www.theoryofgroups.ir

International Journal of Group Theory
ISSN (print): 2251-7650, ISSN (on-line): 2251-7669
Vol. 2 No. 1 (2013), pp. 175-185.

www.ui.ac.ir

THE PRIME GRAPH CONJECTURE FOR INTEGRAL GROUP RINGS OF SOME ALTERNATING GROUPS

MOHAMED A. SALIM

Communicated by Patrizia Longobardi

Abstract. We investigate the classical H. Zassenhaus conjecture for integral group rings of alternating groups A_9 and A_{10} of degree 9 and 10, respectively. As a consequence of our previous results we confirm the Prime Graph Conjecture for integral group rings of A_n for all $n \leq 10$.

1. Introduction and main results

Let G be a finite group and let $V(\mathbb{Z}G)$ denote the group of all normalized units of the integral group ring $\mathbb{Z}G$ of G. In [30], H. Zassenhaus proposed the following conjecture

(**ZC**): *Every torsion unit u in $V(\mathbb{Z}G)$ conjugates to some element g in G within the rational group algebra $\mathbb{Q}G$.*

Let $\pi(H)$ denote the Gruenberg-Kegel (prime) graph of a group H (not necessarily finite); i.e., the graph whose vertices are labeled by primes p for which there exists an element of order p in H and with an edge from p to a distinct prime q if and only if H has an element of order pq. In [25] (see also [23]), the following weaker version of (**ZC**) was proposed. We may call it the Prime Graph Conjecture:

$$(\mathbf{PGC}): \quad \pi(V(\mathbb{Z}G)) = \pi(G) \quad \textit{for any finite group } G.$$

The question about (**ZC**) remains open as no counterexample is known up to date. For nilpotent groups, (**ZC**) has been proved independently by K.W. Roggenkamp and L.L. Scott in [26] and by A. Weiss in [29]. But their method can not be applied to simple groups. However, using a new method based on the partial augmentation of a torsion unit, I.S. Luthar and I.B.S. Passi in [24] confirmed (**ZC**) for the alternating group A_5 of degree 5. Also, in [27, 28, 21] a positive answer for

MSC(2010): Primary: 20C05, 20C10; Secondary: 16U60.
Keywords: integral group ring, Zassenhaus Conjecture, prime graph conjecture, torsion unit.
Received: 22 March 2013, Accepted: 31 March 2013.

(PGC) was given for several new classes of groups, in particular for the alternating groups A_6, A_7 and A_8.

Recently, the (**PGC**) has been investigated in several papers. A positive answer has been given for solvable groups, Frobenius groups and almost for several simple groups in [23], [3] and [2, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 19], respectively. Also, for non simple groups, see [3, 4, 16].

Here we continue our study of (**ZC**) for alternating groups. Our main results are given in the following two theorems.

Theorem 1.1. *Let G denote the alternating group A_9. For a torsion unit u in $V(\mathbb{Z}G)$ of order $|u|$, denote the partial augmentation of u by*

$$P(u) = (\nu_{2a}, \nu_{2b}, \nu_{3a}, \nu_{3b}, \nu_{3c}, \nu_{4a}, \nu_{4b}, \nu_{5a}, \nu_{6a}, \nu_{6b}, \nu_{7a}, \nu_{9a}, \nu_{9b}, \nu_{10a}, \nu_{12a}, \nu_{15a}, \nu_{15b}) \in \mathbb{Z}^{17}.$$

The following hold:

(i) *There are no units of order* 14, 21 *and* 35 *in $V(\mathbb{Z}G)$.*

(ii) *If $|u| \in \{5, 7\}$, then u is rationally conjugate to some $g \in G$.*

(iii) *If $|u| = 2$, then the tuple of the partial augmentations of u belongs to the set*

$$\{\ P(u) \in \mathbb{Z}^{17} \mid (\nu_{2a}, \nu_{2b}) \in \{\ (0,1), (2,-1), (1,0), (-1,2)\ \},\quad \nu_{kx} = 0,\ kx \notin \{2a, 2b\}\ \}.$$

(iv) *If $|u| = 3$, then the tuple of the partial augmentations of u belongs to the set*

$$\{\ P(u) \in \mathbb{Z}^{17} \mid (\nu_{3a}, \nu_{3b}, \nu_{3c}) \in \{\ (0,-1,2), (1,-1,1), (1,0,0), (-1,0,2), (0,0,1), (0,2,-1), (0,-2,3), (0,1,0), (-1,1,1)\ \},\quad \nu_{kx} = 0,\ kx \notin \{3a, 3b, 3c\}\ \}.$$

(v) *If $|u| = 10$, then the tuple of the partial augmentations of u belongs to the set*

$$\{\ P(u) \in \mathbb{Z}^{17} \mid (\nu_{2a}, \nu_{2b}, \nu_{5a}, \nu_{10a}) \in \{\ (0,0,0,1), (1,1,0,-1)\ \},\quad \nu_{kx} = 0,\ kx \notin \{2a, 2b, 5a, 10a\}\ \}.$$

Theorem 1.2. *Let G denote the alternating group A_{10}. For a torsion unit u in $V(\mathbb{Z}G)$ of order $|u|$, denote the partial augmentation of the element u by*

$$P(u) = (\nu_{2a}, \nu_{2b}, \nu_{3a}, \nu_{3b}, \nu_{3c}, \nu_{4a}, \nu_{4b}, \nu_{4c}, \nu_{5a}, \nu_{5b}, \nu_{6a}, \nu_{6b}, \nu_{6c}, \nu_{7a}, \nu_{8a}, \nu_{9a}, \nu_{9b}, \nu_{10a}, \nu_{12a}, \nu_{12b}, \nu_{15a}, \nu_{21a}, \nu_{21b}) \in \mathbb{Z}^{23}.$$

The following hold:

(i) *There are no units of order* 14 *and* 35 *in $V(\mathbb{Z}G)$.*

(ii) *If $|u| \in \{5, 7\}$, then u is rationally conjugate to some $g \in G$.*

(iii) *If* $|u| = 2$, *then the tuple of the partial augmentations of* u *belongs to the set*

$$\{\ P(u) \in \mathbb{Z}^{23} \mid (\nu_{2a}, \nu_{2b}) \in \{\ (0,1), (-2,3), (2,-1), (1,0), (-1,2)\ \},\ \nu_{kx} = 0,\ kx \notin \{2a, 2b\}\ \}.$$

(iv) *If* $|u| = 3$, *then the tuple of the partial augmentations of* u *belongs to the set*

$$\{\ P(u) \in \mathbb{Z}^{23} \mid (\nu_{3a}, \nu_{3b}, \nu_{3c}) \in \{\ (0,-1,2), (0,3,-2), (1,0,0), (0,0,1), (0,2,-1), (-1,2,0), (1,1,-1), (0,1,0), (-1,1,1)\ \},\ \nu_{kx} = 0,\ kx \notin \{3a, 3b, 3c\}\ \}.$$

As an immediate consequence of the first parts of Theorems 1.1, 1.2 and [21, 27, 28] we obtain the solution of the Prime Graph Conjecture for A_n:

Corollary 1.3. *For all* $n \leq 10$, *if* $G = A_n$, *then* $\pi(G) = \pi(V(\mathbb{Z}G))$.

2. Preliminaries

Let G be a finite group and let $\mathcal{C} = \{C_1, C_{kx} \mid x \in \{a, b, \ldots\},\ k \geq 2\}$ be the collection of all conjugacy classes of G, where the first index denotes the order of the elements of this conjugacy class and $C_1 = \{1\}$. Supposing that the torsion unit $u = \sum \alpha_g g \in V(\mathbb{Z}G)$ has order k, denote the partial augmentation of u with respect to the conjugacy class C_{nt} by $\nu_{nt} = \varepsilon_{C_{nt}}(u) = \sum_{g \in C_{nt}} \alpha_g$. Denote the tuple of partial augmentations of the unit u by

$$P(u) = (\nu_{kx} \mid x \in \{a, b, \ldots\}, k \geq 2) \in \mathbb{Z}^l,$$

where $l + 1$ is the number of conjugacy classes of G.

From Higman-Berman's Theorem [1] one knows that $\nu_1 = \alpha_1 = 0$ and

$$\sum_{C_{nt} \in \mathcal{C}} \nu_{nt} = 1.$$

Hence, for any character χ of G, we get that $\chi(u) = \sum \nu_{nt} \chi(h_{nt})$, where h_{nt} is a representative of a conjugacy class C_{nt}. Throughout the paper the p-Brauer character table of the group G will be denoted by $\mathfrak{BCT}(p)$, which can be found using the computational algebra system GAP [18]. Clearly, if $G \in \{A_9, A_{10}\}$, then the prime number p has value $p \in \{2, 3, 5, 7\}$.

Through the proofs of the main results we use the following propositions from [17, 20, 22, 24].

Proposition 2.1. *(see* [24, 22]*) Let either* $p = 0$ *or* p *be a prime divisor of* $|G|$ *and let* F *be the associated prime field. Suppose that* $u \in V(\mathbb{Z}G)$ *has finite order* k *and assume* k *and* p *are coprime in case* $p \neq 0$. *If* z *is a primitive* k*-th root of unity and* χ *is either a classical character or a* p*-Brauer character of* G *then for every integer* l *the number*

$$\mu_l(u, \chi, p) = \frac{1}{k} \sum_{d \mid k} Tr_{F(z^d)/F}\{\chi(u^d) z^{-dl}\} \tag{2.1}$$

is a non-negative integer.

For $p = 0$ we will use the notation $\mu_l(u, \chi, *)$ for $\mu_l(u, \chi, 0)$.

Proposition 2.2. *(see [17]) The order of any unit $u \in V(\mathbb{Z}G)$ is a divisor of the exponent of G.*

Proposition 2.3. *(see [24]) Let u be a torsion unit of $V(\mathbb{Z}G)$. Let C be a conjugacy class of G. If $a \in C$ and p is a prime dividing the order of a but not the order of u then $\varepsilon_C(u) = 0$.*

M. Hertweck ([20], Proposition 3.1; [22], Lemma 5.6) obtained the next results. These already yield that several partial augmentations of the torsion units are zero.

Proposition 2.4. *Let G be a finite group and let u be a torsion unit in $V(\mathbb{Z}G)$. If x is an element of G whose p-part, for some prime p, has order strictly greater than the order of the p-part of u, then $\varepsilon_x(u) = 0$.*

Proposition 2.5. *(see [24]) Let $u \in V(\mathbb{Z}G)$ be of order k. Then u is conjugate in $\mathbb{Q}G$ to an element $g \in G$ if and only if for each d dividing k there is precisely one conjugacy class C with partial augmentation $\varepsilon_C(u^d) \neq 0$.*

3. **Proof of the Theorems**

Proof of Theorem 1.1. Let $G = A_9$. It is well known that $|G| = 181440 = 2^6 \cdot 3^4 \cdot 5 \cdot 7$ and $exp(G) = 1260 = 2^2 \cdot 3^2 \cdot 5 \cdot 7$. The p-Brauer character tables are available for primes $p \in \{2, 3, 5, 7\}$. The group G possesses elements of orders 2, 3, 4, 5, 6, 7, 9, 10, 12 and 15. First we investigate units of orders $2, 3, 5, 7$ and 10. Secondly, according to Proposition 2.2, the order of each torsion unit divides the exponent of G, so the possible orders for units are: 14, 18, 20, 24, 30, 35, 45 and 63. We prove that units of orders 14, 21 and 35 do not appear in $V(\mathbb{Z}G)$.

• Let u be an involution. Then we have $\nu_{2a} + \nu_{2b} = 1$ by Propositions 2.3 and 2.4. According to (2.1) we get the following system of three inequalities:

$$\mu_0(u, \chi_2, *) = \tfrac{1}{2}(4\nu_{2a} + 8) \geq 0; \qquad \mu_1(u, \chi_2, *) = \tfrac{1}{2}(-4\nu_{2a} + 8) \geq 0;$$
$$\mu_0(u, \chi_2, 3) = \tfrac{1}{2}(3\nu_{2a} - \nu_{2b} + 7) \geq 0,$$

which has the four integral solutions listed in part (iii) of the Theorem.

• Let u be a unit of order 3. Then $\nu_{3a} + \nu_{3b} + \nu_{3c} = 1$ by Propositions 2.3 and 2.4. Put $t_1 = 5\nu_{3a} - \nu_{3b} + 2\nu_{3c}$ and $t_2 = 4\nu_{3a} + \nu_{3b} - 2\nu_{3c}$. Then by (2.1) we have that

$$\mu_0(u, \chi_2, *) = \tfrac{1}{3}(2t_1 + 8) \geq 0; \qquad \mu_1(u, \chi_2, *) = \tfrac{1}{3}(-t_1 + 8) \geq 0;$$
$$\mu_0(u, \chi_5, *) = \tfrac{1}{3}(18\nu_{3a} + 27) \geq 0; \quad \mu_1(u, \chi_5, *) = \tfrac{1}{3}(-9\nu_{3a} + 27) \geq 0;$$
$$\mu_0(u, \chi_3, 2) = \tfrac{1}{3}(-2t_2 + 8) \geq 0; \qquad \mu_1(u, \chi_3, 2) = \tfrac{1}{3}(t_2 + 8) \geq 0.$$

From the first two inequalities we get that $t_1 \in \{-4, -1, 2, 5, 8\}$ and from the next two we get that $\nu_{3a} \in \{-1, 0, 1, 2, 3\}$. Considering the last two inequalities, we obtain the 6 non-trivial and 3 trivial integral solutions listed in part (iv) of the Theorem.

• Let u be a unit of order either 5 or 7. Then by Propositions 2.3 and 2.4, the only nonzero partial augmentation is $\nu_{5a} = 1$ or $\nu_{7a} = 1$, respectively. According to Proposition 2.5, such unit u is rationally conjugate to an element $g \in G$.

• Let u be a unit of order 10. Then $\nu_{2a} + \nu_{2b} + \nu_{5a} + \nu_{10a} = 1$ by Propositions 2.3 and 2.4. Put $t_1 = 4\nu_{2a} + 3\nu_{5a} - \nu_{10a}$, $t_2 = \nu_{2a} - 3\nu_{2b} + \nu_{5a} + \nu_{10a}$ and $t_3 = 7\nu_{2a} + 3\nu_{2b} + 2\nu_{5a} + 2\nu_{10a}$. Since u^5 is an involution, we need to consider the following four cases:

$$\chi(u^5) \in \{\chi(2a), \quad \chi(2b), \quad 2\chi(2a) - \chi(2b), \quad -\chi(2a) + 2\chi(2b)\}.$$

Consider each case separately:

Case 1. Let $\chi(u^5) = \chi(2a)$ and $\chi(u^2) = \chi(5a)$. By (2.1) we obtain the following system of inequalities:

$$\mu_1(u, \chi_2, *) = \tfrac{1}{10}(t_1 + 1) \geq 0; \qquad \mu_5(u, \chi_2, *) = \tfrac{1}{10}(-4t_1 + 16) \geq 0;$$
$$\mu_0(u, \chi_3, *) = \tfrac{1}{10}(4t_2 + 26) \geq 0; \quad \mu_5(u, \chi_3, *) = \tfrac{1}{10}(-4t_2 + 24) \geq 0;$$
$$\mu_1(u, \chi_3, *) = \tfrac{1}{10}(t_2 + 19) \geq 0; \quad \mu_1(u, \chi_5, *) = \tfrac{1}{10}(t_3 + 18) \geq 0;$$
$$\mu_0(u, \chi_5, *) = \tfrac{1}{10}(4t_3 + 42) \geq 0; \quad \mu_5(u, \chi_5, *) = \tfrac{1}{10}(-4t_3 + 28) \geq 0.$$

It is easy to check that $t_1 = -1$, $t_2 = 1$ and $t_3 \in \{-8, 2\}$, which has the following integral solution: $(0, 0, 0, 1)$.

Case 2. Let $\chi(u^5) = \chi(2b)$ and $\chi(u^2) = \chi(5a)$. By (2.1) we have

$$\mu_0(u, \chi_2, *) = \tfrac{1}{10}(4t_1 + 20) \geq 0; \quad \mu_2(u, \chi_2, *) = \tfrac{1}{10}(-4t_1 + 5) \geq 0;$$
$$\mu_0(u, \chi_3, *) = \tfrac{1}{10}(4t_2 + 22) \geq 0; \quad \mu_5(u, \chi_3, *) = \tfrac{1}{10}(-4t_2 + 28) \geq 0;$$
$$\mu_1(u, \chi_3, *) = \tfrac{1}{10}(t_2 + 23) \geq 0; \qquad \mu_1(u, \chi_5, *) = \tfrac{1}{10}(t_3 + 22) \geq 0;$$
$$\mu_0(u, \chi_5, *) = \tfrac{1}{10}(4t_3 + 38) \geq 0; \quad \mu_5(u, \chi_5, *) = \tfrac{1}{10}(-4t_3 + 32) \geq 0;$$
$$\mu_0(u, \chi_2, 3) = \tfrac{1}{10}(12\nu_{2a} - 4\nu_{2b} + 8\nu_{5a} - 8\nu_{10a} + 14) \geq 0.$$

It follows that $t_1 \in \{-5, 5\}$, $t_2 \in \{-3, 7\}$ and $t_3 \in \{-2, 8\}$, which has the following integral solution: $(1, 1, 0, -1)$.

Case 3. Let $\chi(u^5) = 2\chi(2a) - \chi(2b)$ and $\chi(u^2) = \chi(5a)$. By (2.1) we have

$$\mu_1(u, \chi_2, *) = \tfrac{1}{10}(t_1 - 3) \geq 0; \qquad \mu_5(u, \chi_2, *) = \tfrac{1}{10}(-4t_1 + 12) \geq 0;$$
$$\mu_0(u, \chi_3, *) = \tfrac{1}{10}(4t_2 + 30) \geq 0; \quad \mu_5(u, \chi_3, *) = \tfrac{1}{10}(-4t_2 + 20) \geq 0;$$
$$\mu_1(u, \chi_3, *) = \tfrac{1}{10}(t_2 + 15) \geq 0; \qquad \mu_1(u, \chi_5, *) = \tfrac{1}{10}(t_3 + 14) \geq 0;$$
$$\mu_0(u, \chi_5, *) = \tfrac{1}{10}(4t_3 + 46) \geq 0; \quad \mu_5(u, \chi_5, *) = \tfrac{1}{10}(-4t_3 + 24) \geq 0;$$
$$\mu_0(u, \chi_7, *) = \tfrac{1}{10}(-20\nu_{2a} + 12\nu_{2b} + 22) \geq 0.$$

It follows that $t_1 = 3$, $t_2 \in \{-5, 5\}$ and $t_3 \in \{-4, 6\}$, which has no integral solution.

Case 4. Let $\chi(u^5) = -\chi(2a) + 2\chi(2b)$ and $\chi(u^2) = \chi(5a)$. Using (2.1) we obtain the following system of inequalities:

$$\mu_0(u, \chi_2, *) = \tfrac{1}{10}(4t_1 + 16) \geq 0; \quad \mu_2(u, \chi_2, *) = \tfrac{1}{10}(-t_1 + 1) \geq 0;$$
$$\mu_0(u, \chi_3, *) = \tfrac{1}{10}(4t_2 + 18) \geq 0; \quad \mu_5(u, \chi_3, *) = \tfrac{1}{10}(-4t_2 + 32) \geq 0;$$
$$\mu_1(u, \chi_3, *) = \tfrac{1}{10}(t_2 + 27) \geq 0; \quad \mu_1(u, \chi_5, *) = \tfrac{1}{10}(t_3 + 26) \geq 0;$$
$$\mu_0(u, \chi_5, *) = \tfrac{1}{10}(4t_3 + 34) \geq 0; \quad \mu_5(u, \chi_5, *) = \tfrac{1}{10}(-4t_3 + 36) \geq 0.$$

It follows that $t_1 = 1$, $t_2 = 3$ and $t_3 \in \{-6, 4\}$, which has no integral solution.

• Let u be a unit of order 14. Then we have $\nu_{2a} + \nu_{2b} + \nu_{7a} = 1$ by Propositions 2.3 and 2.4. Put $t_1 = 4\nu_{2a} + \nu_{7a}$ and $t_2 = \nu_{2a} - 3\nu_{2b}$. Since $\chi(u^7)$ has order 2, according to the previous cases we need to consider the following four cases:

$$\chi(u^7) \in \{\chi(2a), \quad \chi(2b), \quad 2\chi(2a) - \chi(2b), \quad -\chi(2a) + 2\chi(2b)\}.$$

Consider each case separately:

Case 1. Let $\chi(u^7) = \chi(2a)$ and $\chi(u^2) = \chi(7a)$. Then by (2.1) we have that

$$\mu_0(u, \chi_2, *) = \tfrac{1}{14}(6t_1 + 18) \geq 0; \quad \mu_7(u, \chi_2, *) = \tfrac{1}{14}(-6t_1 + 10) \geq 0;$$
$$\mu_0(u, \chi_3, *) = \tfrac{1}{14}(6t_2 + 22) \geq 0; \quad \mu_7(u, \chi_3, *) = \tfrac{1}{14}(-6t_2 + 20) \geq 0;$$
$$\mu_1(u, \chi_3, *) = \tfrac{1}{14}(t_2 + 20) \geq 0,$$

which has no integral solutions.

Case 2. Let $\chi(u^7) = \chi(2b)$ and $\chi(u^2) = \chi(7a)$. Then by (2.1) we have that

$$\mu_0(u, \chi_2, *) = \tfrac{1}{14}(6t_1 + 14) \geq 0; \quad \mu_7(u, \chi_2, *) = \tfrac{1}{14}(-6t_1 + 14) \geq 0;$$
$$\mu_1(u, \chi_2, *) = \tfrac{1}{14}(t_1 + 7) \geq 0;$$
$$\mu_0(u, \chi_3, *) = \tfrac{1}{14}(6t_2 + 26) \geq 0; \quad \mu_7(u, \chi_3, *) = \tfrac{1}{14}(-6t_2 + 16) \geq 0,$$

which has no integral solutions.

Case 3. Let $\chi(u^7) = 2\chi(2a) - \chi(2b)$ and $\chi(u^2) = \chi(7a)$. Then by (2.1) we obtain the following system of inequalities

$$\mu_1(u, \chi_2, *) = \tfrac{1}{14}(t_1 - 1) \geq 0; \quad \mu_7(u, \chi_2, *) = \tfrac{1}{14}(-6t_1 + 6) \geq 0;$$
$$\mu_0(u, \chi_3, *) = \tfrac{1}{14}(6t_2 + 26) \geq 0; \quad \mu_7(u, \chi_3, *) = \tfrac{1}{14}(-6t_2 + 16) \geq 0.$$

It follows that $t_1 = 1$ and $t_2 = -2$ which has no integral solutions.

Case 4. Let $\chi(u^7) = -\chi(2a) + 2\chi(2b)$ and $\chi(u^2) = \chi(7a)$. Then by (2.1) we have that

$$\mu_0(u, \chi_2, *) = \tfrac{1}{14}(6t_1 + 10) \geq 0; \quad \mu_2(u, \chi_2, *) = \tfrac{1}{14}(-t_1 + 3) \geq 0;$$
$$\mu_0(u, \chi_3, *) = \tfrac{1}{14}(6t_2 + 14) \geq 0; \quad \mu_7(u, \chi_3, *) = \tfrac{1}{14}(-6tr_2 + 28) \geq 0.$$

It follows that $t_1 = 3$ and $t_2 = 0$ which has no integral solutions.

• Let u be a unit of order 21. Then $\nu_{3a} + \nu_{3b} + \nu_{3c} + \nu_{7a} = 1$ by Propositions 2.3 and 2.4. Put $t_1 = 5\nu_{3a} - \nu_{3b} + 2\nu_{3c} + \nu_{7a}$ and $t_2 = \nu_{3a} - \nu_{3b}$. Since $\chi(u^7)$ has order 3, according to previous cases we need to consider the following nine cases:

$$\chi(u^7) \in \{\chi(3a), \quad \chi(3b), \quad \chi(3c), \quad -2\chi(3b) + 3\chi(3c), \quad \chi(3a) - \chi(3b) + \chi(3c), \quad -\chi(3a) + \chi(3b) + \chi(3c) \quad -\chi(3a) + 2\chi(3c), \quad 2\chi(3b) - \chi(3c), \quad -\chi(3b) + 2\chi(3c)\}.$$

Consider each case separately:

Case 1. Let $\chi(u^7) = \chi(3a)$ and $\chi(u^3) = \chi(7a)$. Then by (2.1) we have that:

$$\mu_0(u, \chi_2, *) = \tfrac{1}{21}(12t_1 + 24) \geq 0; \quad \mu_7(u, \chi_2, *) = \tfrac{1}{21}(-6t_1 + 9) \geq 0;$$
$$\mu_7(u, \chi_3, *) = \tfrac{1}{21}(18t_2 + 24) \geq 0; \quad \mu_0(u, \chi_3, *) = \tfrac{1}{21}(-36t_2 + 15) \geq 0,$$

which has no integral solutions.

Case 2. Let $\chi(u^7) = \chi(3b)$ and $\chi(u^3) = \chi(7a)$. Then by (2.1) we have the following system of two inequalities:

$$\mu_0(u, \chi_2, *) = \tfrac{1}{21}(12t_1 + 12) \geq 0; \quad \mu_3(u, \chi_2, *) = \tfrac{1}{21}(-2t_1 + 5) \geq 0.$$

Clearly, it has no integral solution.

Case 3. Let $\chi(u^7) = \chi(3c)$ and $\chi(u^3) = \chi(7a)$. Then by (2.1) we have that:

$$\mu_0(u, \chi_2, *) = \tfrac{1}{21}(12t_1 + 18) \geq 0; \quad \mu_7(u, \chi_2, *) = \tfrac{1}{21}(-6t_1 + 12) \geq 0;$$
$$\mu_1(u, \chi_2, *) = \tfrac{1}{21}(t_1 + 5) \geq 0.$$

It has no integral solution.

Case 4. Let $\chi(u^7) = -2\chi(3b) + 3\chi(3c)$ and $\chi(u^3) = \chi(7a)$. According to (2.1) we are able to construct the following system of inequalities:

$$\mu_1(u, \chi_2, *) = \tfrac{1}{21}(t_1 - 1) \geq 0; \quad \mu_7(u, \chi_2, *) = \tfrac{1}{21}(-6t_1 + 6) \geq 0;$$
$$\mu_3(u, \chi_3, *) = \tfrac{1}{21}(6t_2 + 9) \geq 0; \quad \mu_0(u, \chi_3, *) = \tfrac{1}{21}(-36t_2 + 9) \geq 0,$$

which has no integral solutions.

Case 5. Let $\chi(u^7) = \chi(3a) - \chi(3b) + \chi(3c)$ and $\chi(u^3) = \chi(7a)$. Then by (2.1) we have the system:

$$\mu_1(u, \chi_2, *) = \tfrac{1}{21}(t_1 - 1) \geq 0; \quad \mu_7(u, \chi_2, *) = \tfrac{1}{21}(-6t_1 + 6) \geq 0;$$
$$\mu_3(u, \chi_3, *) = \tfrac{1}{21}(6t_2 + 9) \geq 0; \quad \mu_0(u, \chi_3, *) = \tfrac{1}{21}(-36t_2 + 9) \geq 0,$$

which has no integral solutions.

Case 6. Let $\chi(u^7) = -\chi(3b) + 2\chi(3c)$ and $\chi(u^3) = \chi(7a)$. Then by (2.1) we have the system of four inequalities:

$$\mu_0(u, \chi_2, *) = \tfrac{1}{21}(12t_1 + 24) \geq 0; \quad \mu_7(u, \chi_2, *) = \tfrac{1}{21}(-6t_1 + 9) \geq 0;$$
$$\mu_7(u, \chi_3, *) = \tfrac{1}{21}(18t_2 + 24) \geq 0; \quad \mu_0(u, \chi_3, *) = \tfrac{1}{21}(-36t_2 + 15) \geq 0,$$

which has no integral solutions.

Case 7. Let $\chi(u^7) = -\chi(3a)+2\chi(3c)$ and $\chi(u^3) = \chi(7a)$. Then by (2.1) we get the following system of inequalities:

$$\mu_0(u,\chi_2,*) = \tfrac{1}{21}(5t_1+12) \geq 0; \quad \mu_3(u,\chi_2,*) = \tfrac{1}{21}(-12t_1+5) \geq 0,$$

which has no integral solutions.

Case 8. Let $\chi(u^7) = -\chi(3a)+\chi(3b)+\chi(3c)$ and $\chi(u^3) = \chi(7a)$. According to (2.1) we are able to construct the following system of inequalities:

$$\mu_0(u,\chi_2,*) = \tfrac{1}{21}(12t_1+6) \geq 0; \quad \mu_3(u,\chi_2,*) = \tfrac{1}{21}(-2t_1-1) \geq 0,$$

which has no integral solutions.

Case 9. Let $\chi(u^7) = 2\chi(3b)-\chi(3c)$ and $\chi(u^3) = \chi(7a)$. Then by (2.1) we obtain the following unsolvable system of inequalities:

$$\mu_0(u,\chi_2,*) = \tfrac{1}{21}(t_1+6) \geq 0; \quad \mu_3(u,\chi_2,*) = \tfrac{1}{21}(-t_1-1) \geq 0.$$

• Let u be a unit of order 35. Then $\nu_{5a}+\nu_{7a} = 1$ by Propositions 2.3 and 2.4. Clearly $\chi(u^7) = \chi(5a)$ and $\chi(u^5) = \chi(7a)$. Put $t_1 = 3\nu_{5a}+\nu_{7a}$. Then by (2.1) we have the system of inequalities:

$$\mu_0(u,\chi_2,*) = \tfrac{1}{35}(24t_1+26) \geq 0; \quad \mu_7(u,\chi_2,*) = \tfrac{1}{35}(-6t_1+11) \geq 0,$$

which has no integral solution. □

Proof of Theorem 1.2. Let $G = A_{10}$. It is well known that $|G| = 1814400 = 2^7\cdot 3^4\cdot 5^2\cdot 7$ and $exp(G) = 2520 = 2^3\cdot 3^2\cdot 5\cdot 7$. The group G possesses elements of orders 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 15 and 21. First we investigate units of orders $2,3,5$ and 7. Secondly, according to Proposition 2.2, the order of each torsion unit divides the exponent of G, so the possible orders for units are: 14, 18, 20, 24, 30, 35, 45 and 63. We prove that units of orders 14 and 35 do not appear in $V(\mathbb{Z}G)$.

Now we consider each case separately.

• Let u be a unit of order 2. Then $\nu_{2a}+\nu_{2b} = 1$ by Propositions 2.3 and 2.4. Put $t = 5\nu_{2a}+\nu_{2b}$. By (2.1) we obtain that

$$\mu_0(u,\chi_2,*) = \tfrac{1}{2}(t+9) \geq 0; \qquad \mu_1(u,\chi_2,*) = \tfrac{1}{2}(-t+9) \geq 0,$$

which has the 5 integral solutions listed in part (iii) of the Theorem.

• Let u be a unit of order 3. Then we have $\nu_{3a}+\nu_{3b}+\nu_{3c} = 1$ by Propositions 2.3 and 2.4. Put $t_1 = 2\nu_{3a}+\nu_{3b}$, $t_2 = 14\nu_{3a}+2\nu_{3b}-\nu_{3c}$ and $t_3 = 8\nu_{3a}-4\nu_{3b}+2\nu_{3c}$. According to (2.1) we obtain the system of inequalities:

$$\begin{aligned}
\mu_0(u,\chi_2,*) &= \tfrac{1}{3}(6t_1+9) \geq 0; & \mu_1(u,\chi_2,*) &= \tfrac{1}{3}(-3t_1+9) \geq 0;\\
\mu_0(u,\chi_3,*) &= \tfrac{1}{3}(2t_2+35) \geq 0; & \mu_1(u,\chi_3,*) &= \tfrac{1}{3}(-t_2+35) \geq 0;\\
\mu_0(u,\chi_3,2) &= \tfrac{1}{3}(-2t_3+16) \geq 0; & \mu_1(u,\chi_3,2) &= \tfrac{1}{3}(t_3+16) \geq 0.
\end{aligned}$$

It is easy to see that this system has 6 non-trivial and 3 trivial integral solutions, which are listed in Theorem 1.2(iv).

• Let u be a unit of order 5. Then $\nu_{5a} + \nu_{5b} = 1$ by Propositions 2.3 and 2.4. The system of two inequalities constructed by (2.1)

$$\mu_0(u, \chi_2, *) = \tfrac{1}{5}(16\nu_{5a} - 4\nu_{5b} + 9) \geq 0;$$
$$\mu_0(u, \chi_3, 2) = \tfrac{1}{5}(-16\nu_{5a} + 4\nu_{5b} + 16) \geq 0,$$

has only two trivial integral solutions: $(\nu_{5a}, \nu_{5b}) \in \{(0,1),(1,0)\}$.

• Let u be a unit of order 7. Then by Propositions 2.3 and 2.4, $\nu_{7a} = 1$ and $\nu_{kx} = 0$ for $kx \neq 7a$. According to Propositions 2.5, the unit u is rationally conjugate to an element $g \in G$.

• Let u be a unit of order 14. Then $\nu_{2a} + \nu_{2b} + \nu_{7a} = 1$ by Propositions 2.3 and 2.4. Put $t_1 = 5\nu_{2a} + \nu_{2b} + 2\nu_{7a}$ and $t_2 = 11\nu_{2a} + 3\nu_{2b}$. Since

$$\chi(u^7) \in \{\chi(2a), \quad \chi(2b), \quad 2\chi(2a) - \chi(2b), \quad -\chi(2a) + 2\chi(2b), \quad -2\chi(2a) + 3\chi(2b)\}$$

we need to consider the following five cases:

Case 1. Let $\chi(u^7) = \chi(2a)$ and $\chi(u^2) = \chi(7a)$. By (2.1) we obtain that

$$\mu_1(u, \chi_2, *) = \tfrac{1}{14}(t_1 + 2) \geq 0; \qquad \mu_7(u, \chi_2, *) = \tfrac{1}{14}(-6t_1 + 16) \geq 0;$$
$$\mu_0(u, \chi_3, *) = \tfrac{1}{14}(6t_2 + 46) \geq 0; \quad \mu_1(u, \chi_3, *) = \tfrac{1}{14}(t_2 + 24) \geq 0;$$
$$\mu_7(u, \chi_3, *) = \tfrac{1}{14}(-6t_2 + 24) \geq 0.$$

It follows that $t_1 = 5$ and $t_2 = 4$, which has no integral solution.

Case 2. Let $\chi(u^7) = \chi(2b)$ and $\chi(u^2) = \chi(7a)$. According to (2.1)

$$\mu_0(u, \chi_2, *) = \tfrac{1}{14}(6t_1 + 22) \geq 0; \qquad \mu_7(u, \chi_2, *) = \tfrac{1}{14}(-6t_1 + 20) \geq 0;$$
$$\mu_1(u, \chi_2, *) = \tfrac{1}{14}(t_1 + 6) \geq 0.$$

From the first two equations we get $t_1 = 1$, which contradicts the third one. So this system has no integral solutions.

Case 3. Let $\chi(u^7) = 2\chi(2a) - \chi(2b)$ and $\chi(u^2) = \chi(7a)$. By (2.1) we have

$$\mu_1(u, \chi_2, *) = \tfrac{1}{14}(t_1 - 2) \geq 0; \qquad \mu_7(u, \chi_2, *) = \tfrac{1}{14}(-6t_1 + 12) \geq 0;$$
$$\mu_0(u, \chi_3, *) = \tfrac{1}{14}(6t_2 + 54) \geq 0; \quad \mu_1(u, \chi_3, *) = \tfrac{1}{14}(t_2 + 16) \geq 0;$$
$$\mu_7(u, \chi_3, *) = \tfrac{1}{14}(-6t_2 + 16) \geq 0.$$

It follows that $t_1 = 2$ and $t_2 = 2$, which has no integral solution.

Case 4. Let $\chi(u^7) = -\chi(2a) + 2\chi(2b)$ and $\chi(u^2) = \chi(7a)$. By (2.1)

$$\mu_0(u, \chi_2, *) = \tfrac{1}{14}(6t_1 + 18) \geq 0; \quad \mu_2(u, \chi_2, *) = \tfrac{1}{14}(-t_1 + 4) \geq 0;$$
$$\mu_0(u, \chi_3, *) = \tfrac{1}{14}(6t_2 + 30) \geq 0; \quad \mu_1(u, \chi_3, *) = \tfrac{1}{14}(t_2 + 40) \geq 0;$$
$$\mu_7(u, \chi_3, *) = \tfrac{1}{14}(-6t_2 + 40) \geq 0.$$

Clearly, $t_1 = 4$ and $t_2 = 2$. It is easy to check that such system of equations has no integral solution.

Case 5. Let $\chi(u^7) = -2\chi(2a) + 3\chi(2b)$ and $\chi(u^2) = \chi(7a)$. By (2.1) we have the following system of inequalities:

$$\mu_0(u, \chi_2, *) = \tfrac{1}{14}(6t_1 + 14) \geq 0; \quad \mu_2(u, \chi_2, *) = \tfrac{1}{14}(-t_1) \geq 0;$$
$$\mu_0(u, \chi_3, *) = \tfrac{1}{14}(6t_2 + 22) \geq 0; \quad \mu_1(u, \chi_3, *) = \tfrac{1}{14}(t_2 + 48) \geq 0;$$
$$\mu_7(u, \chi_3, *) = \tfrac{1}{14}(-6t_2 + 48) \geq 0.$$

It follows that $t_1 = 0$ and $t_2 = 8$, which has no integral solutions.

- Let u be of order 35. Then by Propositions 2.3 and 2.4 we get that

$$\nu_{5a} + \nu_{5b} + \nu_{7a} = 1.$$

Put $t_1 = 4\nu_{5a} - \nu_{5b} + 2\nu_{7a}$. We consider the following two cases:

Case 1. Let $\chi(u^7) = \chi(5a)$ and $\chi(u^5) = \chi(7a)$. Using (2.1) we obtain that

$$\mu_0(u, \chi_2, *) = \tfrac{1}{35}(24t_1 + 37) \geq 0; \qquad \mu_7(u, \chi_2, *) = \tfrac{1}{35}(-6t_1 + 17) \geq 0,$$

which has no integral solutions.

Case 2. Let $\chi(u^7) = \chi(5b)$ and $\chi(u^5) = \chi(7a)$. By (2.1) we construct the system of two inequalities:

$$\mu_0(u, \chi_2, *) = \tfrac{1}{35}(24t_1 + 17) \geq 0; \qquad \mu_5(u, \chi_2, *) = \tfrac{1}{35}(-4t_1 + 3) \geq 0,$$

which has no integral solutions. □

Acknowledgments

This paper was supported by PPDNF at UAEU.

References

[1] V. A. Artamonov and A. A. Bovdi, Integral group rings: groups of invertible elements and classical K-theory, *Algebra. Topology. Geometry, Vol. 27 (Russian)*, Itogi Nauki i Tekhniki, 3–43, 232, Akad. Nauk SSSR Vsesoyuz. Inst. Nauchn. i Tekhn. Inform., Moscow, 1989. Translated in J. Soviet Math., **57** no. 2 (1991) 2931–2958.

[2] V. Bovdi, A. Grishkov and A. Konovalov, Kimmerle conjecture for the Held and O'Nan sporadic simple groups, *Sci. Math. Jpn.*, **69** no. 3 (2009) 353–361.

[3] V. Bovdi and M. Hertweck, Zassenhaus conjecture for central extensions of S_5, *J. Group Theory*, **11** no. 1 (2008) 63–74.

[4] V. Bovdi, C. Höfert and W. Kimmerle, On the first Zassenhaus conjecture for integral group rings, *Publ. Math. Debrecen*, **65** no. 3-4 (2004) 291–303.

[5] V. Bovdi and A. Konovalov, Integral group ring of the first Mathieu simple group, In *Groups St. Andrews 2005. Vol. 1*, volume 339 of *London Math. Soc. Lecture Note Ser.*, pages 237–245. Cambridge Univ. Press, Cambridge, 2007.

[6] V. Bovdi and A. Konovalov, Integral group ring of the Mathieu simple group M_{23}, *Comm. Algebra*, **36** no. 7 (2008) 2670–2680.

[7] V. Bovdi and A. Konovalov, Integral group ring of Rudvalis simple group, *Ukraïn. Mat. Zh.*, **61** no. 1 (2009) 3–13.

[8] V. Bovdi and A. Konovalov, Integral group ring of the Mathieu simple group M_{24}, *J. Algebra Appl.*, 1250016, **11** no. 1 (2012) 10 pp.

[9] V. Bovdi and A. Konovalov, Torsion units in integral group ring of Higman-Sims simple group, *Studia Sci. Math. Hungar.*, **47** no. 1 (2010) 1–11.

[10] V. Bovdi, A. Konovalov and S. Linton, Torsion units in integral group ring of the Mathieu simple group M_{22}, *LMS J. Comput. Math.*, **11** (2008) 28–39.

[11] V. Bovdi, A. Konovalov and E. Marcos, Integral group ring of the Suzuki sporadic simple group, *Publ. Math. Debrecen*, **72** no. 3-4 (2008) 487–503.

[12] V. Bovdi, A. Konovalov, and S. Siciliano, Integral group ring of the Mathieu simple group M_{12}, *Rend. Circ. Mat. Palermo (2)*, **56** no. 1 (2007) 125–136.

[13] V. A. Bovdi, E. Jespers and A. B. Konovalov, Torsion units in integral group rings of Janko simple groups, *Math. Comp.*, **80** no. 273 (2011) 593–615.

[14] V. A. Bovdi and A. B. Konovalov, Integral group ring of the McLaughlin simple group, *Algebra Discrete Math.*, **2** (2007) 43–53.

[15] V. A. Bovdi, A. B. Konovalov and S. Linton, Torsion units in integral group rings of Conway simple groups, *Internat. J. Algebra Comput.*, **21** no. 4 (2011) 615–634.

[16] M. Caicedo, L. Margolis and Á. del Río, Zassenhaus conjecture for cyclic-by-abelian groups, to appear in *Journal of the London Mathematical Society*, (2012) 1–15.

[17] J. Cohn and D. Livingstone, On the structure of group algebras. I, *Canad. J. Math.*, **17** (1965) 583–593.

[18] GAP – Groups, Algorithms, and Programming, Version 4.4.12. http://www.gap-system.org, 2008.

[19] J. Gildea, Zassenhaus conjecture for integral group ring of simple linear groups, to appear *J. Algebra Appl.*, (2013) 1–13.

[20] M. Hertweck, On the torsion units of some integral group rings, *Algebra Colloq.*, **13** no. 2 (2006) 329–348.

[21] M. Hertweck, Zassenhaus conjecture for A_6, *Proc. Indian Acad. Sci. Math. Sci.*, **118** no. 2 (2008) 189–195.

[22] M. Hertweck, Partial augmentations and Brauer character values of torsion units in group rings, to appear in *Comm. Algebra*, (2007) 1–16. (E-print `arXiv:math.RA/0612429v2`).

[23] W. Kimmerle, On the prime graph of the unit group of integral group rings of finite groups, In *Groups, rings and algebras*, volume 420 of *Contemp. Math.*, pages 215–228. Amer. Math. Soc., Providence, RI, 2006.

[24] I. S. Luthar and I. B. S. Passi, Zassenhaus conjecture for A_5, *Proc. Indian Acad. Sci. Math. Sci.*, **99** no. 1 (1989) 1–5.

[25] Mini-Workshop: Arithmetik von Gruppenringen, Abstracts from the mini-workshop held November 25December 1, 2007, Organized by Eric Jespers, Zbigniew Marciniak, Gabriele Nebe and Wolfgang Kimmerle. Oberwolfach Reports, Vol. 4 no. 4, *Oberwolfach Rep.* **4** no. 4 (2007) 3209–3239.

[26] K. Roggenkamp and L. Scott, Isomorphisms of p-adic group rings, *Ann. of Math. (2)*, **126** no. 3 (1987) 593–647.

[27] M. A. M. Salim, Torsion units in the integral group ring of the alternating group of degree 6, *Comm. Algebra*, **35** no. 12 (2007) 4198–4204.

[28] M. A. M. Salim, Kimmerle's conjecture for integral group rings of some alternating groups, *Acta Math. Acad. Paedagog. Nyházi. (N.S.)*, **27** no. 1 (2011) 9–22.

[29] A. Weiss, Torsion units in integral group rings, *J. Reine Angew. Math.*, **415** (1991) 175–187.

[30] H. Zassenhaus, On the torsion units of finite group rings, In *Studies in mathematics (in honor of A. Almeida Costa) (Portuguese)*, pages 119–126. Instituto de Alta Cultura, Lisbon, 1974.

Mohamed A. Salim

Department of Mathematical Sciences, UAE University, P.O.Box 15551, Al-Ain, United Arab Emirates

`Email: MSalim@uaeu.ac.ae`