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ON THE PROBABILITY OF BEING A 2-ENGEL GROUP

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Abstract. Let G be a finite group and $d_2(G)$ denotes the probability that $[x,y,y]=1$ for randomly chosen elements x,y of G. We will obtain lower and upper bounds for $d_2(G)$ in the case where the sets $E_G(x)=\{y\in G:[y,x,x]=1\}$ are subgroups of G for all $x\in G$. Also the given examples illustrate that all the bounds are sharp.

1. Introduction

For a finite group G, the *commutativity degree* of G, denoted by $d(G)$ (henceforth $d_1(G)$) is defined as the probability that two randomly chosen elements of G commute, that is

$$d_1(G)=\frac{|\{(x,y)\in G\times G:xy=yx\}|}{|G|^2}.$$

It is known that $d_1(G)=k(G)/|G|$, where $k(G)$ is the number of conjugacy classes of G and $d_1(G)\leq 5/8$ if G is non-abelian (see [4, 10, 17]). The commutativity degree and its generalizations are extensively studied in the literature and we may refer the reader to [1, 5, 6, 9, 11, 13].

For a given natural number n we define the *n-Engel degree* of G, denoted by $d_n(G)$, in the same way as the probability that two randomly chosen elements x,y of G satisfy the n-Engel condition $[y,_n x]=1$, that is

$$d_n(G)=\frac{|\{(x,y)\in G\times G:[y,_n x]=1\}|}{|G|^2}.$$

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One can easily see that the Engel degrees of a finite group G are related through the following inequalities:

$$d_1(G) \leq d_2(G) \leq \cdots \leq d_n(G) \leq \cdots .$$

Let $\mathcal{X}$ be the class of all groups G such that the sets $E_G(x) = \{y \in G : [y, x, x] = 1\}$ are subgroups of G for all $x \in G$. We intend to give lower and upper bounds for 2-Engel degree of a finite group G under the assumption that G belongs to the class $\mathcal{X}$, the fact that is fundamental in our investigations. The class $\mathcal{X}$ includes the variety $\mathcal{V}$ of all 3-metabelian groups determined by the law $[[x, y], [x, z]] = 1$, which itself include the variety of all metabelian groups. (See Lemma 2.1)

Recall that the notations $\pi(G)$, $L_n(G)$, $L(G)$, $R_n(G)$ and $R(G)$ denote the set of all prime divisors of the order of G, left n-Engel elements, left Engel elements, right n-Engel elements and right Engel elements of G, respectively, for each natural number n. Also $k_G(X)$ stand for the number of conjugacy classes of G contained in X for each normal subset X of G.

Our aim is to prove the following results:

Theorem A. *Let $G \in \mathcal{X}$ be a finite group which is not a 2-Engel group. If $p = \min \pi(G)$, then*

$$d_2(G) \leq \frac{1}{p} + \left(1 - \frac{1}{p}\right) \frac{|L_2(G)|}{|G|}$$

and if $L_2(G) \leq G$, then

$$d_2(G) \leq \frac{2p - 1}{p^2}.$$

Moreover, both of the upper bounds are sharp at any prime p.

Note that, in spite of the fact that $R_2(G)$ is always a subgroup of G (See [12]), $L_2(G)$ is not necessarily a subgroup even for groups in variety $\mathcal{V}$. The 5th small group of order 54 in the small group library of GAP [8] is the smallest example of such a group. Also, if G is the 138th small group of order 64, not only $L_2(G)$ is not a subgroup of G but also $|L_2(G)| > |G|/2$ and a simple computation with GAP [8] shows that $d_2(G) = 13/16 > 3/4$. Hence, Theorem A cannot be improved further.

Theorem B. *Let $G \in \mathcal{X}$ be a finite group which is not a 2-Engel group. If $p = \min \pi(G)$, then*

$$d_2(G) \geq d_1(G) - (p - 1)\frac{|Z(G)|}{|G|} + (p - 1)\frac{k_G(L(G))}{|G|}$$

and if either G is a p-group, or G belongs to the variety $\mathcal{V}$ such that either G' is a cyclic 2-group or G' is a generalized quaternion 2-group, then

$$d_2(G) \geq pd_1(G) - (p - 1)\frac{|Z(G)|}{|G|}.$$

Moreover, both of the lower bounds are sharp at any prime p.

2. Some basic results

For a given group G and element $x \in G$, let $E_G(x)$ be the set of all elements $y \in G$ such that $[y, x, x] = 1$. We begin with giving a criterion for $E_G(x)$ to be a subgroup of G.

Lemma 2.1. *Let G be a group and $x \in G$. Then the following statements are equivalent:*

(i) $E_G(x) \leq G$;

(ii) $[E_G(x), x, E_G(x), x] = 1$;

(iii) $[[E_G(x), x], [E_G(x), x]] = 1$ *that is* $[E_G(x), x]$ *is abelian.*

Proof. Let $a, b \in E_G(x)$. First we observe that for an element $g \in G$,

$$[g, x, x] = 1 \Leftrightarrow [x^g, x] = 1 \Leftrightarrow [x^{g^{-1}}, x] = 1 \Leftrightarrow [g^{-1}, x, x] = 1.$$

Hence $E_G(x) = E_G(x)^{-1}$.

(i)$\Leftrightarrow$(ii) A simple computation shows that $[ab, x, x] = [a, x, b, x]^{[b,x]}$. Hence $ab \in E_G(x)$ if and only if $[a, x, b, x] = 1$.

(ii)$\Leftrightarrow$(iii) Using the Witt-Hall's identity, we have

$$[a, x, b^{-1}, x]^b \; [b, x^{-1}, [a, x]]^x \; [x, [x, a], b]^{[a,x]} = 1,$$

from which it follows that $[a, x, b^{-1}, x] = 1$ if and only if $[[a, x], [b, x]] = 1$. □

The above lemma indicates that $E_G(x)$ is a subgroup whenever $[E_G(x), x]$ is abelian. A good candidate for such groups is given by the variety $\mathcal{V}$ of all 3-metabelian groups admitting the law $[[x, y], [x, z]] = 1$. Such groups are extensively studied by Macdonald [14] and some further results are recently obtained by the second author [7]. It is shown that a group G in variety $\mathcal{V}$ satisfies the following properties:

(1) $G'^4 = 1$;

(2) $[\gamma_3(G), \gamma_2(G)] = 1$;

(3) $[\gamma_2(G), \gamma_2(G), G] = 1$;

(4) $[[x_1, x_2], [x_3, x_4]] = [[x_{\pi_1}, x_{\pi_2}], [x_{\pi_3}, x_{\pi_4}]]$;

(5) $[[x_1, x_2], [x_3, x_4]] = [x_1, x_2, x_3, x_4][x_1, x_2, x_4, x_3]$,

where $x_1, x_2, x_3, x_4 \in G$ and $\pi \in S_4$, the symmetric group on four letters.

Also, Bachmuth and Lewin [2], Macdonald [15] and Bussman and Jackson [3] had shown that the variety $\mathcal{V}$ is equivalent to varieties defined by either of the laws

$$[x, y, z][y, z, x][z, x, y] = 1,$$

$$[[x, y], [y, z]][[y, z], [z, x]][[z, x], [x, y]] = 1,$$

and

$$[z, y][z, x][y, x] = [y, x][z, x][z, y],$$

respectively.

The construction of non-metabelian 3-metabelian groups of Neumann [16], indicates that the conditions above are in a sense best possible.

In the sequel, we will obtain some numerical and structural results relating to the subsets $E_G(x)$, when the group G belongs to the class $\mathcal{X}$.

Lemma 2.2. *Let $G \in \mathcal{X}$ be a finite group and $x \in G$. Then*

(i) $|E_G(x)| = |C_G(x)||G_G(x) \cap x^G|$,
(ii) $[C_G(x)x^G : C_G(x)] = [G : E_G(x)]$, *and*
(iii) $|C_G(x)x^G| = [G : C_G(x) \cap x^G]$ *divides* $|G|$.

Proof. (i) Clearly $x^a = x^b \in C_G(x)$ if and only if $x^a \in C_G(x)$ and $b \in C_G(x)a$. Hence, the element $x^a \in C_G(x)$ contributes $|C_G(x)|$ elements to $E_G(x)$, from which it follows that $|E_G(x)| = |C_G(x)||G_G(x) \cap x^G|$.

(ii) The result follows according to the fact that

$$\begin{aligned} C_G(x)x^a = C_G(x)x^b &\Leftrightarrow C_G(x^{b^{-1}})x^{ab^{-1}} = C_G(x^{b^{-1}})x \\ &\Leftrightarrow [x^{ab^{-1}}x^{-1}, x^{b^{-1}}] = 1 \\ &\Leftrightarrow [ab^{-1}, x^{-1}, x[x, b^{-1}]] = 1 \\ &\Leftrightarrow [ab^{-1}, x^{-1}, x] = 1 \\ &\Leftrightarrow [ab^{-1}, x, x] = 1 \\ &\Leftrightarrow ab^{-1} \in E_G(x) \\ &\Leftrightarrow E_G(x)a = E_G(x)b \end{aligned}$$

for each $a, b \in G$.

(iii) By (i), $|C_G(x) \cap x^G|$ divides $|G|$. Hence, by (ii), $|C_G(x)x^G| = [G : C_G(x) \cap x^G]$ divides $|G|$, as required. □

Lemma 2.3. *Let $G \in \mathcal{X}$ be a finite group with an element x such that $C_G(x)x^G = G$. Then*

(i) *If $G \in \mathcal{V}$, then $G = [x, G] \rtimes C_G(x)$;*
(ii) *If $x \in L(G)$, then $x \in Z(G)$.*

Proof. (i) Since $C_G(x)x^G = G$, we get $C_G(x)[x, G] = G$ so that

$$[x, G] = [x, C_G(x)[x, G]] = [x, [x, G]] \subseteq [x, G'] \subseteq [x, G].$$

Hence $[x, G] = [x, G'] = [x, [x, G]]$. If $G \in \mathcal{V}$, then $[x, ab] = [x, a][x, b]$ for each $a, b \in [x, G]$, which implies that $[x, G] \trianglelefteq G$. On the other hand, by Lemma 2.2(iii), $|C_G(x) \cap x^G| = 1$ that is $C_G(x) \cap x^G = \{x\}$ and consequently $C_G(x) \cap [x, G] = 1$. Therefore $G = [x, G] \rtimes C_G(x)$.

(ii) Clearly $G = C_G(x)[G, x]$. Hence $[G, x] = [G, x, x]$ and consequently $[G, x] = [G,_n x]$ for each $n \geq 1$. Now if $x \in L(G)$, then $[G, x] = [G,_n x] = 1$ for some positive integer n, which implies that $x \in Z(G)$. □

3. **Proof of main theorems**

Now we are able to prove our main theorems. The sharpness of all the bounds are proved in the next section.

Proof of Theorem A. Since $E_G(x) \leq G$ for each element $x \in G$, we observe that $|E_G(x)| = |G|$ if $x \in L_2(G)$ and $|E_G(x)| \leq |G|/p$ if $x \in G \setminus L_2(G)$. Hence

$$\begin{aligned} d_2(G) &= \frac{1}{|G|^2}\sum_{x\in G}|E_G(x)| \\ &= \frac{1}{|G|^2}\sum_{x\in L_2(G)}|E_G(x)| + \frac{1}{|G|^2}\sum_{x\in G\setminus L_2(G)}|E_G(x)| \\ &\leq \frac{|L_2(G)|}{|G|} + \frac{|G|-|L_2(G)|}{p|G|} \\ &= \frac{1}{p} + \left(1-\frac{1}{p}\right)\frac{|L_2(G)|}{|G|}. \end{aligned}$$

In particular, if $L_2(G) \leq G$, then $|L_2(G)| \leq |G|/p$ and the result follows. □

Proof of Theorem B. Let T be a complete set of representatives of conjugacy classes of G. Then

$$\begin{aligned} d_2(G) &= \frac{1}{|G|^2}\sum_{x\in G}|E_G(x)| \\ &= \frac{1}{|G|^2}\sum_{x\in G}\frac{|C_G(x)||G|}{|C_G(x)x^G|} \qquad \text{(by Lemma 2.2(ii))} \\ &= \sum_{x\in G}\frac{1}{|x^G||C_G(x)x^G|} \\ &= \sum_{x\in T}\frac{1}{|C_G(x)x^G|}. \end{aligned}$$

If $C_G(x)x^G \neq G$, then $|C_G(x)x^G| \leq |G|/p$ for by Lemma 2.2(iii), $|C_G(x)x^G|$ divides $|G|$. Now if $C_G(x)x^G = G$, then either $x \in L(G)$, from which in conjunction with Lemma 2.3(ii), we get $x \in Z(G)$, or $x \in G \setminus L(G)$. Hence

$$\begin{aligned} d_2(G) &= \sum_{x\in T}\frac{1}{|C_G(x)x^G|} \\ &\geq \sum_{x\in Z(G)}\frac{1}{|G|} + \sum_{x\in T\cap L(G)\setminus Z(G)}\frac{p}{|G|} + \sum_{x\in T\setminus L(G)}\frac{1}{|G|} \\ &\geq \frac{|Z(G)|}{|G|} + \frac{p(k_G(L(G))-|Z(G)|)}{|G|} + \frac{k(G)-k_G(L(G))}{|G|} \\ &= d_1(G) - (p-1)\frac{|Z(G)|}{|G|} + (p-1)\frac{k_G(L(G))}{|G|}. \end{aligned}$$

Now we prove the second inequality. If G is a p-group, then $G = L(G)$ that is $k_G(L(G)) = k(G)$. Hence

$$d_2(G) \geq d_1(G) - (p-1)\frac{|Z(G)|}{|G|} + (p-1)d_1(G) = pd_1(G) - (p-1)\frac{|Z(G)|}{|G|}.$$

Finally, assume that G belongs to the variety $\mathcal{V}$ such that G' is either a cyclic 2-group or a generalized quaternion 2-group. Let $x \in G \setminus Z(G)$ such that $C_G(x)x^G = G$. Then, by Lemma 2.3(i), $G = [x,G] \rtimes C_G(x)$. Clearly, G' has a unique involution, say a. Since $1 \neq [x,G] \leq G'$, we should have $a \in [x,G]$. On the other hand, $[x,G] \trianglelefteq G$ and x acts by conjugation on $[x,G]$. Thus we should have $a^x = a$ and consequently $a \in [x,G] \cap C_G(x) = \{1\}$, which is a contradition. Thus, $C_G(x)x^G \neq G$ for all $x \in G \setminus Z(G)$ and, by invoking Lemma 2.2(iii), we obtain

$$\begin{aligned} d_2(G) &= \sum_{x\in T} \frac{1}{|C_G(x)x^G|} \\ &\geq \sum_{x\in Z(G)} \frac{1}{|G|} + \sum_{x\in T\setminus Z(G)} \frac{p}{|G|} \\ &= \frac{|Z(G)|}{|G|} + \frac{k(G)-|Z(G)|}{|G|} \\ &= pd_1(G) - (p-1)\frac{|Z(G)|}{|G|}. \end{aligned}$$

The proof is complete. □

4. **Examples**

This section is devoted to the evaluation of $d_2(G)$ for some classes of finite groups, which in part proves the sharpness of all bounds in Theorems A and B.

Example 1. *Let $G = D_{2n} = \langle a, b : a^n = b^2 = 1, a^b = a^{-1}\rangle$ be a dihedral group of order $2n$. Then*

$$d_2(G) = \begin{cases} \frac{n+1}{2n}, & n \text{ odd}, \\ \frac{n+2}{2n}, & n = 2m, m \text{ odd}, \\ \frac{n+4}{2n}, & n = 4m. \end{cases}$$

Proof. If $n = 1, 2$ or 4, then clearly $d_2(G) = 1$. Now assume that $n \neq 1, 2, 4$. It can be easily verified that $L_2(G) = \langle a\rangle$ and

$$E_G(b) = \begin{cases} \langle b\rangle, & n \text{ odd}, \\ \langle b, a^m\rangle, & n = 2m, m \text{ odd}, \\ \langle b, a^m\rangle, & n = 4m. \end{cases}$$

On the other hand, $|E_G(a^ib)| = |E_G(b)|$, for all $i = 0, \ldots, n-1$. Now the result follows from the fact that

$$d_2(G) = \frac{|G||L_2(G)| + |G \setminus L_2(G)||E_G(b)|}{|G|^2} = \frac{2n^2 + n|E_G(b)|}{4n^2}.$$

□

Example 2. *Let $G = Q_{4n} = \langle a, b : a^{2n} = 1, a^n = b^2, a^b = a^{-1}\rangle$ be a generalized quaternion group of order $4n$. Then*

$$d_2(G) = \begin{cases} \frac{n+1}{2n}, & n \text{ odd,} \\ \frac{n+2}{2n}, & n \text{ even.} \end{cases}$$

Proof. If $n = 1, 2$, then $d_2(G) = 1$. Now assume that $n \neq 1, 2$. It can be easily verified that $L_2(G) = \langle a\rangle$ and

$$E_G(b) = \begin{cases} \langle b\rangle, & n \text{ odd,} \\ \langle b, a^{\frac{n}{2}}\rangle, & n \text{ even.} \end{cases}$$

On the other hand, $|E_G(a^i b)| = |E_G(b)|$, for all $i = 0, \ldots, n-1$ and the result follows from the fact that

$$d_2(G) = \frac{|G||L_2(G)| + |G \setminus L_2(G)||E_G(b)|}{|G|^2} = \frac{8n^2 + 2n|E_G(b)|}{16n^2}.$$

□

Example 3. *Let $p > 2$ be a prime and*

$$G = \langle a, b, c : a^{p^2} = b^p = c^p = 1, [a, b] = a^p, [a, c] = b, [b, c] = 1\rangle.$$

Then

$$d_2(G) = \frac{2p-1}{p^2}.$$

Proof. A simple computation shows that $L_2(G) = \langle a^p, b, c\rangle$ is a subgroup of G of order p^3 and for each $x = a^i b^j c^k \in G \setminus L_2(G)$, we have

$$E_G(x) = \langle a^p, b, a^i c^k\rangle$$

is a subgroup of G of order p^3. Thus

$$d_2(G) = \frac{|G||L_2(G)| + |G \setminus L_2(G)||E_G(x)|}{|G|^2} = \frac{p^7 + (p^4 - p^3)p^3}{p^8} = \frac{2p-1}{p^2}.$$

□

Note that the group $G = D_{16}$ of Example 1 and $G = Q_{16}$ of Example 2 show that the upper bounds in Theorem A and second lower bound in Theorem B are sharp at $\min \pi(G) = 2$. Also, the groups $G = D_{2n}$ (n odd) and $G = D_{2^n}$ ($n > 3$) of Example 1 show that the first and second lower bounds in Theorem B are sharp at $\min \pi(G) = 2$, respectively.

Finally, the groups of Example 3 illustrates that all bounds in Theorems A and B are sharp at $\min \pi(G) > 2$.

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