International Journal of Group Theory
ISSN (print): 2251-7650, ISSN (on-line): 2251-7669
Vol. 2 No. 4 (2013), pp. 21-29.

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ON SUPERSOLVABILITY OF FINITE GROUPS WITH ℙ-SUBNORMAL SUBGROUPS

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Communicated by Evgeny Vdovin

Abstract. In this paper we find systems of subgroups of a finite group, which ℙ-subnormality guarantees supersolvability of the whole group.

1. Introduction

We consider finite groups only. The Huppert theorem [3, VI.9.5] asserts that a finite group G is supersolvable if and only if every maximal subgroup of G has prime index. It follows that if H is a proper subgroup of a supersolvable group G, then there exists the chain of subgroups

$$H = H_0 \subset H_1 \subset \ldots \subset H_n = G \tag{1}$$

such that $|H_{i+1} : H_i|$ is prime for all i.

A. F. Vasilyev, T. I. Vasilyeva and V. N. Tyutyanov in [8] introduced the following definition. Let $\mathbb{P}$ be the set of all prime numbers. A subgroup H of a group G is called $\mathbb{P}$-*subnormal* in G whenever either $H = G$ or there is a chain of subgroups (1) such that $|H_i : H_{i-1}|$ is prime for all i.

It is clear that a group with ℙ-subnormal maximal subgroups is supersolvable. Group, in which 2-maximal subgroups are ℙ-subnormal, can be nonsupersolvable; such groups are studied in [4]. Groups with ℙ-subnormal Sylow subgroups can be also nonsupersolvable, see [5], [8].

In connection with these results, the following problem is natural: for every finite group G finding systems of subgroups with the following property: G is supersolvable if the subgroups in these systems are ℙ–subnormal. In this direction the following results are obtained:

MSC(2010): Primary: 20D20; Secondary: 20E34.

Keywords: Finite group, supersolvable group, ℙ-subnormal subgroup.

Received: 1 December 2012, Accepted: 30 April 2013.

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A group is supersolvable if and only if the normalizers of all of its Sylow subgroups are $\mathbb{P}$-subnormal.

A group is supersolvable if and only if all of its Hall subgroups are $\mathbb{P}$-subnormal.

A group is supersolvable if and only if all of its primary subgroups and all of its biprimary noncyclic z-subgroups are $\mathbb{P}$-subnormal.

Recall that z-group is a group with cyclic Sylow subgroups. A group G is called primary if $|\pi(G)| = 1$ and biprimary if $|\pi(G)| = 2$. Here, $|\pi(G)|$ is the total number of distinct prime divisors of $|G|$.

2. Preliminary results

We use the standart notation of [3]. The set of all prime divisors of $|G|$ is denoted $\pi(G)$. We write $[A]B$ for a semidirect product with a normal subgroup A. If H is a subgroup of G, then $\mathrm{Core}_G H = \bigcap_{x\in G} x^{-1}Hx$ is called the core of H in G. If a group G contains a maximal subgroup M with trivial core, then G is said to be primitive and M is its primitivator, see [2]. The general properties of a primitive group are described in [3, II], [2, I].

Let G be a group with $|G| = p_1^{a_1} p_2^{a_2} \dots p_k^{a_k}$, where $p_1 > p_2 > \dots > p_k$, $a_i \in \mathbb{N}$. We say that G has an ordered Sylow tower of supersolvable type if there exists a chain of subgroups

$$1 = G_0 \subset G_1 \subset G_2 \subset \dots \subset G_{k-1} \subset G_k = G$$

such that G_i is a normal subgroup of G and G_i/G_{i-1} is isomorphic to a Sylow p_i-subgroup of G for all i. We write $\mathfrak{U}$ and $\mathfrak{D}$ to denote the classes of all supersolvable groups and all groups which have an ordered Sylow tower of supersolvable type, respectively. It's well known that $\mathfrak{U}$ and $\mathfrak{D}$ are saturated hereditary formations, and $\mathfrak{U} \subseteq \mathfrak{D}$.

A nonnilpotent finite group whose proper subgroups are all nilpotent is called a Schmidt group.

Lemma 2.1. *(*[3, II.3.2], [2, I.3–I.8]*) Let G be a solvable primitive group and M is a primitivator of G. The following then hold:*

1) $\Phi(G) = 1$*;*

2) *G contains a unique minimal normal subgroup N, furthermore, $N = C_G(N)$ and $G = [N]M$;*

3) *$N = F(G) = O_p(G)$ for some $p \in \pi(G)$ and $O_p(M) = 1$.*

Lemma 2.2. *Let $\mathfrak{F}$ be a saturated formation and let G be a solvable group. Assume that G does not belong to $\mathfrak{F}$, but $G/K \in \mathfrak{F}$ for all nontrivial normal subgroups K of G. Then G is a primitive group.*

Proof. First assume that $\Phi(G) \neq 1$. By hypothesis, $G/\Phi(G) \in \mathfrak{F}$, and since $\mathfrak{F}$ is a saturated formation, we deduce that $G \in \mathfrak{F}$, this is a contradiction. Now let $\Phi(G) = 1$. Suppose that N_1 N_2 are distinct minimal normal subgroups of G. It is obvious that $N_1 \cap N_2 = 1$. By hypothesis, $G/N_1 \in \mathfrak{F}$ and $G/N_2 \in \mathfrak{F}$, and since $\mathfrak{F}$ is a formation, we have $G \simeq G/(N_1 \cap N_2) \in \mathfrak{F}$, contradicting the assumption that G does not belong to $\mathfrak{F}$. Therefore, G contains a unique minimal normal subgroup N. Since $\Phi(G) = 1$, there exists a maximal subgroup M of G such that $N \not\subseteq M$. It is clear that $G = MN$. Suppose that $\mathrm{Core}_G M \neq 1$. It follows from the fact that N is a unique minimal normal subgroup that

$N \subseteq \mathrm{Core}_G M$, and hence $G = MN = M$, this is a contradiction. Thus $\mathrm{Core}_G M = 1$, G is a primitive group, and M is its primitivator. □

Lemma 2.3. *Let H be a subgroup of a solvable group G, and assume that $|G : H|$ is a prime number. Then $G/\mathrm{Core}_G H$ is supersolvable.*

Proof. By hypothesis, $|G : H| = p$, where p is prime. If $H = \mathrm{Core}_G H$, then $G/\mathrm{Core}_G H$ is cyclic of order p, and hence $G/\mathrm{Core}_G H$ is supersolvable, as required. Assume now that $H \neq \mathrm{Core}_G H$, it follows that H is not normal in G. In this case, $G/\mathrm{Core}_G H$ contains a maximal subgroup $H/\mathrm{Core}_G H$ with trivial core. Hence, $G/\mathrm{Core}_G H$ is primitive and the Fitting subgroup $F/\mathrm{Core}_G H$ of $G/\mathrm{Core}_G H$ has prime order p. Since

$$F/\mathrm{Core}_G H = C_{G/\mathrm{Core}_G H}(F/\mathrm{Core}_G H),$$

we have

$$(G/\mathrm{Core}_G H)/(F/\mathrm{Core}_G H) \simeq H/\mathrm{Core}_G H.$$

It follows that $H/\mathrm{Core}_G H$ is isomorphic to a cyclic group of order dividing $p-1$. Thus, $G/\mathrm{Core}_G H$ is supersolvable. □

Example 1. The subgroup $H = A_4$ of the alternating group $G = A_5$ is $\mathbb{P}$-subnormal. If $x \in G \setminus H$, then H^x is $\mathbb{P}$-subnormal in G. The subgroup $D = H \cap H^x$ is a Sylow 3-subgroup of G and D is not $\mathbb{P}$-subnormal in H. Therefore, an intersection of two $\mathbb{P}$-subnormal subgroups is not necessarily $\mathbb{P}$-subnormal. Moreover, if H is $\mathbb{P}$-subnormal in G and K is an arbitrary subgroup of G, in general, their intersection $H \cap K$ is not $\mathbb{P}$-subnormal in K.

However, in solvable groups such situations are not feasible. The following lemma is formulated without a proof in [8, 1.4 (3)] with a link to the properties of $\mathfrak{U}$-subnormal subgroups. We provide a proof that doesn't use formation theory.

Lemma 2.4. *([8, 1.4 (3)]) Let H be a $\mathbb{P}$-subnormal subgroup of a solvable group G, and suppose that K is an arbitrary subgroup. Then $(H \cap K)$ is $\mathbb{P}$-subnormal in K.*

Proof. If $H = G$, there is nothing to prove. So we can assume that $H \neq G$. The definition of $\mathbb{P}$-subnormality applies, and there exists a chain of subgroups

$$H = H_0 \subset H_1 \subset \ldots \subset H_n = G$$

such that $|H_i : H_{i-1}|$ is a prime number for all $i = 1, \ldots, n$.. We will use induction by n.

First, consider the case when $n = 1$. Then $H = H_{n-1}$ is a maximal subgroup of the group G of prime index. By Lemma 2.3, G/N is supersolvable, where $N = \mathrm{Core}_G H$. Since every subgroup of a supersolvable group is $\mathbb{P}$-subnormal, we deduce that

$$H/N \cap KN/N = N(H \cap K)/N$$

is $\mathbb{P}$-subnormal in KN/N. It is clear that $N(H \cap K)$ is $\mathbb{P}$-subnormal in KN. This means that there exists a chain of subgroups

$$N(H \cap K) = A_0 \subset A_1 \subset \ldots \subset A_{m-1} \subset A_m = NK$$

such that $|A_i : A_{i-1}| \in \mathbb{P}$ for all $i = 1, \ldots, m$. Since

$$N(H \cap K) \subseteq A_i \subseteq NK,$$

we have $A_i = N(A_i \cap K)$ and $H \cap K \subseteq A_i \cap K$ for all $i = 1, \ldots, m$. We introduce the notation $B_i = A_i \cap K$. It is obvious that

$$B_{i-1} \subseteq B_i,\ A_i = N(A_i \cap K) = NB_i,\ N \cap B_i = N \cap A_i \cap K = N \cap K$$

for all $i = 1, \ldots, m$. Since $N \subseteq H$, we have

$$B_0 = A_0 \cap K = N(H \cap K) \cap K = (N \cap K)(H \cap K) = H \cap K,$$

$$B_m = A_m \cap K = KN \cap K = K.$$

Moreover,

$$|A_i : A_{i-1}| = \frac{|NB_i|}{|NB_{i-1}|} = \frac{|N||B_i||N \cap B_{i-1}|}{|N \cap B_i||N||B_{i-1}|} = \frac{|B_i : B_{i-1}|}{|N \cap B_i : N \cap B_{i-1}|} =$$

$$= \frac{|B_i : B_{i-1}|}{|N \cap K : N \cap K|} = |B_i : B_{i-1}|.$$

We have a chain of subgroups

$$H \cap K = B_0 \subset B_1 \subset \ldots \subset B_{m-1} \subset B_m = K,\ |B_i : B_{i-1}| \in \mathbb{P},\ 1 \leq i \leq m.$$

This proves that the subgroup $H \cap K$ is $\mathbb{P}$-subnormal in K.

Let $n > 1$. Since H_{n-1} is maximal subgroup of G with prime index, and G is solvable, thus, as it was proved, $H_{n-1} \cap K$ is $\mathbb{P}$-subnormal in K. The subgroup H is $\mathbb{P}$-subnormal in the solvable group H_{n-1} and the induction is applicable to them. By induction,

$$H \cap (H_{n-1} \cap K) = H \cap K$$

is $\mathbb{P}$-subnormal in $H_{n-1} \cap K$, and thus $H \cap K$ is $\mathbb{P}$-subnormal in K. □

Lemma 2.5. *Let A be a p-subgroup of a group G. If $|G : N_G(A)| = p^a$, $a \in \mathbb{N}$, then A is subnormal in G.*

Proof. Let P be a Sylow p-subgroup of G containing A. Then

$$G = N_G(A)P,\ A^G = A^{N_G(A)P} = A^P \subseteq P,$$

thus $A^G \subseteq O_p(G)$. It is clear that A is subnormal in G. □

Lemma 2.6. *Let $P \subseteq G$ be a Sylow subgroup for the largest prime divisor p of $|G|$. Assume that P is not normal in G, and that $H, K \subseteq G$ are subgroups with $N_G(P) \subseteq K \subseteq H$. Then $|H : K|$ is not a prime number.*

Proof. Clearly, $N_G(P) = N_K(P) = N_H(P)$, and P is a Sylow p-subgroup of K and H. We have

$$|H : N_H(P)| = |H : K|\ |K : N_K(P)|,$$

and, by Sylow's theorem,

$$|H : N_H(P)| = 1 + hp,\ |K : N_K(P)| = 1 + kp,\ h, k \in \mathbb{N} \cup \{0\}.$$

Let $|H : K| = t$. Then, $1 + hp = t(1 + kp),\quad t = 1 + (h - tk)p$. We see that p divides $t - 1$, and so $t > p$. If t is a prime number, then this contradicts the maximality of p. □

Lemma 2.7. *Let $A \subseteq G$ be a p-subgroup for the largest prime divisor p of $|G|$. If A is a $\mathbb{P}$-subnormal subgroup of G, then A is subnormal in G.*

Proof. Let $|A| = p^\alpha$. Because A is $\mathbb{P}$-subnormal in G, there exists a series

$$A = A_0 \subset A_1 \subset \ldots \subset A_{t-1} \subset A_t = G,\ |A_i : A_{i-1}| \in \mathbb{P},\ 1 \le i \le t.$$

Since $|A_1 : A_0| \in \mathbb{P}$, we have

$$|A_1| = p^{1+\alpha} \quad or \quad |A_1| = p^\alpha q, \quad p \neq q.$$

If $|A_1| = p^{1+\alpha}$, then A is a normal subgroup of A_1. If $|A_1| = p^\alpha q$, then $p > q$ and again we have A is normal in A_1. Suppose we already know that A is subnormal in A_j. It follows that $A \subseteq O_p(A_j)$. Since $|A_{j+1} : A_j| \in \mathbb{P}$, we obtain $|A_{j+1}| = p|A_j|$ or $|A_{j+1}| = q|A_j|, p \neq q$. If $|A_{j+1}| = p|A_j|$, then by Lemma 2.5, $O_p(A_j) \subseteq O_p(A_{j+1})$, and we conclude that A is subnormal in A_{j+1}. If $|A_{j+1}| = q|A_j|$, $p \neq q$, then $p > q$. Consider the set of left cosets of A_j in A_{j+1}. We know that $A_{j+1}/\mathrm{Core}_{A_{j+1}}A_j$ is isomorphic to a subgroup of the symmetric group S_q, and so any Sylow p-subgroup of A_{j+1} is contained in $\mathrm{Core}_{A_{j+1}}A_j$. Since A is subnormal in A_j, it follows that A is subnormal in $\mathrm{Core}_{A_{j+1}}A_j$. Since $\mathrm{Core}_{A_{j+1}}A_j$ is normal in A_{j+1}, we deduce that A is subnormal in A_{j+1}. Therefore, A is subnormal in A_i for all i. This implies that A is subnormal in G. □

Lemma 2.8. *Suppose that a Sylow p-subgroup of a finite group G is $\mathbb{P}$-subnormal, and let N be a normal subgroup of G. Then a Sylow p-subgroup of G/N is $\mathbb{P}$-subnormal.*

Proof. Let A/N be a Sylow p-subgroup of G/N. If G_p is a Sylow p-subgroup of G, then G_pN/N is a Sylow p-subgroup of G/N. By hypothesis, G_p is $\mathbb{P}$-subnormal in G, and hence G_pN/N is $\mathbb{P}$-subnormal in G/N. By Sylow's theorem, the subgroups A/N and G_pN/N are conjugate in G/N, and we deduce that A/N is $\mathbb{P}$-subnormal in G/N. □

Lemma 2.9. *Let G be a group of order $p_1^{a_1}p_2^{a_2}\ldots p_n^{a_n}$, $p_1 < p_2 < \ldots < p_{n-1} < p_n$. If every Sylow p_i-subgroup of G is $\mathbb{P}$-subnormal for all $i > 1$, then G has an ordered Sylow tower of supersolvable type.*

Proof. We proceed by induction of $|G|$. By Lemma 2.7, a Sylow p_n-subgroup P_n of G is normal, and by Lemma 2.8, a Sylow p_i-subgroup of G/P_n is $\mathbb{P}$-subnormal for all $i > 1$. By the unductive hypothesis, G/P_n has an ordered Sylow tower of supersolvable type. Therefore, G also has an ordered Sylow tower of supersolvable type and the proof is complete. □

Lemma 2.10. *Suppose that a Sylow p-subgroup P of a group G is normal. Then $\Phi(P) = P \cap \Phi(G)$.*

Proof. By [3, III.3.3], we have $\Phi(P) \subseteq \Phi(G)$. Suppose that $P \cap \Phi(G)$ is different from $\Phi(P)$. It follows that $(P \cap \Phi(G))/\Phi(P)$ is a nonidentity normal subgroup of $G/\Phi(P)$. The Sylow p-subgroup $P/\Phi(P)$ of $G/\Phi(P)$ is normal and elementary abelian. It follows by Maschke's theorem that $G/\Phi(P)$ has a normal subgroup $P_1/\Phi(G)$ such that

$$P/\Phi(P) = ((P \cap \Phi(G))/\Phi(P)) \times (P_1/\Phi(P)).$$

Let H be a complement for P in G. In this case $G = [P]H$. Since P_1 is normal in G and $P_1 \neq P$, we conclude that P_1H is a proper subgroup of G and

$$G = [P]H = [(P \cap \Phi(G))P_1]H.$$

It follows from [3, III.3.2] that $G = P_1H$, this is a contradiction. Thus our assumption is false and $\Phi(P) = P \cap \Phi(G)$. □

Lemma 2.11. *Let $G/\Phi(G)$ be a p-solvable group with cyclic Sylow p-subgroup. Then a Sylow p-subgroup of G is cyclic. In particular, if $G/\Phi(G)$ is a z-group, then G is also a z-group.*

Proof. It is obvious that G is p-solvable, and by [3, VI.6.4;VI.6.6] we have

$$l_p(G) = l_p(G/\Phi(G)) = 1.$$

It follows from [3, III.3.4] that

$$\Phi(G)O_{p'}(G)/O_{p'}(G) \subseteq \Phi(G/O_{p'}(G)),$$

and thus $(G/O_{p'}(G))/\Phi(G/O_{p'}(G))$ has a cyclic Sylow p-subgroup. If $O_{p'}(G) \neq 1$, then by the inductive hypothesis, a Sylow p-subgroup of $G/O_{p'}(G)$ is cyclic, and hence a Sylow p-subgroup of G is cyclic. We can assume, therefore, that $O_{p'}(G) = 1$. It follows that a Sylow p-subgroup P of G is normal. By Lemma 2.10, $\Phi(P) = \Phi(G) \cap P$. By hypothesis,

$$P\Phi(G)/\Phi(G) \simeq P/P \cap \Phi(G) = P/\Phi(P),$$

is cyclic, and thus P is cyclic.

Suppose now that $G/\Phi(G)$ is a z-group. It follows from [3, IV.2.11] that G is supersolvable and the lemma applies to every Sylow subgroup of G. Therefore, G is a z-group. The lemma is proved. □

Lemma 2.12. *Let G be a nonsupersolvable group whose proper subgroups are supersolvable. Then the following hold:*

1) $|\pi(G)| \leq 3$ *and G contains a unique normal Sylow subgroup P, where $P = G^{\mathfrak{U}}$;*

2) *$P/\Phi(P)$ is a minimal normal subgroup of $G/\Phi(P)$, and $P/\Phi(P)$ is noncyclic;*

3) *if D is a complement for P in G, then $D/D \cap \Phi(G)$ is either a minimal nonabelian group or a cyclic primary group;*

4) *if $|\pi(D)| > 1$, then D is a biprimary noncyclic z-subgroup.*

Proof. Assertions 1–3 were proved in [3, VI.9.16].

4. Let $|\pi(D)| > 1$. Any minimal nonabelian group is either a primary group or a Schmidt group whose proper subgroups are abelian. Furthermore, $\pi(G) = \pi(G/\Phi(G))$, see [3, III.3.8]. Suppose that

$D\Phi(G)/\Phi(G)$ is a p-group, it follows that D is a p-group, this contradicts the choice of D. Therefore, $D\Phi(G)/\Phi(G)$ is a Schmidt group. Besides, this group is supersolvable because D is supersolvable as a proper subgroup of G. It follows from the properties of Schmidt groups [3, III.5.2], [6, Lemma 1] that $D\Phi(G)/\Phi(G)$ is a biprimary noncyclic z-group, and thus D is a biprimary noncyclic group. It follows by Lemma 2.11, that D is a z-group. □

3. The supersolvability criteria

Theorem 3.1. *A group is supersolvable if and only if the normalizers of all of its Sylow subgroups are $\mathbb{P}$-subnormal.*

Proof. Suppose that a group G is supersolvable. It follows that every proper subgroup of G is $\mathbb{P}$-subnormal, in particular, the normalizers of all Sylow subgroups of G are $\mathbb{P}$-subnormal.

Conversely, suppose that the normalizer of every Sylow subgroup of a group G is $\mathbb{P}$-subnormal. Let P be a Sylow p-subgroup of G, where p is the largest prime divisor of $|G|$. Assume that P is not normal in G. It follows that $N_G(P)$ is a proper subgroup of G, and by hypothesis, it is $\mathbb{P}$-subnormal in G. By the definition of $\mathbb{P}$-subnormality, we know that there exists a subgroup $H_1 \subseteq G$ with the properties $N_G(P) \subset H_1$ and $|H_1 : N_G(P)| \in \mathbb{P}$. This contradicts Lemma 2.6, so we conclude that P is normal in G.

Let K be a nonidentity normal subgroup of G, and let A/K be a Sylow p-subgroup of G/K. Suppose that G_p is a Sylow p-subgroup of G. Then G_pK/K is a Sylow p-subgroup of G/K, and $N_{G/K}(G_pK/K) = N_G(G_p)K/K$ is the normalizer of G_pK/K. By hypothesis, $N_G(G_p)$ is $\mathbb{P}$-subnormal in G, and hence $N_{G/K}(G_pK/K)$ is $\mathbb{P}$-subnormal in G/K. Since the Sylow p-subgroups A/K and G_pK/K of G/K are conjugate by Sylow's theorem, it follows that their normalizers are also conjugate. We deduce that $N_{G/K}(A/K)$ is $\mathbb{P}$-subnormal in G/K. Therefore, we conclude that the normalizers of all Sylow p-subgroups of G/K for any given prime p are $\mathbb{P}$-subnormal. By the inductive hypothesis, G/K is supersolvable, in particular, G/P is supersolvable, and hence G is solvable. Since the class of all supersolvable groups is a saturated formation, we see by Lemma 2.2 that G is a primitive group, and by Lemma 2.1, G contains a unique minimal normal subgroup N, $N = C_G(N)$ and $G = [N]M$, where M is a primitivator of G. Because P is normal in G and N is a unique minimal normal subgroup of G, it follows that N is the Fitting subgroup of G, $N = P$ and $G = [P]M$. Since $G/P \simeq M$ and G/P is supersolvable, then M is also supersolvable. Let q be the largest prime divisor of $|M|$. By [3, VI.9.1], a Sylow q-subgroup Q of M is normal in M. As M is a maximal subgroup of G, we have $N_G(Q) = M$. Since M is a Hall subgroup of G, it follows that Q is a Sylow q-subgroup of G, $q \neq p$. By hypothesis, $M = N_G(Q)$ is $\mathbb{P}$-subnormal in G. Because M is maximal in G, we have $|G : M| = |P| = p$. We see that G is solvable, $|G : M|$ is a prime number and $\mathrm{Core}_G(M) = 1$. We conclude by Lemma 2.3, that G is supersolvable. The theorem is proved. □

Theorem 3.2. *A group is supersolvable if and only if all of its Hall subgroups are $\mathbb{P}$-subnormal.*

Proof. Suppose that a group G is supersolvable. It follows that every proper subgroup of G is $\mathbb{P}$-subnormal, in particular, every Hall subgroup of G is $\mathbb{P}$-subnormal.

Conversely, suppose that every Hall subgroup of a group G is $\mathbb{P}$-subnormal. In particular, every Sylow subgroup of G is $\mathbb{P}$-subnormal. By Lemma 2.9, the group G has an ordered Sylow tower of supersolvable type, in particular, G is solvable. Let N be a normal subgroup of the group G, and let K/N be a π-Hall subgroup of G/N. If L is a π-Hall subgroup of G, then LN/N is a π-Hall subgroup of G/N. By hypothesis, L is $\mathbb{P}$-subnormal in G, and thus LN/N is $\mathbb{P}$-subnormal in G/N. Since G is solvable, we conclude that LN/N and K/N are conjugate. Therefore, K/N is $\mathbb{P}$-subnormal in G/N. Consequently, the conditions of the theorem hold for all quotient groups of G and so we can apply the inductive hypothesis to G/N. We conclude that G/N is supersolvable. Since the class of all supersolvable groups is a saturated formation, we know by Lemma 2.2 that G is a primitive group, and it follows by Lemma 2.1 that G has a unique minimal normal subgroup $N = F(G)$, $G = [N]M$, where M is a primitivator of G. Let P be a Sylow p-subgroup of G, where p is the largest prime divisor of $|G|$. Since G has an ordered Sylow tower of supersolvable type, we deduce that P is normal in G, and thus $P = N$. Therefore, M is a Hall subgroup of the group G. By hypothesis, M is $\mathbb{P}$-subnormal in G. It follows by the maximality of M that $|N| = p$. Since G is solvable, $|G : M|$ is prime and $\mathrm{Core}_G(M) = 1$, we deduce from Lemma 2.3 that G is supersolvable. The theorem is proved. □

Theorem 3.3. *A group is supersolvable if and only if all of its primary subgroups and all of its biprimary noncyclic z-subgroups are $\mathbb{P}$-subnormal.*

Proof. If a group G is supersolvable, then the assertion is true.

Conversely, suppose that every primary subgroup of G and every biprimary noncyclic z-subgroup of G is $\mathbb{P}$-subnormal. Because every Sylow subgroup of G is $\mathbb{P}$-subnormal, it follows by Lemma 2.9 that G has an ordered Sylow tower of supersolvable type. Let H be an arbitrary proper subgroup of G, and let $R \subseteq H$ and $B \subseteq H$, respectively, be a p-subgroup and a biprimary noncyclic z-subgroup of H. By hypothesis, R and B are $\mathbb{P}$-subnormal in G. Since G is solvable, it follows by Lemma 2.4 that $R = R \cap H$ and $B = B \cap H$ are $\mathbb{P}$-subnormal in H. Consequently, the conditions of the theorem hold for all subgroups of G, and so by the inductive hypothesis, every own subgroup of the group G is supersolvable. Therefore, G is a minimal nonsupersolvable group and applying Lemma 2.12 (1) to the group G, we conclude that $|\pi(G)| \leq 3$. Let $P = G^{\mathfrak{U}}$, a let $D \subseteq G$ be a complement for P in G, so we have $G = [P]D$.

Suppose that $|\pi(G)| = 3$. We conclude by Lemma 2.12 that D is a noncyclic biprimary z-subgroup. By hypothesis, D is $\mathbb{P}$-subnormal in G. If $|\pi(G)| = 2$, then D is a p-subgroup and by hypothesis, D is $\mathbb{P}$-subnormal in G. Thus in any case, the subgroup D is $\mathbb{P}$-subnormal in G. Since $\Phi(P)$ is normal in G, we see that $\Phi(P)D/\Phi(P)$ is $\mathbb{P}$-subnormal in $G/\Phi(P)$, and $\Phi(P)D$ is $\mathbb{P}$-subnormal in G. By Lemma 2.12 (2), the subgroup $\Phi(P)D$ is maximal in G and it has nonprime index. This is a contradiction and the proof is complete. The theorem is proved. □

Finite groups with $\mathbb{P}$-subnormal primary cyclic subgroups were studied by the authors in [5]. These groups have an ordered Sylow tower of supersolvable type and every biprimary subgroup with a cyclic

Sylow subgroup of these groups is supersolvable. Moreover, the class of groups whose primary cyclic subgroups are $\mathbb{P}$-subnormal is a hereditary saturated formation.

Example 2. In general, a group with $\mathbb{P}$-subnormal primary cyclic subgroups and $\mathbb{P}$-subnormal noncyclic biprimary z-subgroups can be nonsupersolvable. The group

$$G = [E_{5^2}](\langle a\rangle\langle b\rangle),\ |a| = |b| = 4,$$

is a minimal nonsupersolvable group of order 400, and it has the desired properties [7, Theorem 1 (3)]. The number of this group in the library of SmallGroups [1] is [400,129]. All subgroups of the group G except for the noncyclic maximal subgroup $\langle a\rangle\langle b\rangle$ of order 16 are $\mathbb{P}$-subnormal in G.

Example 3. In general, a group with $\mathbb{P}$-subnormal p-subgroups and $\mathbb{P}$-subnormal Schmidt subgroups can be nonsupersolvable. For example, the group

$$G = [E_{7^2}]([\langle a\rangle]\langle b\rangle),\ |a| = 3^2,\ |b| = 2.$$

has the desired properties [7, Theorem 1 (6)]. It is minimal nonsupersolvable group of order $882 = 2 \cdot 3^2 \cdot 7^2$. The number of this group in the library of SmallGroups [1] is [882,17]. All subgroups of the group G except for the maximal subgroup $[\langle a\rangle]\langle b\rangle$ are $\mathbb{P}$-subnormal. The subgroup $[\langle a\rangle]\langle b\rangle$ of order $3^2 \cdot 2$ is a biprimary noncyclic z-group, but it is not a Schmidt group.

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