(P)$ is s-permutable in $G/\Phi(P)$, a contradiction. If $K = P$, then $\langle x \rangle = \langle x \rangle_{\overline{s}G}$ is s-permutable in G and so $X/\Phi(P) = \langle x \rangle\Phi(P)/\Phi(P)$ is s-permutable in $G/\Phi(P)$, a contradiction too. This final contradiction completes the proof of the theorem. □

Now, we can prove that:

Theorem 3.6. *Let $\mathcal{F}$ be a saturated formation containing the class of all nilpotent groups $\mathcal{N}$. Suppose that every cyclic subgroup of $G^{\mathcal{F}}$ of order 4 is partially S-embedded in G. Then $G \in \mathcal{F}$ if and only if every minimal subgroup of $G^{\mathcal{F}}$ lies in the $\mathcal{F}$-hypercenter $Z_{\mathcal{F}}(G)$ of G.*

Proof. The necessity is obvious, we need to prove only the sufficiency. Let $\langle x\rangle$ be a minimal subgroup of $G^{\mathcal{F}}$, then $\langle x\rangle \leq Z_{\mathcal{F}}(G) \cap G^{\mathcal{F}}$ which is contained in $Z(G^{\mathcal{F}})$ by [6, IV, 6.10]. From Lemma 2.4, we know that every cyclic subgroup of $G^{\mathcal{F}}$ with order 4 is partially S-embedded in $G^{\mathcal{F}}$. Theorem 3.5 implies that $G^{\mathcal{F}}$ is nilpotent and so it is soluble. If $G^{\mathcal{F}} \leq \Phi(G)$, then $G \in \mathcal{F}$. Thus we may assume that there exists a maximal subgroup M of G such that $G = MG^{\mathcal{F}} = MF(G)$. By [6, IV, 1.17], we know $M^{\mathcal{F}} \leq G^{\mathcal{F}}$. Hence every minimal subgroup of $M^{\mathcal{F}}$ is contained in $Z_{\mathcal{F}}(G) \cap M \leq Z_{\mathcal{F}}(M)$ and every cyclic subgroup of $M^{\mathcal{F}}$ of order 4 is partially S-embedded in M. Therefore, M satisfies the hypotheses of the theorem. Then $M \in \mathcal{F}$ by induction. From [1, Theorem 1 and Proposition 1], we know $G^{\mathcal{F}}$ is a p-group for some prime p; $G^{\mathcal{F}}/\Phi(G^{\mathcal{F}})$ is a minimal normal subgroup of $G/\Phi(G^{\mathcal{F}})$; $G^{\mathcal{F}}$ has exponent p if $p > 2$ and exponent at most 4 if $p = 2$.

If $exp(G^{\mathcal{F}}) = p$, then $G^{\mathcal{F}} = \Omega_1(G^{\mathcal{F}}) \leq Z_{\mathcal{F}}(G)$ by the hypotheses, this would imply that $G \in \mathcal{F}$. Thus we may assume that $p = 2$ and $exp(G^{\mathcal{F}}) = 4$. If there exists some $x \in G^{\mathcal{F}} \backslash \Phi(G^{\mathcal{F}})$ such that $o(x) = 2$, let $H = \langle x\rangle^G$. Then $H \trianglelefteq G$ and $H \leq \Omega_1(G^{\mathcal{F}}) \leq Z_{\mathcal{F}}(G)$. On the other hand, $G^{\mathcal{F}} = H\Phi(G^{\mathcal{F}}) = H$ as $G^{\mathcal{F}}/\Phi(G^{\mathcal{F}})$ is a minimal normal subgroup of $G/\Phi(G^{\mathcal{F}})$. In this case, $G \in \mathcal{F}$. Next we assume that every $x \in G^{\mathcal{F}} \backslash \Phi(G^{\mathcal{F}})$ is of order 4, then $\langle x\rangle$ is partially S-embedded in G. Let $X/\Phi(G^{\mathcal{F}})$ be an arbitrary minimal subgroup of $G^{\mathcal{F}}/\Phi(G^{\mathcal{F}})$ and $x \in X\backslash\Phi(G^{\mathcal{F}})$. Then there exists a normal subgroup K of G contained in $G^{\mathcal{F}}$ such that $\langle x\rangle K$ is s-permutable in G and $\langle x\rangle \cap K \leq \langle x\rangle_{\overline{s}G}$. Since $\langle x\rangle_{\overline{s}G} \leq G^{\mathcal{F}} \leq O_2(G)$ is subnormal in G, $\langle x\rangle_{\overline{s}G}$ is s-permutable in G by Lemma 2.3. Since $G^{\mathcal{F}}/\Phi(G^{\mathcal{F}})$ is a chief factor of G, $K \leq \Phi(G^{\mathcal{F}})$ or $K = G^{\mathcal{F}}$. If $K \leq \Phi(G^{\mathcal{F}})$, then $X/\Phi(P) = \langle x\rangle K\Phi(G^{\mathcal{F}})/\Phi(G^{\mathcal{F}})$ is s-permutable in $G/\Phi(G^{\mathcal{F}})$. If $K = G^{\mathcal{F}}$, then $\langle x\rangle = \langle x\rangle_{\overline{s}G}$ is s-permutable in G and we also deduce that $X/\Phi(G^{\mathcal{F}}) = \langle x\rangle\Phi(G^{\mathcal{F}})/\Phi(G^{\mathcal{F}})$ is s-permutable in $G/\Phi(G^{\mathcal{F}})$. This means that every minimal subgroup of $G^{\mathcal{F}}/\Phi(G^{\mathcal{F}})$ is s-permutable in $G/\Phi(G^{\mathcal{F}})$. Then by [16, Lemma 2.11] we know $G^{\mathcal{F}}/\Phi(G^{\mathcal{F}})$ has a maximal subgroup which is normal in $G/\Phi(G^{\mathcal{F}})$. Since $G^{\mathcal{F}}/\Phi(G^{\mathcal{F}})$ is a chief factor of G, $|G^{\mathcal{F}}/\Phi(G^{\mathcal{F}})| = 2$. Thus, we know $G^{\mathcal{F}}/\Phi(G^{\mathcal{F}}) \leq Z(G/\Phi(G^{\mathcal{F}}))$. Since $(G/\Phi(G^{\mathcal{F}}))/(G^{\mathcal{F}}/\Phi(G^{\mathcal{F}})) \cong G/G^{\mathcal{F}} \in \mathcal{F}$, Lemma 2.8 implies that $G/\Phi(G^{\mathcal{F}}) \in \mathcal{F}$. Since $\Phi(G^{\mathcal{F}}) \leq \Phi(G)$ and $\mathcal{F}$ is an saturated formation, $G \in \mathcal{F}$, as desired. □

Theorem 3.7. *A group G is nilpotent if and only if every minimal subgroup of $F^*(G^{\mathcal{N}})$ lies in $Z_\infty(G)$ and every cyclic subgroup of $F^*(G^{\mathcal{N}})$ with order 4 is partially S-embedded in G.*

Proof. The necessity is obvious, we need to prove only the sufficiency. Suppose that the result is false and let G be a counterexample of minimal order. Then

(1) Every proper normal subgroup of G is nilpotent.

Let M be a proper normal subgroup of G. Since $M/(M \cap G^{\mathcal{N}}) \cong MG^{\mathcal{N}}/G^{\mathcal{N}} \leq G/G^{\mathcal{N}}$ is nilpotent, $M^{\mathcal{N}} \trianglelefteq M \cap G^{\mathcal{N}} \trianglelefteq G^{\mathcal{N}}$. Then by Lemma 2.9, we know $F^*(M^{\mathcal{N}}) \leq F^*(M \cap G^{\mathcal{N}}) \leq F^*(G^{\mathcal{N}})$. Moreover,

$M \cap Z_\infty(G) \leq Z_\infty(M)$. Thus, M satisfies the hypotheses of the theorem. The minimal choice of G implies that M is nilpotent.

(2) $F(G)$ is the unique maximal normal subgroup of G.

Let M be a maximal normal subgroup of G, then M is nilpotent by (1). Since the class of all nilpotent groups formed a Fitting class, the nilpotency of M implies that $M = F(G)$ is the unique maximal normal subgroup of G.

(3) $G^{\mathcal{N}} = G = G'$ and $F^*(G) = F(G) < G$.

If $G^{\mathcal{N}} < G$, then $G^{\mathcal{N}}$ is nilpotent. Thus, by Lemma 2.9 we know $F^*(G^{\mathcal{N}}) = G^{\mathcal{N}}$. Now Theorem 3.5 implies immediately that G is nilpotent, a contradiction. Hence, we have $G^{\mathcal{N}} = G$. Since $G^{\mathcal{N}} \leq G'$, it follows that $G' = G$. Hence $G/F(G)$ cannot be cyclic of prime order. Thus $G/F(G)$ is a non-abelian simple group. If $F(G) < F^*(G)$, then $F^*(G^{\mathcal{N}}) = F^*(G) = G$ by (2). Again by Theorem 3.5, we can deduce that G is nilpotent, which is a contradiction.

(4) The final contradiction.

Since $F(G) = F^*(G) \neq 1$, we may choose the smallest prime divisor p of $|F(G)|$ such that $O_p(G) \neq 1$. For any Sylow q-subgroup Q of G ($q \neq p$), we consider the subgroup $G_0 = O_p(G)Q$. It is clear that $G_0^{\mathcal{N}} \leq O_p(G)$ and $G_0 \cap Z_\infty(G) \leq Z_\infty(G_0)$. Hence, every minimal subgroup of $G_0^{\mathcal{N}}$ lies in $Z_\infty(G_0)$ and every cyclic subgroup of $G_0^{\mathcal{N}}$ with order 4 is partially S-embedded in G_0. By Theorem 3.5, we know that G_0 is nilpotent. Hence, $G_0 = O_p(G) \times Q$ and $Q \leq C_G(O_p(G))$. Consequently, $G/C_G(O_p(G))$ is a p-group. Thus we have $C_G(O_p(G)) = G$ by (3), namely $O_p(G) \leq Z(G)$. Now we consider the factor group $\overline{G} = G/O_p(G)$. First we have $F^*(\overline{G}) = F^*(G)/O_p(G)$ by Lemma 2.9(4). For any element $\overline{x}$ of odd prime order in $F^*(\overline{G})$, since $O_p(G)$ is the Sylow p-subgroup of $F^*(G)$, $\overline{x}$ can be viewed as the image of an element x of odd prime order in $F^*(G)$. It follows that x lies in $Z_\infty(G)$ and $\overline{x}$ lies in $Z_\infty(\overline{G})$, for $Z_\infty(G/O_p(G)) = Z_\infty(G)/O_p(G)$. This shows that $\overline{G}$ satisfies the hypotheses of the theorem. Thus $\overline{G}$ is nilpotent by the minimal choice of G. Consequently, G is also nilpotent. This final contradiction completes the proof of the theorem. □

Theorem 3.8. *Let $\mathcal{F}$ be a saturated formation containing $\mathcal{N}$. Then a group $G \in \mathcal{F}$ if and only if every minimal subgroup of $F^*(G^{\mathcal{F}})$ lies in $Z_{\mathcal{F}}(G)$ and every cyclic subgroup of $F^*(G^{\mathcal{F}})$ with order 4 is partially S-embedded in G.*

Proof. Only the sufficiency needs to be verified. By [6, IV, 6.10], $G^{\mathcal{F}} \cap Z_{\mathcal{F}}(G) \leq Z(G^{\mathcal{F}}) \leq Z_\infty(G^{\mathcal{F}})$. Consequently, every minimal subgroup of $F^*(G^{\mathcal{F}})$ is contained in $Z_\infty(G^{\mathcal{F}})$. By the hypotheses and Lemma 2.4, every cyclic subgroup of $F^*(G^{\mathcal{F}})$ with order 4 is partially S-embedded in $G^{\mathcal{F}}$. By Theorem 3.7, we see that $G^{\mathcal{F}}$ is nilpotent and so $F^*(G^{\mathcal{F}}) = G^{\mathcal{F}}$. Now by Theorem 3.6, we can deduce that $G \in \mathcal{F}$, as required. □

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