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THE n-ARY ADDING MACHINE AND SOLVABLE GROUPS

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Abstract. We describe under various conditions abelian subgroups of the automorphism group $\mathrm{Aut}(T_n)$ of the regular n-ary tree T_n, which are normalized by the n-ary adding machine $\tau = (e, \ldots, e, \tau)\sigma_\tau$ where σ_τ is the n-cycle $(0, 1, \ldots, n-1)$. As an application, for $n = p$ a prime number, and for $n = 4$, we prove that every soluble subgroup of $\mathrm{Aut}(T_n)$, containing τ is an extension of a torsion-free metabelian group by a finite group.

1. Introduction

Adding machines have played an important role in dynamical systems, and in the theory of groups acting on trees : see [1, 6, 7, 3, 5].

An element α in the automorphism group $\mathcal{A}_n = \mathrm{Aut}(T_n)$ of the n-ary tree T_n, is represented as $\alpha = \alpha|_\phi = (\alpha|_0, \ldots, \alpha|_{n-1})\,\sigma_\alpha$ where ϕ is the empty sequence from the free monoid $\mathcal{M}$ generated by $Y = \{0, 1, \ldots, n-1\}$, where $\alpha|_i \in \mathcal{A}_n$, for $i \in Y$, are called 1st level states of α and where σ_α (the activity of α) is a permutation in the symmetric group Σ_n on Y extended 'rigidly' to act on the tree; if $\sigma_\alpha = e$, we say that α is inactive.

In applying the same representation to $\alpha|_0$ we produce $\alpha|_{0i}$ for all $i \in Y$ and we produce in general $\{\alpha|_u \mid u \in \mathcal{M}\}$ the set of *states* of α. Following this notation, the n-ary adding machine is represented as $\tau = (e, \ldots, e, \tau)\sigma_\tau$ where e is the identity automorphism and σ_τ is the regular permutation $\sigma = (0, 1, \ldots, n-1)$. In this sense, the adding machine is an infinite variant of the regular permutation which appears often in geometric and combinatorial contexts.

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A characteristic feature of τ is that its n-th power τ^n is the diagonal automorphism of the tree $(\tau, \ldots, \tau)$. This fact implies that the centralizer of the cyclic group $\langle\tau\rangle$ in $\mathcal{A}_n$ is equal to its topological closure $\overline{\langle\tau\rangle}$ in the group $\mathcal{A}_n$ when considered as a topological group with respect to the the natural topology induced by the tree. The pro-cyclic group $\overline{\langle\tau\rangle}$ is isomorphic to $\mathbb{Z}_n$, the ring of n-adic integers $\xi = \sum_{i\geq 0} a_i n^i$ $(0 \leq a_i \leq n-1$ for all $i)$.

A large variety of subgroups of $\mathcal{A}_n$ which contain τ have been constructed, including groups which are torsion-free and just non-solvable without free subgroups of rank 2 (see, [2, 8] and generalizations thereof [10]). Furthermore, the free group of rank 2 has been represented on the binary tree as a group generated by two conjugates of the adding machine τ each having a finite number of states [11]. On the other hand, the restricted structure of its centralizer indicate that solvable groups which contain τ have restricted structure. For nilpotent groups we show

Proposition. *Let G be a nilpotent subgroup of $\mathcal{A}_n$ which contains the n-adic adding machine τ. Then G is a subgroup of $\overline{<\tau>}$.*

The most visible examples of solvable groups containing τ are conjugate to subgroups of those belonging to the infinite sequence of groups

$$\begin{aligned}\Gamma_0 &= N_{\mathcal{A}_n}(\overline{\langle\tau\rangle}),\\ \Gamma_{i+1} &= (\times_n \Gamma_i) \rtimes G_{i+1}\ (i \geq 0)\end{aligned}$$

where $\times_n \Gamma_i$ is a direct product of n copies of Γ_i (seen as a subgroup of the 1st level stabilizer of the tree) and where G_i is a solvable subgroup of the symmetric group Σ_n in its canonical action on the tree and containing the cycle σ_τ. We observe that for all i, the groups Γ_i are metabelian by 'finite solvable subgroups of Σ_n'. It was shown by the second author that for $n = 2$, solvable groups which contain the binary adding machine are conjugate to some subgroups of Γ_i acting on the binary tree [9]. This appears to be the general pattern. However, the description for degrees $n > 2$ requires a classification of solvable subgroups of Σ_n which contain the cycle $\sigma = (0, 1, \ldots, n-1)$[4]. This in itself is an open problem, even for metabelian groups. On the other hand, the answer for primitive solvable subgroups of Σ_n is simple and classical. For then, n is a prime number p or $n = 4$. In case $n = p$, the solvable subgroups G_i can all be taken to be the normalizer $F = N_{\Sigma_n}(\langle\sigma\rangle)$ of order $p(p-1)$ and in case $n = 4$, the G_i's can all be taken to be the symmetric group Σ_4.

Given this background, the main theorem of this paper is

Theorem A. *Let $n = p$, a prime number, or $n = 4$. Then any solvable subgroup of $\mathcal{A}_n$which contains the n-ary machine τ is conjugate to a subgroup of Γ_i for some i.*

The result follows first from general analysis of the conditions $[\beta, \beta^{\tau^x}] = e$ (for some $\beta \in \mathcal{A}_n$ and all $x \in \mathbb{Z}$), then their impact on the 1st level states of the subgroup $\langle\beta, \tau\rangle$ and on how these in turn translate successively to conditions on states at lower levels. It is somewhat surprising that the process converges to a clear global description for trees of degrees p and 4.

The first step of this analysis leads to the following description of the normal closure of $\langle\beta\rangle$ under the action of τ.

Theorem B. *Let B be an abelian subgroup of $\mathcal{A}_n$ normalized by τ, let $\beta = (\beta|_0, \beta|_1, \ldots, \beta|_{n-1})\sigma_\beta \in B$ and define the subgroup $H = \langle \beta|_i \, (i \in Y), \tau \rangle$ generated by the first level states of β and τ.*
(I) Suppose $\sigma_\beta = (\sigma_\tau)^s$ for some integer s. Then H is metabelian-by-finite. More precisely, let $m = \frac{n}{\gcd(n,s)}$, define the product $\pi_i = \beta|_i\beta|_{i+s}\beta|_{i+2s}\cdots\beta|_{i+(m-1)s}$ (the notation $\beta|_j$ means $\beta|_{\bar{j}}$, where $\bar{j}$ is the representative of j in Y modulo n) and define the subgroup

$$K = \left\langle \; [\beta|_i, \tau^k], \; \pi_i \mid k \in \mathbb{Z}, \; i \in Y \; \right\rangle$$

Then K is an abelian group and H affords the normal series

$$H \trianglerighteq K \langle \tau \rangle \, (= O) \trianglerighteq K$$

where the quotient group $\frac{H}{O}$ is a homomorphic image of a subgroup of the wreath product $C_m \wr C_n$ of the cyclic groups C_m, C_n.
(II) Let n be an even number. Then H is a metabelian group if $s = \frac{n}{2}$ or if σ_β is a transposition.

Part (I) of Theorem B will be proven in Sections 4 and 5 and part (II) in Section 7.

Let P be a subgroup of Σ_n. The *layer closure* of P in $\mathcal{A}_n$ is the group $L(P)$ formed by elements of $\mathcal{A}_n$ whose states have activities in P. The following result is yet another characterization of the adding machine.

Theorem C. *Let n be an odd number and let $L = L(\langle \sigma \rangle)$, the layer closure of $\langle \sigma \rangle$ in A_n. Let s be an integer which is relatively prime to n and let $\beta = (\beta|_0, \beta|_1, \ldots, \beta|_{n-1})\sigma^s \in L$ be such* that $[\beta, \beta^{\tau^x}] = e$ *for all $x \in \mathbb{Z}$. Then β is a conjugate of τ in L.*

2. **Preliminaries**

We start by introducing definitions and notation. The n-ary tree T_n can be identified with the free monoid $\mathcal{M} =< 0, 1, \ldots, n-1 >^*$ of finite sequences from $Y = \{0, 1, \ldots, n-1\}$, ordered by $v \leq u$ provided u is an initial subword of v.

The identity element of $\mathcal{M}$ is the empty sequence ϕ. The level function for T_n, denoted by $|m|$ is the length of $m \in \mathcal{M}$; the root vertex ϕ has level 0.

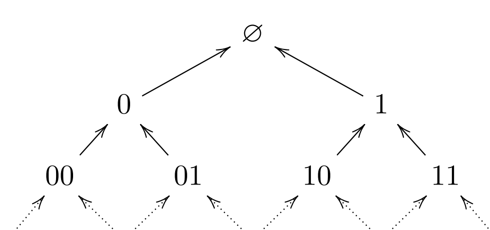

FIGURE 1. The Binary Tree

The action $\rho : i \to j$ of a permutation $\rho \in \Sigma_n$ will be from the right and written as $(i)\rho = j$ or as $i^\rho = j$. If i, j are integers then the action of ρ on i is to be identified with its action on its representatives $\bar{i}$ in Y, modulo n . Permutations σ in Σ_n are extended 'rigidly' to automorphisms of $\mathcal{A}_n$ by

$$(y.u)\rho = (y)\sigma.u, \ \forall \, y \in Y, \ \forall \, u \in \mathcal{M}.$$

An automorphism $\alpha \in \mathcal{A}_n$ induces a permutation σ_α on the set Y. Consequently, α affords the representation $\alpha = \alpha'\sigma_\alpha$ where α' fixes Y point-wise and for each $i \in Y$, α' induces $\alpha|_i$ on the subtree whose vertices form the set $i \cdot \mathcal{M}$. If j is an integer the $\alpha|_j$ will be understood as $\alpha|_{\bar{j}}$ where $\bar{j}$ is the representative of j in Y modulo n.

Given i in Y, we use the canonical isomorphism $i \cdot u \mapsto u$ between $i \cdot \mathcal{M}$ and the tree T_n, and thus identify $\alpha|_i$ with an automorphism of T_n; therefore, $\alpha' \in \mathcal{F}(Y, \mathcal{A}_n)$, the set of functions from Y into $\mathcal{A}_n$, or what is the same, the 1st level stabilizer $Stab(1)$ of the tree. This provides us with the factorization $\mathcal{A}_n = \mathcal{F}(Y, \mathcal{A}_n) \cdot \Sigma_n$.

Let $\alpha, \beta, \gamma \in \mathcal{A}_n$. Then the following formulas hold

$$\sigma_{\alpha^{-1}} = (\sigma_\alpha)^{-1}, \ \sigma_\alpha \sigma_\beta = \sigma_{\alpha\beta}, \tag{2.1}$$

$$(\alpha^{-1})|_u = \left(\alpha|_{(u)^{\alpha^{-1}}}\right)^{-1}, \tag{2.2}$$

$$(\alpha\beta)|_u = (\alpha|_u)\,(\gamma|_u) \ \text{ where } \gamma|_u = \beta|_{(u)^\alpha} \tag{2.3}$$

$$\gamma = \alpha^{-1}\beta\alpha \Leftrightarrow \left(\sigma_\gamma = \sigma_\alpha^{-1}\sigma_\beta\sigma_\alpha \text{ and } \gamma|_{(i)\sigma_\alpha} = \alpha|_i^{-1}\beta|_i\alpha|_{(i)\sigma_\beta}, \forall i \in Y\right). \tag{2.4}$$

$$\theta = [\beta, \alpha] = \beta^{-1}\beta^\alpha \Leftrightarrow \begin{cases} \sigma_\theta = [\sigma_\beta, \sigma_\alpha], \\ \theta|_{(i)\sigma_{\alpha\beta}} = \left(\beta|_{(i)\sigma_\alpha}\right)^{-1} (\alpha|_i)^{-1} (\beta|_i) \left(\alpha|_{(i)\sigma_\beta}\right), \forall i \in Y. \end{cases} \tag{2.5}$$

$$(\alpha^m)\,|_i = (\alpha|_i)\left(\alpha|_{(i)\sigma_\alpha}\right)\left(\alpha|_{(i)\sigma_\alpha^2}\right) \cdots \left(\alpha|_{(i)\sigma_\alpha^{m-1}}\right) \tag{2.6}$$

$$(\beta^\alpha)\,|_u = \left(\beta|_{(u)\alpha^{-1}}\right)^{\alpha|_{(u)\alpha^{-1}}}, \text{where } \beta \in Stab(k) \text{ and } |u| \leq k. \tag{2.7}$$

An automorphism $\alpha \in \mathcal{A}_n$ corresponds to an input-output automaton with alphabet Y and with set of states $\mathrm{Q}(\alpha) = \{\alpha|_u \mid u \in \mathcal{M}\}$. The automaton α transforms the letters as follows: if the automaton is in state $\alpha|_u$ and reads a letter $i \in Y$ then it outputs the letter $j = (i)\,\alpha|_u$ and its state changes to $\alpha|_{ui}$; these operations can be best described by the labeled edge $\alpha|_u \xrightarrow{i|j} \alpha|_{ui}$. Following terminology of automata theory, every automorphism $\alpha|_u$ is called the *state* of α at u.

The tree T_n is a topological space which is the direct limit of its truncations at the n-th levels. Thus the group $\mathcal{A}_n$ is the inverse limit of the permutation groups it induces on the n-th level vertices. This transforms $\mathcal{A}_n$ into a topological group. An infinite product of elements $\mathcal{A}_n$ is a well-defined element of $\mathcal{A}_n$ provided that for any given level l, only finitely many of the elements in the product have non-trivial action on vertices at level l. Thus, if $\alpha \in \mathcal{A}_n$ and $\xi = \sum_{i \geq 0} a_i n^i \in \mathbb{Z}_n$ then $\alpha^{\xi} = \alpha^{a_0} \cdot \alpha^{na_1} \cdots \alpha^{n^i a_i} \cdots$ is a well defined element of $\mathcal{A}_n$. The notation $\alpha|_{\xi}$ is to be understood as $\alpha|_i$ where $i = a_0$.

The topological closure of a subgroup H in $\mathcal{A}_n$ will be indicated by $\overline{H}$. We note that if H is abelian then

$$\overline{H} = \{h^{\xi} \mid h \in H, \xi \in \mathbb{Z}_n \}.$$

One of the characterizing aspects of the n-ary adding machine is that the centralizer of τ is a pro-cyclic group; namely,

$$C_{\mathcal{A}_n}(\tau) = \overline{\langle \tau \rangle} = \{\tau^{\xi} \mid \xi \in \mathbb{Z}_n\}.$$

Let $v = yu$ where $y \in Y, u \in \mathcal{M}$. The image of v under the action of α is

$$(v)\alpha = (yu)\alpha = (y)\,\sigma_{\alpha}.(u)\alpha|_y.$$

The action extends to infinite sequences (or boundary points of the tree) in the same manner. A boundary point of the tree $c = c_0 c_1 c_2 \ldots$,where $c_i \in Y$ for all i, corresponds also to the n-adic integer $\xi = \sum\{c_i n^i | i \geq 0\} \in \mathbb{Z}_n$. Thus the action of the tree automorphism α can thus be translated to an action on the ring of n-adic integers. We will indicate c_0 by $\overline{\xi}$ which is ξ modulo n. In the case of the automorphism $\tau = (e, e, \ldots, e, \tau)\sigma$, the action of τ on c is

$$(c)\,\tau = \begin{cases} (c_0 + 1)\, c_1 c_2 \ldots & \text{if } 0 \leq c_0 \leq n-2, \\ 0(c_1 c_2 \ldots)\tau, & \text{if } c_0 = n-1, \end{cases}$$

which translates to the n-ary addition

$$\xi^{\tau} = \xi + 1.$$

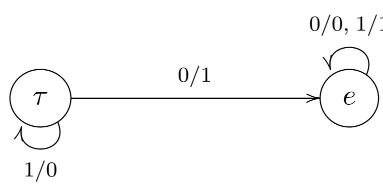

FIGURE 2. The binary adding machine

3. Normalizer of the topological closure $\overline{\langle \tau \rangle}$

An element $\xi = \sum_{i \geq 0} a_i n^i \in \mathbb{Z}_n$ is a unit in $\mathbb{Z}_n$ if and only if $\overline{\xi}\,(= a_0)$ is a unit in $\mathbb{Z}$ modulo n. The group of automorphisms of $\mathbb{Z}_n$ is isomorphic to the multiplicative group of units $U(\mathbb{Z}_n)$. The subgroup of $U(\mathbb{Z}_n)$ consisting of elements ξ with $\overline{\xi} = 1$ is denoted by by $\mathbb{Z}_n^1$. This subgroup has the transversal $\{j \mid 1 \leq j \leq n-1, \gcd(j, n) = 1\}$ in $U(\mathbb{Z}_n)$ and therefore has index $\left[U(\mathbb{Z}_n) : \mathbb{Z}_n^1\right] = \varphi(n)$ where φ is the Euler function.

Given $\alpha \in \mathcal{A}_n$ we denote the diagonal automorphism $(\alpha, ..., \alpha)$ by $\alpha^{(1)}$ and define inductively $\alpha^{(i+1)} = \left(\alpha^{(i)}\right)^{(1)}$ for all $i \geq 1$.

3.1. **Powers of τ.** Let $\xi = \sum_{i \geq 0} a_i n^i \in \mathbb{Z}_n$. Then $\sum_{i\geq 1} a_i n^{i-1} = \frac{\xi - \overline{\xi}}{n}$.

Lemma 3.1. *Let $\xi \in \mathbb{Z}_n$. Then*

$$\tau^{\xi} = (\tau^{\frac{\xi-a_0}{n}}, \ldots, \tau^{\frac{\xi-a_0}{n}}, \underbrace{\tau^{\frac{\xi-a_0}{n}+1}, \ldots, \tau^{\frac{\xi-a_0}{n}+1}}_{a_0 \text{ terms}})\sigma_\tau^{a_0}.$$

Proof. For j an integer with $1 \leq j \leq n-1$, we have

$$\tau^j = \left(e, \ldots, e, \underbrace{\tau, \ldots, \tau}_{j \text{ terms}} \right) \sigma_\tau^j$$

and $\tau^n = (\tau, \ldots, \tau) = \tau^{(1)}$.

Given $\xi = \sum_{i\geq 0} a_i n^i$, then

$$\tau^{a_0} = (e, \ldots, e, \underbrace{\tau, \ldots, \tau}_{a_0 \text{ terms}})\sigma_\tau^{a_0}, \tag{3.1}$$

$$\tau^{a_j n^j} = \tau^{(a_j n^{j-1})n} = \left(\tau^{a_j n^{j-1}}\right)^{(1)}, \tag{3.2}$$

$$\tau^{\xi} = (\tau^{\frac{\xi-a_0}{n}}, \ldots, \tau^{\frac{\xi-a_0}{n}}, \underbrace{\tau^{\frac{\xi-a_0}{n}+1}, \ldots, \tau^{\frac{\xi-a_0}{n}+1}}_{a_0 \text{ terms}})\sigma_\tau^{a_0} \tag{3.3}$$

$$= (\tau^{\frac{\xi-\overline{\xi}}{n}}, \ldots, \tau^{\frac{\xi-\overline{\xi}}{n}}, \underbrace{\tau^{\frac{\xi-\overline{\xi}}{n}+1}, \ldots, \tau^{\frac{\xi-\overline{\xi}}{n}+1}}_{\overline{\xi} \text{ terms}})\sigma_\tau^{\overline{\xi}}. \tag{3.4}$$

□

As we have seen, the description of τ^ξ involves the partition of the interval $[0, \ldots, n-1]$ into two subintervals. It is convenient to use here the carry 2-cocycle

$\delta : \mathbb{Z}_n \times \mathbb{Z}_n \to \{0, 1\}$ defined by

$$\delta(\eta, \kappa) = \frac{\overline{\eta} + \overline{\kappa} - \overline{\eta + \kappa}}{n} = \begin{cases} 0, & \text{if } \overline{\eta} + \overline{\kappa} < n \\ 1, & \text{otherwise} \end{cases}.$$

We call this 2-valued function by *Delta-2 (* later on we will introduce a 3-valued function *Delta-3)*.

Using *Delta-2*, the notation for the power of τ becomes

$$\tau^{\xi} = \left(\tau^{\frac{\xi - \overline{\xi}}{n} + \delta(i,\xi)} \right)_{0 \leq i \leq n-1} \sigma_\tau^{\overline{\xi}}. \tag{3.5}$$

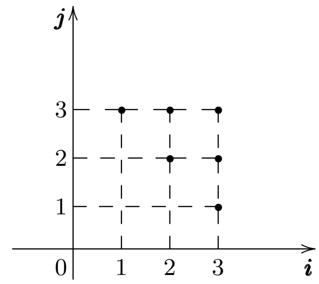

FIGURE 3. Delta 2 function for $n = 4$.

3.2. **Centralizer of τ.**

Lemma 3.2. $C_{\mathcal{A}_n}(\tau) = \overline{\langle \tau \rangle}$.

Proof. Let $\alpha \in \mathcal{A}_n$ commute with τ. Then, $[\sigma_\alpha, \sigma_\tau] = e$ and therefore $\sigma_\alpha = (\sigma_\tau)^{s_0}$ for some integer $0 \leq s_0 \leq n-1$. Therefore, $\beta = \alpha\tau^{-s_0} = (\beta|_0, ..., \beta|_{n-1})$ commutes with τ and $\sigma_\beta = e$. Now,

$$\beta^\tau = \left((\beta|_{n-1})^\tau, \beta|_0, \ldots, \beta|_{n-2}\right) = \beta$$

implies $\beta|_i = \beta|_0$ for all $0 \leq i \leq n-1$ and $\beta|_0$ commutes with τ. Therefore $\beta = (\beta|_0)^{(1)}$ and $\beta|_0$ replaces α in the previous argument. Hence, there exists an integer s_1 such that $0 \leq s_1 \leq n-1$ and $\gamma = \beta|_0\tau^{-s_1} = (\gamma|_0)^{(1)}$. From this we conclude

$$\begin{aligned} \alpha &= \beta\tau^{s_0} = (\beta|_0)^{(1)}\tau^{s_0} \\ &= \left((\gamma|_0)^{(1)}\tau^{s_1}, \ldots, (\gamma|_0)^{(1)}\tau^{s_1}\right)\tau^{s_0} \\ &= (\gamma|_0)^{(2)}\tau^{ns_1}\tau^{s_0} = (\gamma|_0)^{(2)}\tau^{ns_1+s_0}. \end{aligned}$$

We obtain the desired form inductively, $\alpha = \tau^\xi$ where

$$\xi = s_0 + s_1 n + s_2 n^2 + \ldots$$

□

The characterization of nilpotent groups which contain τ, announced in the introduction, follows.

Proposition 3.3. *Let G be a nilpotent subgroup of $\mathcal{A}_n$ which contains the n-adic adding machine τ. Then G is a subgroup of $\overline{<\tau>}$.*

Proof. Suppose G is a nilpotent group of class $k > 1$ which contains τ. Then, the center $Z(G)$ is contained in $\overline{\langle\tau\rangle}$. Let j be the maximum index such that $Z_j(G) \leq \overline{\langle\tau\rangle}$ and let $\alpha \in Z_{j+1}(G) \setminus \overline{\langle\tau\rangle}$. Then $[\tau, \alpha] \in Z_j(G)$ and therefore $[\tau, \alpha] = \tau^\xi$ for some $\xi \in \mathbb{Z}_n \setminus \{0\}$. Therefore

$$\begin{aligned} [\tau, 2\alpha] &= [\tau, \alpha, \alpha] = \left[\tau^\xi, \alpha\right] \\ &= [\tau, \alpha]^\xi = \tau^{\xi^2} \in Z_{j-1}(G) \end{aligned}$$

and more generally, for $l \geq 1$, we have $[\tau, l\alpha] = \tau^{\xi^l} \in Z_{j-l+1}(G)$. It follows that $\tau^{\xi^{j-1}} \in Z_0(G) = \{e\}$. Thus, $\xi^{j-1} = 0$ and $\xi = 0$; a contradiction. □

3.3. **Normalizer of $\overline{\langle\tau\rangle}$.**

Lemma 3.4. *The group $\Gamma_0 = N_{\mathcal{A}_n}\left(\overline{\langle\tau\rangle}\right)$ is metabelian. Indeed, the derived subgroup Γ_0' is contained in $\overline{\langle\tau\rangle}$.*

Proof. Let $\alpha, \beta \in \Gamma_0$, then $\tau^\alpha = \tau^\xi$ and $\tau^\beta = \tau^\eta$ for some $\eta, \xi \in U(\mathbb{Z}_n)$. Therefore,

$$\tau^\alpha = \tau^\xi, \tau = (\tau^\xi)^{\alpha^{-1}} = (\tau^{\alpha^{-1}})^\xi,$$
$$\tau^{\alpha^{-1}} = \tau^{\xi^{-1}}.$$

Likewise, $\tau^{\beta^{-1}} = \tau^{\eta^{-1}}$. Thus, $\tau^{[\alpha,\beta]} = \tau$ and $\Gamma_0' \le C_{\mathcal{A}_n}(\tau) = \overline{\langle\tau\rangle}$ follows. □

We present properties of the Delta-2 function which we will use in the sequel.

Lemma 3.5. *For all $0 \le i, j < n$ and $\xi \in \mathbb{Z}_n$ we have*

$$\sum_{i=0}^{n-1} \delta(i,j) = \overline{j},$$
$$\delta(i, j\xi) = j\left(\frac{\xi - \overline{\xi}}{n}\right) - \frac{j\xi - \overline{j\xi}}{n} + \sum_{k=0}^{j-1} \delta(i + k\xi, \xi).$$

Proof. The first assertion is easy to verify.

The second is obtained from

$$(\tau^\xi)^j|_i = (\tau^\xi)|_i (\tau^\xi)|_{i+\xi} \cdots (\tau^\xi)|_{i+(j-1)\xi},$$

by substituting

$$(\tau^\xi)|_i = \tau^{\frac{\xi-\overline{\xi}}{n} + \delta(i,\xi)}$$

in its right hand side and

$$\tau^{\xi j}|_i = \tau^{\frac{j\xi - \overline{j\xi}}{n} + \delta(i, j\xi)}$$

in its left. □

Proposition 3.6. *Given $\alpha \in \mathcal{A}_n$ and $\xi \in U(\mathbb{Z}_n)$. Then the condition $\tau^\alpha = \tau^\xi$ is equivalent to conditions (i), (ii) and (iii) below.*
(i)

$$\alpha|_i = (\alpha|_0)\, \tau^{\mu_i} \ \ (1 \le i \le n-1)$$

where

$$\mu_i = i\frac{(\xi - \overline{\xi})}{n} + \sum_{k=0}^{i-1} \delta((v(\alpha) + k)\xi, \xi)$$

and where $v(\alpha)$ is defined by

$$0 \le v(\alpha) \le n-1,$$
$$(0)\,\sigma_\alpha = \overline{v(\alpha)\xi};$$

(ii) (recursion)

$$\tau^{\alpha|_0} = \tau^{\xi};$$

(iii)

$$(j)\sigma_\alpha = \overline{(v(\alpha)+j)\xi} \ \ (0 \le j \le n-1)\,.$$

Furthermore, if $\xi \in \mathbb{Z}_n^1$*, then* $v(\alpha) = 0,\ (j)\sigma_\alpha = \overline{j\xi} = j$ *and* $\mu_i = i\frac{\xi-1}{n}$.

Proof. Since $\sigma_\tau^{\sigma_\alpha} = \sigma_\tau^{\xi}$, we have an equality between the permutations

$$((0)\,\sigma_\alpha, (1)\,\sigma_\alpha, \ldots, (n-1)\sigma_\alpha) = (0, \overline{\xi}, \overline{2\xi}, \ldots, \overline{(n-1)\xi}).$$

Therefore, there exists $v(\alpha) \in Y$ such that $(0)\,\sigma_\alpha = \overline{v(\alpha)\xi}$ and so,

$$(j)\sigma_\alpha = \overline{(v(\alpha)+j)\xi},\ \forall j \in Y.$$

Now, $\tau^\alpha = \tau^\xi$ is equivalent to $\alpha = \tau^{-s}\alpha\tau^{s\xi}$ for every $s \in \mathbb{Z}$, which in turn is equivalent to

$$\alpha|_{(i)\sigma_\tau^s} = ((\tau^s)|_i)^{-1}\,(\alpha|_i)\,(\tau^{\xi s})|_{(i)\sigma_\alpha}, \forall i \in Y, \forall s \in \mathbb{Z}.$$

The latter conditions are equivalent to

$$\begin{aligned}
\alpha|_0 &= \alpha|_{(0)\sigma_\tau^n} = ((\tau^n)|_0)^{-1}\,(\alpha|_0)\,(\tau^{\xi n})|_{(0)\sigma_\alpha},\\
\alpha|_i &= \alpha|_{(0)\sigma_\tau^i} = \left((\tau^i)|_0\right)^{-1}\alpha|_0(\tau^{\xi i})|_{(0)\sigma_\alpha}\ \forall i \in Y\backslash\{0\}
\end{aligned}$$

and these in turn are equivalent to

$$\begin{aligned}
\alpha|_i &= \alpha|_0\tau^{\frac{\xi i-\overline{\xi i}}{n}+\delta(v(\alpha)\xi,\xi i)} = \alpha|_0\tau^{\mu_i},\\
\mu_i &= i\left(\frac{\xi-\overline{\xi}}{n}\right) + \sum_{k=0}^{i-1}\delta((v(\alpha)+k)\xi,\xi)\ \forall i \in Y\backslash\{0\}.
\end{aligned}$$

If $\xi \in \mathbb{Z}_n^1$, then $\sum_{k=0}^{i-1}\delta(k\xi,\xi) = \sum_{k=0}^{i-1}\delta(k,1) = 0$. The rest of the assertion follows directly. □

Corollary 3.7. *Let* $\xi \in U(\mathbb{Z}_n), \sigma_\alpha$ *and* μ_i *be as above. Then* $\alpha = (\alpha)^{(1)}(e, \tau^{\mu_1}, \ldots, \tau^{\mu_{n-1}})\,\sigma_\alpha$ *conjugates* τ *to* τ^ξ*. In particular, if* $\xi \in \mathbb{Z}_n^1$*, then* $\alpha = (\alpha)^{(1)}(e, \tau^{\frac{\xi-1}{n}}, \tau^{2\frac{\xi-1}{n}}, \ldots, \tau^{(n-1)\frac{\xi-1}{n}})$ *(denoted by* λ_ξ*) conjugates* τ *to* τ^ξ.

Although we have computed above an automorphism which inverts τ, we give another with a simpler description. Define the permutation

$$\varepsilon = (0, n-1)\,(1, n-2)\ldots\left(\left[\frac{n-2}{2}\right], \left[\frac{n+1}{2}\right]\right).$$

Then ε inverts $\sigma_\tau = (0, 1, \ldots, n-1)$ and

$$\iota = \iota^{(1)}\varepsilon$$

inverts τ.

Define

$$\begin{aligned}
\Lambda &= \{\lambda_\xi \mid \xi \in \mathbb{Z}_n^1\},\\
\Psi &= \{\lambda_\xi\tau^t \mid \xi \in \mathbb{Z}_n^1,\ t \in \mathbb{Z}_n\}
\end{aligned}$$

and call Ψ the *monic normalizer* of $\overline{\langle\tau\rangle}$.

Proposition 3.8. *(i) Λ is an abelian group isomorphic to $\mathbb{Z}_n^1$;*
(ii) $\Psi = \Lambda \ltimes \overline{\langle\tau\rangle} \cong \mathbb{Z}_n^1 \ltimes \mathbb{Z}_n$;
(iii) on letting Ψ' denote the derived subgroup of Ψ, we have $\Psi' = \overline{\langle\tau^n\rangle}$.

Proof. (i) Let $\xi, \theta \in \mathbb{Z}_n^1$. Then, as $\lambda_\xi, \lambda_\theta$ and $\lambda_{\xi\theta}$ are inactive, it follows that

$$(\lambda_\xi\lambda_\theta\lambda_{\xi\theta}^{-1})|_i = (\lambda_\xi)|_i(\lambda_\theta)|_i\left((\lambda_{\xi\theta})|_i\right)^{-1}$$

$$= \lambda_\xi\tau^{i\frac{\xi-1}{n}}\lambda_\theta\tau^{i\frac{\theta-1}{n}}\left(\lambda_{\xi\theta}\tau^{i\frac{\xi\theta-1}{n}}\right)^{-1} = \lambda_\xi\lambda_\theta\lambda_\theta^{-1}\tau^{i\frac{\xi-1}{n}}\lambda_\theta\tau^{i\frac{\theta-1}{n}}\tau^{-i\frac{\xi\theta-1}{n}}\lambda_{\xi\theta}^{-1}$$

$$= \lambda_\xi\lambda_\theta\left(\tau^{i\theta\frac{\xi-1}{n}}\tau^{i\frac{\theta-1}{n}}\tau^{-i\frac{\xi\theta-1}{n}}\right)\lambda_{\xi\theta}^{-1} = \lambda_\xi\lambda_\theta\lambda_{\xi\theta}^{-1}, \forall i \in \{0, \ldots, n-1\}.$$

Therefore, $\lambda_\xi\lambda_\theta = \lambda_{\xi\theta}$. In addition, $\lambda_\xi = e$ if and only if $\xi = 1$.

(ii) This factorization is clear.

(iii) Let $\theta = 1 + \theta' n, \eta \in \mathbb{Z}_n$. Then

$$\begin{aligned} [\tau^\eta, \lambda_\theta] &= \tau^{-\eta}\lambda_{\theta^{-1}}\tau^\eta\lambda_\theta = \\ \tau^{-\eta}\tau^{\eta\theta} &= \tau^{\eta(\theta-1)} = (\tau^n)^{\eta\theta'} . \end{aligned}$$

□

We prove below the existence of conjugates τ^α of τ in $N_{\mathcal{A}_n}\left(\overline{\langle\tau\rangle}\right)$, which lie outside $\overline{\langle\tau\rangle}$. This fact allows us to construct the first important type of metabelian groups $\overline{\langle\tau\rangle}\langle\tau^\alpha\rangle$ containing τ.

Proposition 3.9. *Given $\xi, \rho \in \mathbb{Z}_n^1$ with $\xi \neq 1$. Then for all n odd and for all n even such that $2n \mid (\xi - 1)$, an element $\alpha = (\alpha|_0, \ldots, \alpha|_{n-1})$ in A_n satisfies $\tau^\alpha = \lambda_\xi\tau^\rho$ if and only if*

$$\begin{cases} \alpha|_{i+1} = (\alpha|_0)\,\lambda_{\xi^{i+1}}\tau^{\frac{1}{n}\left[\rho\frac{\xi^{i+1}-1}{\xi-1}-(i+1)\right]} & (0 \le i \le n-2), \\ \tau^{\alpha|_0} = \lambda_{\xi^n}\tau^{\frac{1}{n}\left[\rho\frac{\xi^n-1}{\xi-1}\right]}. \end{cases}$$

Proof. From $\tau^\alpha = \lambda_\xi\tau^{1+\kappa n}$, we obtain using (2.4),

$$\begin{cases} \lambda_\xi\tau^{i\frac{\xi-1}{n}+\kappa} = \left(\alpha|_i^{-1}\right)\alpha|_{i+1}, & \text{if } i \in Y - \{n-1\} \\ \lambda_\xi\tau^{(n-1)\frac{\xi-1}{n}+\kappa+1} = \left(\alpha|_{n-1}^{-1}\right)\tau\left(\alpha|_0\right). \end{cases}$$

Therefore,

$$\begin{aligned} &\alpha|_{i+1} = (\alpha|_0)\,\lambda_\xi\tau^\kappa\lambda_\xi\tau^{\frac{\xi-1}{n}+\kappa}\cdots\lambda_\xi\tau^{i\frac{\xi-1}{n}+\kappa}, \ \text{ for } i = 0, 1, \ldots, n-2, \\ &\alpha|_0 = \tau^{-1}\left(\alpha|_{n-1}\right)\lambda_\xi\tau^{(n-1)\frac{\xi-1}{n}+\kappa+1}. \end{aligned}$$

The first equations can be expressed as

$$\begin{aligned} \alpha|_{i+1} &= (\alpha|_0)\,\lambda_{\xi^{i+1}}\tau^{\kappa\left(\sum_{j=0}^{i}\xi^j\right)+\frac{\xi-1}{n}\xi^i\left(\sum_{j=1}^{i}j(\xi^{-1})^j\right)} \\ &= (\alpha|_0)\,\lambda_{\xi^{i+1}}\tau^{\frac{1}{n}\left[(1+\kappa n)\frac{\xi^{i+1}-1}{\xi-1}-(i+1)\right]} \end{aligned}$$

and the last as

$$\begin{aligned}\alpha|_0 &= \tau^{-1}(\alpha|_0)\,\lambda_{\xi^n}\tau^{\frac{\xi}{n}\left[(1+\kappa n)\frac{\xi^{n-1}-1}{\xi-1}-(n-1)\right]}\tau^{(n-1)\frac{\xi-1}{n}+\kappa+1}\\ &= \tau^{-1}(\alpha|_0)\,\lambda_{\xi^n}\tau^{\frac{1}{n}\left[(1+\kappa n)\frac{\xi^n-1}{\xi-1}\right]}.\end{aligned}$$

Now, we need to show that $\tau^{\alpha|_0} = \lambda_{\xi^n}\tau^{\frac{1}{n}\left[(1+\kappa n)\frac{\xi^n-1}{\xi-1}\right]}$ satisfies the same conditions as those for α; that is, both ξ^n, $\rho' = \frac{1}{n}\left[(1+\kappa n)\frac{\xi^n-1}{\xi-1}\right] \in \mathbb{Z}_n^1$.

Of course, $\xi^n \in \mathbb{Z}_n^1$, so let us consider $\rho(\xi^n-1)/n(\xi-1)$. Since $\xi \in \mathbb{Z}_n^1$, we can write $\xi = 1+\ell n$, and then

$$\frac{\xi^n-1}{\xi-1} \equiv n + \binom{n}{2}\ell n \pmod{n^2},$$

by using the *Binomial Theorem*. Since $\rho \equiv 1 \pmod{n}$, it follows that

$$\frac{\rho(\xi^n-1)}{n(\xi-1)} \equiv 1 + \binom{n}{2}\ell \pmod{n},$$

and consequently, $\rho(\xi^n-1)/n(\xi-1) \in \mathbb{Z}_n^1$ if and only if $n \mid \binom{n}{2}\ell$ (that is, if and only if $(n-1)\ell$ is even). So $\rho(\xi^n-1)/n(\xi-1) \in \mathbb{Z}_n^1$ holds for odd n, and for even n provided that $2n \mid (\xi-1)$. □

4. Abelian groups B normalized by τ

Let B be an abelian subgroup of $\mathcal{A}_n$ normalized by τ. For a fixed $\beta \in B$, we define the '1st level state closure' of $\langle \beta, \tau\rangle$ as the group

$$H = \langle \beta|_i \ (i \in Y), \tau\rangle .$$

We will be dealing frequently with the following subgroups of H ,

$$\begin{aligned} N &= \left\langle [\beta|_i, \tau^{k_i}] \mid k_i \in \mathbb{Z}, i \in Y \right\rangle \\ M &= N\langle\tau\rangle . \end{aligned}$$

When $\sigma_\beta = (\sigma_\tau)^s$ *for some integer* s, $m = \frac{n}{\gcd(n,s)}$ and

$$\pi_i = \beta|_i\beta|_{i+s}\beta|_{i+2s}\cdots\beta|_{i+(m-1)s}$$

we will also be dealing with the subgroups

$$\begin{aligned} K &= \langle N,\ \pi_i \mid i \in Y\rangle ,\\ O &= K\langle\tau\rangle . \end{aligned}$$

We show below that when n is a power of a prime number p^k, the activity range of β narrows down to a Sylow p-subgroup of Σ_n. This is used to restrict the location of an abelian group B normalized by τ, within $\mathcal{A}_n$.

Proposition 4.1. *Let $n = p^k$, $\sigma = (0, 1, ..., n-1)$ and P be a Sylow p-subgroup P of Σ_n which contains σ. Then*
(i) P is isomorphic to $((...(...C_p)\wr)C_p)\wr C_p$, a wreath product of the cyclic group C_p of order p iterated $k-1$ times; the normalizer of P in Σ_n is $N_{\Sigma_n}(P) = P\langle c\rangle$ where c is cyclic of order $p-1$;
(ii) P is the unique Sylow p-subgroup P of Σ_n which contains σ;
(iii) if W is an abelian subgroup of Σ_n normalized by σ then W is contained in P.

Proof. (i) The structure of P as an iterated wreath product is well-known. The center of P is $Z = \left\langle z\left(=\sigma^{p^{k-1}}\right)\right\rangle$ and $C_{\Sigma_n}(z) = P$. Therefore, $N_{\Sigma_n}(P) = N_{\Sigma_n}(Z) = P\langle c\rangle$ where c is cyclic of order $p-1$.

(ii) If $\sigma \in P^g$ for some $g \in \Sigma_n$ then $z^g \in C_{\Sigma_n}(\sigma) = \langle\sigma\rangle$ and therefore $\langle z^g\rangle = \langle z\rangle$, $P^g = P$. Thus, P is the unique Sylow p-subgroup of Σ_n to contain σ.

(iii) Let W be an abelian subgroup of Σ_n normalized by σ. Let $V = W\langle\sigma\rangle$ and V_0 be the stabilizer of 0 in V. Then, since σ is a regular cycle, it follows that $V = V_0\langle\sigma\rangle$, $V_0 \cap \langle\sigma\rangle = \{e\}$. Suppose that there exists a prime q different from p which divides the order of W and let Q be the unique Sylow q-subgroup of W. Then Q is the unique Sylow q-subgroup of V and $Q \leq V_0$. Therefore, $Q = \{e\}$ and W is a p-group. As $\sigma \in V$, we conclude $W \leq P$. □

Lemma 4.2. *(a) Let $\gamma \in \mathcal{A}_n$. Conditions (i), (ii) below are equivalent:*
(i) $[\gamma, \gamma^{\tau^k}] = e$ for all $k \in \mathbb{Z}$;
(ii) $[\tau^k, \gamma, \gamma] = e$ for all $k \in \mathbb{Z}$.
Condition (i) implies
(iii) $\left\langle [\gamma, \tau^k] \mid k \in \mathbb{Z}\right\rangle$ is a commutative group.
Condition (iii) implies
$\left\langle [\gamma|_u, \tau^k] \mid k \in \mathbb{Z}\right\rangle$ is a commutative group for all indices u.
(b) Let $n = p^k$. Then any abelian subgroup B normalized by τ is contained in the layer closure $L = L(N_{\Sigma_n}(P))$.

Proof. (a) First,

$$\begin{aligned}[\gamma, \gamma^{\tau^k}] &= \gamma^{-1}\left(\tau^{-k}\gamma^{-1}\tau^k\right)\gamma\left(\tau^{-k}\gamma\tau^k\right) \\ &= \gamma^{-1}\left(\tau^{-k}\gamma^{-1}\tau^k\gamma\right)\gamma\left(\gamma^{-1}\tau^{-k}\gamma\tau^k\right) \\ &= [\tau^k, \gamma]^\gamma[\gamma, \tau^k]\end{aligned}$$

and so,

$$[\gamma, \gamma^{\tau^k}] = e \Leftrightarrow [\gamma, \tau^k]^\gamma = [\gamma, \tau^k].$$

Furthermore, since

$$[\gamma, \tau^{k_1}]^{\tau^{k_2}} = [\gamma, \tau^{k_2}]^{-1}[\gamma, \tau^{k_1+k_2}] \tag{4.1}$$

for all integers k_1, k_2, condition (ii) implies

$$\begin{aligned} [\gamma, \tau^{k_1}]^{[\gamma,\tau^{k_2}]} &= [\gamma, \tau^{k_1}]^{\gamma^{-1}\tau^{-k_2}\gamma\tau^{k_2}} = [\gamma, \tau^{k_1}]^{\tau^{-k_2}\gamma\tau^{k_2}} \\ &= \left([\gamma, \tau^{-k_2}]^{-1}[\gamma, \tau^{k_1-k_2}]\right)^{\gamma\tau^{k_2}} = \left([\gamma, \tau^{-k_2}]^{-1}[\gamma, \tau^{k_1-k_2}]\right)^{\tau^{k_2}} \\ &= [\gamma, \tau^{k_1}]. \end{aligned}$$

Finally, we note that by (2.5),

$$\begin{aligned} ([\gamma, \tau^{nk}])|_{(i)\sigma_\gamma} &= (\gamma^{-1})|_{(i)\sigma_\gamma} (\tau^{-nk})|_i \, (\gamma|_i)\, (\tau^{nk})|_{(i)\sigma_\gamma} \\ &= \left(\gamma|_i^{-1}\right) \tau^{-k} \, (\gamma|_i)\, \tau^k \\ &= [\gamma|_i, \tau^k]. \end{aligned}$$

Since $[\gamma, \tau^{kn}]$ is inactive for all $k \in \mathbb{Z}$, we obtain $\{[\gamma|_i, \tau^k] \mid k \in \mathbb{Z}\}$ is a commutative set for all i. The rest of the assertion follows by induction on the tree level.

(b) Let $\beta \in B$. Since the normal closure of $\langle \sigma_\beta \rangle$ under the action of $\langle \sigma_\tau \rangle$ is an abelian subgroup, it follows that $\sigma_\beta \in P$. Furthermore, as $\left\langle [\beta|_u, \tau^k] \mid k \in \mathbb{Z} \right\rangle$ is an abelian group normalized by τ, it follows that $[\sigma_{\beta|_u}, \sigma] \in P$ and therefore $\sigma^{\sigma_{\beta|u}} \in P$. Thus, we conclude $\sigma_{\beta|_u} \in N_{\Sigma_n}(P)$ and $\beta \in L$. □

Proposition 4.3. *Let $l \geq 1$ and suppose $\alpha, \gamma \in \mathrm{Stab}(l)$ satisfy $[\alpha, \gamma^{\tau^x}] = e$ for all $x \in \mathbb{Z}$. Then*

$$[\alpha|_u, (\gamma|_v)^{\tau^x}] = e, \forall u, v \in \mathcal{M} \text{ having } |u| = |v| \leq l \text{ and } \forall x \in \mathbb{Z}.$$

Proof. We start with the case $l = 1$. Write $x = r + kn$ where $r = \overline{x}$.

By (2.4),

$$\begin{aligned} \left(\gamma^{\tau^x}\right)|_{(i)\tau^x} &= \left((\tau^x)|_i^{-1}\right) (\gamma|_i)\, (\tau^x)|_i, \\ \left(\gamma^{\tau^x}\right)|_i &= \tau^{-k-\delta(i-r,r)} \, (\gamma|_{i-r})\, \tau^{k+\delta(i-r,r)}. \end{aligned}$$

As $[\alpha, \gamma^{\tau^x}] = e$ and $\alpha, \gamma^{\tau^x} \in \mathrm{Stab}(1)$,we have, for all $i, j, r \in Y$ and all $k, x \in \mathbb{Z}$,

$$\begin{aligned} [\alpha|_i, (\gamma^{\tau^x})|_i] &= e, \ [\alpha|_i, (\gamma|_{i-r})^{\tau^{k+\delta(i-r,r)}}] = e, \\ [\alpha|_i, (\gamma|_j)^{\tau^x}] &= e. \end{aligned}$$

The general case $l \geq 1$ follows by induction. □

We apply the above proposition to $\beta \in B$.

Corollary 4.4. *Let $\sigma_\beta = e$. Then for all $i, j \in Y$ and for all $x \in \mathbb{Z}$*

$$[(\beta|_i), (\beta|_j)^{\tau^x}] = e.$$

We derive further relations in $H = \langle \beta|_i \ (i \in Y), \tau \rangle$.

Proposition 4.5. *Let $\beta \in B$. Then the following relations hold in H for all $v \in \mathbb{Z}$ and for all $i \in Y$:*

(I)

$$\begin{aligned} & \left(\tau^v|_{(i)\sigma_\tau^{-v}}\right)^{-1}\left(\beta|_{(i)\sigma_\tau^{-v}}\right)\left(\tau^v|_{(i)\sigma_\tau^{-v}\sigma_\beta}\right)\left(\beta|_{(i)\sigma_\tau^{-v}\sigma_\beta\sigma_\tau^v}\right) \\ = \quad & (\beta|_i)\left(\tau^v|_{(i)\sigma_\beta\sigma_\tau^{-v}}\right)^{-1}\left(\beta|_{(i)\sigma_\beta\sigma_\tau^{-v}}\right)\left(\tau^v|_{(i)\sigma_\beta\sigma_\tau^{-v}\sigma_\beta}\right), \end{aligned}$$

$$[\sigma_\beta, \sigma_\beta^{\sigma_\tau^v}] = e;$$

(II)

$$[\beta|_i, \tau^v]^{\beta|_{(i)\sigma_\beta}} = [\beta|_{(i)\sigma_\beta}, \tau^v];$$

(III)

$$\left(\beta|_{(i)\sigma_\beta}\right)\left(\beta|_{(i)\sigma_\beta^2}\right)\cdots\left(\beta|_{(i)\sigma_\beta^{s_i}}\right) \text{ commutes with } [\beta|_i, \tau^v]$$

where s_i is the size of the orbit of i under the action of $\langle\sigma_\beta\rangle$.

Proof. (I) Clearly $[\beta, \beta^{\tau^v}] = e$ implies $[\sigma_\beta, \sigma_\beta^{\sigma_\tau^v}] = e$. It also implies

$$\left(\beta|_{(i)\sigma_{\beta^{\tau^v}}}\right)^{-1}\left(\beta^{\tau^v}|_i\right)^{-1}(\beta|_i)\left(\beta^{\tau^v}|_{(i)\sigma_\beta}\right) = e,$$

$$\left(\beta^{\tau^v}|_i\right)\left(\beta|_{(i)\sigma_{\beta^{\tau^v}}}\right) = (\beta|_i)\left(\beta^{\tau^v}|_{(i)\sigma_\beta}\right),$$

$$\begin{aligned} & \left(\tau^v|_{(i)\sigma_{\tau^v}^{-1}}\right)^{-1}\left(\beta|_{(i)\sigma_{\tau^v}^{-1}}\right)\left(\tau^v|_{(i)(\sigma_{\tau^v}^{-1})(\sigma_\beta)}\right)\left(\beta|_{(i)\sigma_{\beta^{\tau^v}}}\right) \\ = \quad & (\beta|_i)\left(\tau^v|_{(i)(\sigma_\beta)(\sigma_{\tau^v}^{-1})}\right)^{-1}\left(\beta|_{(i)(\sigma_\beta)(\sigma_{\tau^v}^{-1})}\right)\left((\tau^v)|_{(i)\sigma_\beta(\sigma_{\tau^v}^{-1})\sigma_\beta}\right). \end{aligned}$$

(II) On changing v to nv in (I), we obtain:

$$\tau^{-v}(\beta|_i)\tau^v\left(\beta|_{(i)\sigma_\beta}\right) = (\beta|_i)\tau^{-v}\left(\beta|_{(i)\sigma_\beta}\right)\tau^v,$$

$$\begin{aligned} & \left(\beta|_{(i)\sigma_\beta}\right)^{-1}\left(\left(\beta|_i^{-1}\right)\tau^{-v}(\beta|_i)\tau^v\right)\left(\beta|_{(i)\sigma_\beta}\right) \\ = \quad & \left(\left(\beta|_{(i)\sigma_\beta}\right)^{-1}\left(\beta|_i^{-1}\right)\right)(\beta|_i)\tau^{-v}\left(\beta|_{(i)\sigma_\beta}\right)\tau^v. \end{aligned}$$

(III) From (II), we derive

$$[\beta|_i, \tau^v]^{\left(\left(\beta|_{(i)\sigma_\beta}\right)\left(\beta|_{(i)\sigma_\beta^2}\right)\cdots\left(\beta|_{(i)\sigma_\beta^{s_i}}\right)\right)} = [\beta|_{(i)\sigma_\beta}, \tau^v]^{\left(\left(\beta|_{(i)\sigma_\beta^2}\right)\cdots\left(\beta|_{(i)\sigma_\beta^{s_i}}\right)\right)} = \ldots = [\beta|_i, \tau^v].$$

□

5. The case $\beta \in B$ with $\sigma_\beta \in \langle \sigma_\tau \rangle$

This section is devoted to the proof of the second part of Theorem B. For this purpose, we introduce the following 3-variable combination of Delta-2 functions

$$\Delta_s(i,t) = \delta(i, t-i) - \delta(i-s, t-i)$$

which we call the *Delta-3* function.

Lemma 5.1. *Let $\beta \in \mathcal{A}_n$ such that $[\beta, \beta^{\tau^x}] = e$ for any $x \in \mathbb{Z}$ and let $\sigma_\beta = \sigma_\tau^s$ for some $s \in Y$. Then,*

$$\begin{aligned} & \tau^{\Delta_s(i,t)} \left(\beta|_{i-s}\right) [\beta|_{i-s}, \tau^z] \left(\beta|_t\right) \\ = \; & \left(\beta|_{t-s}\right) \left(\beta|_i\right) [\beta|_i, \tau^z] \tau^{\Delta_s(i+s,t+s)}, \end{aligned}$$

for all $i, t \in \{0, 1, \ldots, n-1\}, z \in \mathbb{Z}$.

Proof. Since $\sigma_\beta = \sigma_\tau^s$, we have $\sigma_{\beta^{\tau^x}} = \sigma_\beta = \sigma_\tau^s$.

From (2.4) and (2.5), we obtain

$$\begin{aligned} & \tau^{-\frac{x-\overline{x}}{n} - \delta(j-x,x)} \left(\beta|_{j-x}\right) \tau^{\frac{x-\overline{x}}{n} + \delta(j-x+s,x)} \left(\beta|_{j+s}\right) \\ = \; & \left(\beta|_j\right) \tau^{-\frac{x-\overline{x}}{n} - \delta(j+s-x,x)} \left(\beta|_{j+s-x}\right) \tau^{\frac{x-\overline{x}}{n} + \delta(j+2s-x,x)} \end{aligned} \tag{5.1}$$

Setting $k = \dfrac{x - \overline{x}}{n}$ and $r = \overline{x}$ and using (5.1), we have

$$\begin{aligned} & \tau^{-k-\delta(j-r,r)} \left(\beta|_{j-r}\right) \tau^{k+\delta(j+s-r,r)} \left(\beta|_{j+s}\right) \\ & = \left(\beta|_j\right) \tau^{-k-\delta(j+s-r,r)} \left(\beta|_{j+s-r}\right) \tau^{k+\delta(j+2s-r,r)}, \end{aligned} \tag{5.2}$$

for all $r, j \in Y$ and all $k \in \mathbb{Z}$.

Also on setting $t = \overline{j+s}, i = \overline{j+s-r}$ and $z = k + \delta(j+s-r, r)\,(= k + \delta(i, t-i))$ and using (5.2), we obtain

$$\begin{aligned} & \tau^{-z+\delta(i,t-i)-\delta(i-s,t-i)} \beta|_{i-s} \tau^z \beta|_t \\ = \; & \beta|_{t-s} \tau^{-z} \beta|_i \tau^{z-\delta(i,t-i)+\delta(i+s,t-i)}, \end{aligned}$$

for all $t, i \in \{0, 1, \ldots, n-1\}$ and all $z \in \mathbb{Z}$.

It follows that

$$\begin{aligned} & \tau^{\delta(i,t-i)-\delta(i-s,t-i)} \left(\beta|_{i-s}\right) [\beta|_{i-s}, \tau^z] \left(\beta|_t\right) \\ = \; & \left(\beta|_{t-s}\right) \left(\beta|_i\right) [\beta|_i, \tau^z] \tau^{-\delta(i,t-i)+\delta(i+s,t-i)} \end{aligned}$$

for all $t, i \in \{0, 1, \ldots, n-1\}$ and all $z \in \mathbb{Z}$. □

We develop below some properties of the Δ_s function to be used in the sequel.

Proposition 5.2. *The Delta-3 function satisfies*

(i) $\Delta_s(i,t) = \delta(i,-s) - \delta(t,-s) = \begin{cases} 0, & \text{if } \overline{t}, \overline{i} \geq \overline{s} \text{ or } \overline{t}, \overline{i} < \overline{s} \\ 1, & \textit{if } \overline{t} < \overline{s} \leq \overline{i} \\ -1, & \textit{if } \overline{i} < \overline{s} \leq \overline{t} \end{cases}$,

(ii) $\Delta_s(i,t) = -\Delta_s(t,i)$,

(iii) $\Delta_s(i+s,t+s) = -\Delta_{-s}(i,t)$,

(iv) $\Delta_s(i,t) = \Delta_s(i,z) + \Delta_s(z,t)$,

(v) $\sum_{k=0}^{\frac{n}{(s,n)}-1} \Delta_s(i+ks,t+ks) = 0$,

(vi) $\sum_{k=0}^{n-1} \Delta_s(k,t) = \begin{cases} n-\overline{s}, & \textit{if } \overline{t} < \overline{s} \\ -\overline{s} & \textit{if } \overline{t} \geq \overline{s} \end{cases}$

for all $i,t,z \in \mathbb{Z}$.

Proof.

(i) Using the definition $\delta(i,j) = \frac{\overline{i}+\overline{j}-\overline{i+j}}{n}$ we have

$$\begin{aligned}
\Delta_s(i,t) &= \frac{\overline{i}+\overline{t-i}-\overline{t}}{n} - \frac{\overline{i-s}+\overline{t-i}-\overline{t-s}}{n} \\
&= \frac{\overline{i}+\overline{-s}-\overline{i-s}}{n} - \frac{\overline{t}+\overline{-s}-\overline{t-s}}{n} \\
&= \delta(i,-s) - \delta(t,-s) \\
&= \begin{cases} 0, & \text{if } \overline{t},\overline{i} \geq \overline{s} \text{ or } \overline{t},\overline{i} < \overline{s} \\ 1, & \text{if } \overline{t} < \overline{s} \leq \overline{i} \\ -1, & \text{if } \overline{i} < \overline{s} \leq \overline{t} \end{cases}.
\end{aligned}$$

(ii) Follows from (i).

(iii) Calculate

$$\begin{aligned}
\Delta_s(i+s,t+s) &= \delta(i+s,t-i) - \delta(i,t-i) \\
&= -\left(\delta(i,t-i) - \delta(i+s,t-i)\right) \\
&= -\Delta_{-s}(i,t).
\end{aligned}$$

(iv) This part follows from (i).

(v) From the definition of the *Delta-2* function

$$\sum_{k=0}^{\frac{n}{(n,s)}-1} \delta(i+ks,t-i) = \sum_{k=0}^{\frac{n}{(n,s)}-1} \delta(i+(k-1)s,t-i).$$

(vi) Finally, we have

$$\begin{aligned}
\sum_{k=0}^{n-1} \Delta_s(k,t) &= \textstyle\sum_{k=0}^{\overline{s}-1} \Delta_s(k,t) + \sum_{k=\overline{s}}^{n-1} \Delta_s(k,t) \\
&\overset{(i)}{=} \begin{cases} n-\overline{s}, & \text{if } \overline{t} < \overline{s} \\ -\overline{s}, & \text{if } \overline{t} \geq \overline{s} \end{cases}.
\end{aligned}$$

□

With the use of the *Delta-3* function we obtain

Proposition 5.3. *The following relations are verified in* $H = \langle \beta|_i \ (i \in Y), \tau \rangle$, *for all* $x, z \in \mathbb{Z}$ *and all* $i, t \in Y$:

(I) $\tau^{\Delta_s(i,t)} (\beta|_{i-s}) (\beta|_t) = (\beta|_{t-s}) (\beta|_i) \tau^{\Delta_s(i+s,t+s)}$;

(II) $[(\beta|_{i-s}), \tau^z]^{(\beta|_t)\tau^{-\Delta_s(i+s,t+s)}} = [\beta|_i, \tau^z]$;

(III) $[[\beta|_i, \tau^z], [\beta|_t, \tau^x]] = e$.

Proof. Returning to Lemma 5.1, we have

$$\begin{aligned} & \tau^{\Delta_s(i,t)} (\beta|_{i-s}) [\beta|_{i-s}, \tau^z] (\beta|_t) \\ = \ & (\beta|_{t-s}) (\beta|_i) [\beta|_i, \tau^z] \tau^{\Delta_s(i+s,t+s)}. \end{aligned}$$

Consequently,

$$\tau^{\Delta_s(i,t)} (\beta|_{i-s}) (\beta|_t) = (\beta|_{t-s}) (\beta|_i) \tau^{\Delta_s(i+s,t+s)} \tag{5.3}$$

and

$$[\beta|_{i-s}, \tau^z]^{(\beta|_t)\tau^{-\Delta_s(i+s,t+s)}} = [\beta|_i, \tau^z], \tag{5.4}$$

for all $t, i \in Y$ and all $z \in \mathbb{Z}$.

From (5.4) and (4.1), $N = \left\langle [\beta|_i, \tau^{k_i}] \mid k_i \in \mathbb{Z}, i \in Y \right\rangle$ is a normal subgroup of H. Moreover, by applying alternately the above equations, we obtain

$$\begin{aligned} [\beta|_i, \tau^z]^{[\beta|_t, \tau^k]} &= [\beta|_i, \tau^z]^{(\beta|_t^{-1})\tau^{-k}(\beta|_t)\tau^k} \\ &= [\beta|_i, \tau^z]^{\left(\tau^{-\Delta_s(i+s,t+s)}\tau^{\Delta_s(i+s,t+s)}(\beta|_t^{-1})\tau^{-k}(\beta|_t)\tau^k\right)} \\ &\overset{(4.1)}{=} \left([\beta|_i, \tau^{-\Delta_s(i+s,t+s)}]^{-1}.[\beta|_i, \tau^{z-\Delta_s(i+s,t+s)}]\right)^{\left(\tau^{\Delta_s(i+s,t+s)} (\beta|_t^{-1}) \tau^{-k} (\beta|_t) \tau^k\right)} \\ &\overset{(5.4)}{=} \left([\beta|_{i-s}, \tau^{-\Delta_s(i+s,t+s)}]^{-1}.[\beta|_{i-s}, \tau^{z-\Delta_s(i+s,t+s)}]\right)^{\tau^{-k} (\beta|_t) \tau^k} \\ &\overset{(4.1)}{=} \begin{pmatrix} \left([\beta|_{i-s}, \tau^{-k}]^{-1}.[\beta|_{i-s}, \tau^{-k-\Delta_s(i+s,t+s)}]\right)^{-1} \\ \left([\beta|_{i-s}, \tau^{-k}]^{-1}.[\beta|_{i-s}, \tau^{-k+z-\Delta_s(i+s,t+s)}]\right) \end{pmatrix}^{\beta|_t \tau^k} \\ &= \left([\beta|_{i-s}, \tau^{-k-\Delta_s(i+s,t+s)}]^{-1}.[\beta|_{i-s}, \tau^{-k+z-\Delta_s(i+s,t+s)}]\right)^{(\beta|_t) \tau^k} \\ &\overset{(5.4)}{=} \left([\beta|_i, \tau^{-k-\Delta_s(i+s,t+s)}]^{-1}.[\beta|_i, \tau^{-k+z-\Delta_s(i+s,t+s)}]\right)^{\tau^{k+\Delta_s(i+s,t+s)}} \\ &\overset{(4.1)}{=} [\beta|_i, \tau^z]. \end{aligned}$$

□

Corollary 5.4. *Let $\beta \in A_n$ such that $[\beta, \beta^{\tau^x}] = e$ for every $x \in \mathbb{Z}$ with $\sigma_\beta = \sigma_\tau^s$ for some $s \in \{0, 1, \ldots, n-1\}$. Then*

$$M = \left\langle [\beta|_i, \tau^{k_i}], \tau \mid k_i \in \mathbb{Z}, 0 \leq i \leq n-1 \right\rangle$$

is a normal metabelian subgroup of H.

Proof. By Proposition 5.3, $N = \langle [\beta|_i, \tau^{k_i}] \mid k_i \in \mathbb{Z}, 0 \leq i \leq n-1 \rangle$ is abelian and normal in H. Since $N\tau \in Z(H/N)$, it follows that $M = N\langle \tau \rangle$ is a normal subgroup of H and is clearly metabelian. □

We are ready to prove part *(I) of* Theorem B.

Theorem 5.5. *Let $\beta \in \mathcal{A}_n$ be such that $[\beta, \beta^{\tau^x}] = e, \forall x \in \mathbb{Z}$ and $\sigma_\beta = \sigma_\tau^s$ for some $s \in Y$ and $H = \langle \beta|_0, \ldots, \beta|_{n-1}, \tau \rangle$. Recall $\pi_j = \beta|_j \beta|_{j+s} \beta|_{j+2s} \cdots \beta|_{j+(m-1)s}$. Then,*
(i) the group $K = \langle [\beta|_i, \tau^x], \pi_j \mid i, j \in Y, x \in \mathbb{Z}_n \rangle$ is an abelian normal subgroup of H and the group $O = K\langle \tau \rangle$ is a metabelian normal subgroup of H;
(ii) the quotient group H/O is a homomorphic image of a subgroup of $C_m \wr C_n$.

In particular, H is metabelian-by-finite.

Proof. (i) Recall

$$\begin{aligned} N &= \left\langle [\beta|_i, \tau^{k_i}] \mid k_i \in \mathbb{Z}, i \in Y \right\rangle, \\ K &= N \langle \pi_j \mid j \in Y \rangle \end{aligned}$$

where $m = \frac{n}{\gcd(n,s)}$. Then, by Proposition 5.3, N is an abelian normal subgroup of H.

By (5.4), we have

$$\begin{aligned} & [\beta|_i, \tau^z]^{\pi_j} \\ = \; & [\beta|_{i+s}, \tau^z]^{\tau^{\Delta_s(i+2s, j+s)} \beta|_{j+s} \cdots \beta|_{j+(m-1)s}} \\ = \; & [\beta|_{i+2s}, \tau^z]^{\tau^{\Delta_s(i+2s,j+s) + \Delta_s(i+3s,j+2s)} \beta|_{j+2s} \cdots \beta|_{j+(m-1)s}} \\ = \; & [\beta|_i, \tau^z]^{\tau^{\left(\sum_{k=0}^{m-1} \Delta_s(i+(k+1)s, j+ks)\right)}} \\ \overset{\text{Prop.5.2}(v)}{=} \; & [\beta|_i, \tau^z]. \end{aligned}$$

Thus,

$$[[\beta|_i, \tau^z], (\beta^m)|_j] = e, \forall i, j \in Y, \forall z \in \mathbb{Z} \tag{5.5}$$

Since $\sigma_{\beta^m} = e$, we have by Corollary 4.4

$$[(\beta^m)|_i, (\beta^m)|_j] = e, \forall i, j \in Y. \tag{5.6}$$

Moreover,

$$(\beta^m)|_i^\tau = ((\beta^m)|_i)\,[(\beta^m)|_i, \tau]. \tag{5.7}$$

Since $[\beta, \beta^{\tau^x}] = e, \forall x \in \mathbb{Z}$, it follows that $[\beta^m, \beta^{\tau^x}] = e, \forall x \in \mathbb{Z}$.

Therefore, by (2.5),

$$e = (\beta^m)|^{-1}_{(i)\sigma_{\left(\beta^{\tau^x}\right)}} (\beta^{\tau^x})|^{-1}_i (\beta^m)|_i (\beta^{\tau^x})|_{(i)\sigma_{\beta^m}}, \forall x \in \mathbb{Z}, \forall i \in Y.$$

Now, as $\sigma_\beta = \sigma_\tau^s$ and $\sigma_{\beta^m} = e$, we reach

$$(\beta^m)|_{i+s} = (\beta^m)|_i^{(\beta^{\tau^x})|_i}, \forall x \in \mathbb{Z}, \forall i \in Y. \tag{5.8}$$

By (2.4), the following

$$(\beta^{\tau^x})|_i = (\tau^x)|^{-1}_{(i)\sigma^{-1}_{\tau^x}} \left(\beta|_{(i)\sigma^{-1}_{\tau^x}}\right) (\tau^x)|_{(i)\left(\sigma^{-1}_{\tau^x}\right)\sigma_\beta} = (\tau^x)|^{-1}_{i-x} \left(\beta|_{i-x}\right) (\tau^x)|_{i-x+s}$$

holds for all $i \in Y$ and all $x \in \mathbb{Z}$.

From which we derive

$$(\beta^{\tau^x})|_i = \tau^{-\frac{x-\overline{x}}{n} - \delta(i-x,x)} \left(\beta|_{i-x}\right) \tau^{\frac{x-\overline{x}}{n} + \delta(i-x+s,x)} \tag{5.9}$$

for all $i \in Y$ and all $x \in \mathbb{Z}$.

Therefore, by (5.8) and (5.9),

$$(\beta^m)|_{i+s} = (\beta^m)|_i^{\left(\tau^{-\frac{x-\overline{x}}{n} - \delta(i-x,x)} (\beta|_{i-x}) \tau^{\frac{x-\overline{x}}{n} + \delta(i-x+s,x)}\right)},$$

for all $i \in Y$ and all $x \in \mathbb{Z}$..

On writing $x = kn + \overline{x} = kn + r, r \in \mathbb{Z}$ in the above equation, we obtain

$$\begin{aligned}
(\beta^m)|_{i+s} &= (\beta^m)|_i^{\tau^{-k-\delta(i-r,r)} \left(\beta|_{i-r}\right) \tau^{k+\delta(i-r+s,r)}} \\
\Rightarrow (\beta^m)|_{i+s}^{\tau^{-k-\delta(i-r+s,r)}} &= (\beta^m)|_i^{\left(\beta|_{i-r}\right) \tau^{-k-\delta(i-r,r)} [\tau^{-k-\delta(i-r,r)}, \beta|_{i-r}]} \\
\Rightarrow (\beta^m)|_{i+s}^{\tau^{-k-\delta(i-r+s,r)} [\beta|_{i-r}, \tau^{-k-\delta(i-r,r)}] \tau^{k+\delta(i-r,r)}} &= (\beta^m)|_i^{(\beta|_{i-r})}
\end{aligned}$$

for all $i, r \in Y$ and all $k \in \mathbb{Z}$.

By (5.5), (5.7) and using the fact that N is abelian and normal in H, we find

$$\begin{aligned}
(\beta^m)|_{i+s}^{\tau^{\delta(i-r,r)-\delta(i-r+s,r)}} &= (\beta^m)|_i^{(\beta|_{i-r})} \\
\Rightarrow (\beta^m)|_{i+s}^{\tau^{\Delta_{-s}(i-r,i-i)}} &= (\beta^m)|_i^{(\beta|_{i-r})}
\end{aligned}$$

for all $i, r \in Y$.

On setting $j = \overline{i-r}$, we get

$$(\beta^m)|_{i+s}^{\tau^{\Delta_{-s}(j,i)}} = (\beta^m)|_i^{(\beta|_j)} \tag{5.10}$$

for all $i, j \in Y$.

Further, by using equations (5.5),(5.6) (5.7), (5.10) and

$$(\beta^m)|_i = \pi_i, \tag{5.11}$$

we conclude that also K is an abelian normal subgroup of H.

Now, $O = K\langle\tau\rangle$ is metabelian. Moreover it is normal in H, because

$$\tau^{\beta|_i} = \tau\tau^{-1}\tau^{\beta|_i} = \tau[\tau, \beta|_i] \in O$$

for all $i \in Y$.

(ii) Consider the following Fibonacci type group

$$X = \left\langle \begin{array}{c} b_0, \dots, b_{n-1} \mid b_i b_{\overline{j+s}} = b_j b_{\overline{i+s}}, \\ b_i b_{\overline{i+s}} \cdots b_{\overline{i+(n-1)s}} = e, \forall i, j \in Y \end{array} \right\rangle$$

where the bar notation indicates ´modulo n ´.

The Equation (5.3) shows that $\dfrac{H}{O}$ is a homomorphic image of X. We will prove that X is isomorphic to a subgroup of the wreath product $C_m \wr C_n$.

As a matter of fact the group $C_m \wr C_n$ has the presentation

$$\left\langle u, a \mid u^m = e, a^n = e, u^{a^i} u^{a^j} = u^{a^j} u^{a^i} \right\rangle.$$

On defining $b = a^s u^{-1}$, we have

$$\begin{aligned} & u^m = e \\ \Rightarrow\ & (a^{-s}b)^m = e \\ \Rightarrow\ & (\underbrace{a^{-s}b \cdots a^{-s}b}_{m \text{ terms}})^{a^{-s+i}} = e \\ \Rightarrow\ & b^{a^i} b^{a^{i+s}} \cdots b^{a^{i+(m-1)s}} = e. \end{aligned}$$

Also, the commutation relation

$$u^{a^i} u^{a^j} = u^{a^j} u^{a^i}$$

implies

$$\begin{aligned} & (b^{-1}a^s)^{a^i}(b^{-1}a^s)^{a^j} = (b^{-1}a^s)^{a^j}(b^{-1}a^s)^{a^i} \\ \Rightarrow\ & (a^{-s}b)^{a^j}(a^{-s}b)^{a^i} = (a^{-s}b)^{a^i}(a^{-s}b)^{a^j} \\ \Rightarrow\ & b^{a^j} a^{-s} b^{a^i} = b^{a^i} a^{-s} b^{a^j} \\ \Rightarrow\ & b^{a^j} b^{a^{i+s}} = b^{a^i} b^{a^{j+s}}. \end{aligned}$$

By using Tietze transformations we conclude that $C_m \wr C_n$ has the presentation

$$\left\langle a, b \mid a^n = e, b^{a^j} b^{a^{i+s}} = b^{a^i} b^{a^{j+s}}, b^{a^i} b^{a^{i+s}} \cdots b^{a^{i+(m-1)s}} = e, \forall i, j \in Y \right\rangle.$$

Then, on introducing $b_i = b^{a^i}, i = 0, \dots, n-1$, the above presentation is expressed as

$$\left\langle a, b_0, \dots, b_{n-1} \mid a^n = e, b_i = b_0^{a^i},\ b_j b_{\overline{i+s}} = b_i b_{\overline{j+s}},\ b_i b_{\overline{i+s}} \cdots b_{\overline{i+(m-1)s}} = e, \right.$$
$$\left. \forall i, j \in Y \right\rangle.$$

□

The next results leads to a proof of Theorem C.

Lemma 5.6. *Let $\sigma = (0, 1, ..., n-1) \in \Sigma_n$ and let L be the layer closure of $\langle\sigma\rangle$ in $\mathcal{A}_n$. Suppose $\beta = (\beta|_0, \beta|_1, \ldots, \beta|_{n-1})\sigma_\beta \in L$ satisfies $[\beta, \beta^{\tau^x}] = e$ for all $x \in \mathbb{Z}$. Write $\sigma_\beta = \sigma^s$ and $\sigma_{\beta|_i} = \sigma^{m_i}$ for all $i \in Y$. Then for all $i, t \in Y$, the following congruence holds*

$$\Delta_s(i,t) + m_{\overline{i-s}} + m_t \equiv m_{\overline{t-s}} + m_i + \Delta_s(i+s, t+s) \bmod n. \tag{5.12}$$

Proof. Since $\sigma_{\beta|_i} = \sigma^{m_i}$, we conclude by (5.3),

$$\sigma^{\Delta_s(i,t)+m_{\overline{i-s}}+m_t} = \sigma^{m_{\overline{t-s}}+m_i+\Delta_s(i+s,t+s)}$$

and therefore, $\Delta_s(i,t) + m_{\overline{i-s}} + m_t \equiv m_{\overline{t-s}} + m_i + \Delta_s(i+s, t+s) \bmod n$. □

Lemma 5.7. *Maintain the notation of the previous lemma and let $s = 1$. Then,*

$$\sigma_{(\beta^n)|_0} = \sigma_{(\beta|_0)(\beta|_1)\cdots(\beta|_{n-1})} = \sigma.$$

Proof. The case $n = 2$ is covered by Proposition 6 of [9].

Now let n be an odd prime. From

$$\Delta_1(i,t) + m_{\overline{i-1}} + m_t \equiv m_{\overline{t-1}} + m_i + \Delta_1(i+1, t+1) \bmod n$$

we conclude

$$\begin{aligned} &\sum_{i=0}^{n-2}\sum_{t=i+1}^{n-1} \left(\Delta_1(i,t) + m_{\overline{i-1}} + m_t\right) \\ \equiv\ &\sum_{i=0}^{n-2}\sum_{t=i+1}^{n-1} \left(m_{\overline{t-1}} + m_i + \Delta_1(i+1,t+1)\right) \bmod n. \end{aligned}$$

Now,

$$\begin{aligned} \sum_{i=0}^{n-2}\sum_{t=i+1}^{n-1} \Delta_1(i,t) &\overset{\text{Prop.5.2(i)}}{=} \sum_{t=1}^{n-1}\Delta_1(0,t) \overset{\text{Prop.5.2(ii)}}{=} \sum_{t=0}^{n-1}\Delta_1(0,t) \\ &\overset{\text{Prop.5.2(ii)}}{=} \sum_{t=0}^{n-1} -\Delta_1(t,0) \overset{\text{Prop.5.2(vi)}}{=} -(n-1), \end{aligned}$$

$$\begin{aligned} \sum_{i=0}^{n-2}\sum_{t=i+1}^{n-1} \Delta_1(i+1,t+1) &\overset{\text{Prop.5.2(i)}}{=} \sum_{i=0}^{n-2}\Delta_1(i+1,0) \overset{\text{Prop.5.2(ii)}}{=} \sum_{i=0}^{n-1}\Delta_1(i,0) \\ &\overset{\text{Prop.5.2(vi)}}{=} (n-1), \end{aligned}$$

$$\sum_{i=0}^{n-2}\sum_{t=i+1}^{n-1} \left(m_{\overline{i-1}} + m_t\right) = 2(n-1)m_{n-1} + (n-2)\sum_{k=0}^{n-2} m_k$$

and

$$\sum_{i=0}^{n-2}\sum_{t=i+1}^{n-1} \left(m_{\overline{t-1}} + m_i\right) = n\sum_{k=0}^{n-2} m_k.$$

Since n is odd, we have

$$\sum_{k=0}^{n-1} m_k \equiv 1 \bmod n$$

and therefore, $\sigma_{(\beta|_0)\cdots(\beta|_{n-1})} = \sigma^{m_0+\cdots+m_{n-1}} = \sigma$. □

Now we prove Theorem C.

Theorem 5.8. *Let n be an odd number, $\sigma = (0, \ldots, n-1) \in \Sigma_n$ and let L be the layer closure of $\langle\sigma\rangle$ in $\mathcal{A}_n$. Let s be an integer relatively prime to n and $\beta = (\beta|_0, \beta|_1, \ldots, \beta|_{n-1})\sigma^s \in L$ be such that $[\beta, \beta^{\tau^x}] = e$ for all $x \in \mathbb{Z}$. Then β is a conjugate of τ in $\mathcal{A}_n$.*

Proof. We start with the case $s = 1$. The element

$$\alpha(1) = (e, \beta|_0^{-1}, ((\beta|_0)(\beta|_1))^{-1}, \ldots, ((\beta|_0)\cdots(\beta|_{n-2}))^{-1}) \in \mathrm{Stab}(1)$$

conjugates β to

$$\beta^{\alpha(1)} = (e, \ldots, e, (\beta|_0)\cdots(\beta|_{n-1}))\sigma.$$

By Lemma 5.7 we find $\sigma_{(\beta|_0)\cdots(\beta|_{n-1})} = \sigma$. Moreover by Proposition 4.3,

$$[(\beta^n)|_0, ((\beta^n)|_0)^{\tau^x}] = [(\beta|_0)\cdots(\beta|_{n-1}), ((\beta|_0)\cdots(\beta|_{n-1}))^{\tau^x}] = e,$$

for all integers x. Therefore $(\beta|_0)\cdots(\beta|_{n-1})$ satisfies the hypothesis of the theorem. The process can be repeated until we obtain a sequence $(\alpha(k))_{k\in\mathbb{N}}$ such that $\beta^{\alpha(1)\alpha(2)\cdots\alpha(k)\cdots} = \tau$, where $\alpha(k) \in \mathrm{Stab}(k)$ satisfies $\alpha(k)|_u = \alpha(k)|_v$ for all $u, v \in \mathcal{M}$ with $|u| = |v| = k-1$.

Now, suppose more generally s is such that $\gcd(s, n) = 1$ and let k be the minimum positive integer for which $sk \equiv 1 \bmod(n)$. Then β^k satisfies the hypothesis of the first part and so, there exists $\alpha \in L$ such that $(\beta^k)^\alpha = \tau$. Since k is invertible in $\mathbb{Z}_n$, there exists $\gamma \in \mathcal{A}_n$ such that $\tau^\gamma = \tau^{k^{-1}}$. Thus, $\beta^{\alpha\gamma^{-1}} = \tau$. □

6. Solvable groups for $n = p$, a prime number

We will prove in this section the case $n = p$ of Theorem A.

Let B be an abelian subgroup of $Aut(T_p)$ normalized by τ and let $\beta \in B$. By Proposition 4.1, $\sigma_\beta \in \langle\sigma_\tau\rangle$ and therefore we have in effect two cases, namely, $\sigma_\beta = e, \sigma_\tau$.

Proposition 6.1. *Suppose $\sigma_\beta \in \langle\sigma_\tau\rangle$. Then, $\sigma_{(\beta|_i)} \in \langle\sigma_\tau\rangle$ for all $i \in Y$.*

Proof. By Theorem 5.5, K is an abelian normal subgroup of H and $\frac{H}{O}$ is homomorphic to a subgroup of $C_p \wr C_p$ for $O = K\langle\tau\rangle$.

By Proposition 4.1, K is a subgroup of $\langle\sigma_\tau\rangle$ modulo $Stab(1)$. So the same is true for $O = K\langle\tau\rangle$.

Therefore, H is a p-group modulo $Stab(1)$. Since H is a p-group modulo $Stab(1)$ and since $\tau \in H$, it follows that H coincides with $\langle\sigma_\tau\rangle$ modulo $Stab(1)$, by Proposition 4.1. Hence, necessarily we have $\sigma_{(\beta|_i)} \in \langle\sigma_\tau\rangle$. □

Theorem 6.2. *Let p be a prime number and $\beta \in \mathrm{Aut}(T_p)$ such that $\sigma_\beta = \sigma_\tau^s$ for some integer s relatively prime to p. Suppose $[\beta, \beta^{\tau^x}] = e$ for all $x \in \mathbb{Z}$. Then β is conjugate to τ in $\mathrm{Aut}(T_p)$.*

Proof. As the second author showed the case p=2 in [9], we will show the case p odd.

Suppose $s = 1$. Recall that

$$\alpha(1) = (e, \beta|_0^{-1}, (\beta|_0\beta|_1)^{-1}, \dots, (\beta|_0 \cdots \beta|_{p-2})^{-1}) \in \mathrm{Stab}(1)$$

conjugates β to its normal form

$$\beta^{\alpha(1)} = (e, \dots, e, \beta|_0 \cdots \beta|_{p-1})\sigma.$$

By Lemma 5.7 we have $\sigma_{\beta|_0\beta|_1\cdots\beta|_{p-1}} = \sigma_\tau$. Moreover by Proposition 4.3,

$$[\beta^p|_0, (\beta^p|_0)^{\tau^x}] = [\beta|_0\beta|_1 \cdots \beta|_{p-1}, (\beta|_0\beta|_1 \cdots \beta|_{p-1})^{\tau^x}] = e,$$

for all integers x. Therefore $\beta|_0\beta|_1 \cdots \beta|_{p-1}$ satisfies the condition of the theorem. This process can be repeated to produce a sequence $(\alpha(k))_{k\in\mathbb{N}}$ such that $\beta^{\alpha(1)\alpha(2)\cdots\alpha(k)\cdots} = \tau$, where $\alpha(k) \in \mathrm{Stab}(k)$ satisfies $\alpha(k)|_u = \alpha(k)|_v$ for all $u, v \in \mathcal{M}$ where $|u| = |v| = k - 1$.

In the general case, s is such that $\gcd(p, s) = 1$. Let k be the minimum positive integer which is the inverse of s modulo p. Then, $\sigma|_{\beta^k} = \sigma_\tau$ and β^k satisfies the hypotheses. Thus there exists $\alpha \in \mathcal{A}_p$ such that $\left(\beta^k\right)^\alpha = \tau$. Let k^{-1} be the inverse of k in $U(\mathbb{Z}_n)$; then $\beta^\alpha = \tau^{k^{-1}}$. There exists $\gamma \in N_{\mathcal{A}_p}(\overline{<\tau>})$ which conjugates τ to $\tau^{k^{-1}}$ and so, $(\beta^\alpha)^{\gamma^{-1}} = \tau$. □

Lemma 6.3. *Let p be a prime number and $\beta \in \mathrm{Aut}(T_p)$ such that $[\beta, \beta^{\tau^x}] = e$ for all $x \in \mathbb{Z}$. Then, there exists a tree level m and a conjugate μ of τ such that $\beta \in \times_{p^m}\overline{\langle\mu\rangle}$ and there exists an index u of length m such that $\beta|_u = \mu$.*

Proof. Let m be the minimum tree level such that $\sigma_{\beta|_u} \neq e$ for some $|u| = m$. Therefore, $\beta \in Stab(m)$ and $\sigma_{\beta|_u} = \sigma_\tau^s$ for some integer s such that $\gcd(p, s) = 1$. By Proposition 4.3, $[\beta|_u, \beta|_v^{\tau^k}] = e$ for all indices v such that $|v| = m$ and for all $k \in \mathbb{Z}$. So, by Theorem 6.2, $\mu = \beta|_u$ is conjugate to τ in $\mathrm{Aut}(T_p)$ and $\beta|_v \in \overline{\langle\mu\rangle}$ for all v such that $|v| = m$, by Lemma 3.2. □

Theorem 6.4. *Let p be a prime number, $\sigma = (0, 1, \dots, p-1) \in \Sigma_p$, $F = N_{\Sigma_p}(\langle\sigma\rangle)$, $\Gamma_0 = N_{\mathcal{A}}(\overline{<\tau>})$. Let G be a solvable subgroup of $Aut(T_p)$ which contains the p-adic adding machine τ. Then, there exists an integer $t \geq 1$ such that G is conjugate to a subgroup of*

$$\times_p (\cdots (\times_p (\times_p \Gamma_0 \rtimes F) \rtimes) \cdots) \rtimes F,$$

where $\times_p$ appears t times.

Proof. We may suppose G has derived length $d \geq 2$. Let B be the $(d-1)$-th term of the derived series of G. By Lemma 6.3, there exists a level t such that B is a subgroup of $V = \times_{p^t}\overline{\langle\mu\rangle}$ where $\mu = \tau^\alpha$ for some $\alpha \in Aut(T_p)$.

We will show that G is a subgroup of

$$\dot{J} = \times_p (\cdots (\times_p (\times_p (\Gamma_0)^\alpha \rtimes \Sigma_p) \rtimes \Sigma_p) \cdots) \rtimes \Sigma_p,$$

where $\times_p$ appears t times.

Let $\gamma \in G\backslash\dot{J}$. Then there exists an index w of length t such that $\gamma|_w \notin (\Gamma_0)^\alpha$. Since B is an abelian subgroup normalized by τ and τ is transitive on all levels of the tree, by Lemma 6.3, there exists $\beta \in B$ such that $\beta|_w = \mu^\eta$ for some $\eta \in U(\mathbb{Z}_p)$.

Write $v = w^\gamma$. Then,

$$(\beta^\gamma)|_v \overset{(2.7)}{=} \left(\beta|_{v^{\gamma^{-1}}}\right)^{\left(\gamma|_{v^{\gamma^{-1}}}\right)} = (\beta|_w)^{\gamma|_w} \notin \overline{\langle\mu\rangle},$$

and this implies $\beta^\gamma \notin B \leq \times_{p^t}\overline{\langle\mu\rangle}$ and $\gamma \notin G$. Hence, G is a subgroup of $\dot{J}$.

Now, since G is a solvable group containing τ, there exist G_i $(0 \leq i \leq t)$ solvable subgroups of Σ_p containing $\sigma = (0, 1, \ldots, p-1)$ such that G is a subgroup of

$$R_t(\alpha) = \times_p(\cdots(\times_p(\times_p(\Gamma_0)^\alpha \rtimes G_1) \rtimes G_2)\cdots) \rtimes G_t.$$

Since for all i, we have $G_i \leq F$ we may substitute every the G_i by F. Finally, $R_t(\alpha)$ is a conjugate of $R_t(1)$ by the diagonal automorphism $\alpha^{(t)}$. □

7. Two cases for n even

We prove in this section part *(II)* of Theorem B.

7.1. The case $\sigma_\beta = (\sigma_\tau)^{\frac{n}{2}}$.

Theorem 7.1. *Let n be an even number, $\beta \in \mathcal{A}_n$ such that $\sigma_\beta = \sigma_\tau^{\frac{n}{2}}$ and $[\beta, \beta^{\tau^x}] = e$ for all $x \in \mathbb{Z}$. Then $H = \langle \beta|_i \ (0 \leq i \leq n-1), \tau\rangle$ is a metabelian subgroup of $\mathcal{A}_n$.*

Proof. Denote $\Delta_{\frac{n}{2}}(i,j)$ by $\Delta(i,j)$.

Define the subgroup

$$R = \left\langle [\beta|_t, \tau^k],\ \beta|_i\beta|_{i+\frac{n}{2}},\ \beta|_j^2\tau^{-\Delta(j,j+\frac{n}{2})} \mid k \in \mathbb{Z} \text{ and } i,j,t \in Y\right\rangle.$$

We will prove that R is an abelian normal subgroup of H.

(I) R is normal in H :

– $\left\langle[\beta|_i, \tau^k]\right\rangle^H \leq R$:

$$[\beta|_{i+\frac{n}{2}}, \tau^k]^{\beta|_j} \overset{(5.4)}{=} [\beta|_i, \tau^k]^{\tau^{\Delta(j,i)}};$$

– $\left\langle \beta|_i\beta|_{i+\frac{n}{2}}\right\rangle^H \leq R$:

$$\begin{aligned}(\beta|_i\beta|_{i+\frac{n}{2}})^{\tau^k} &= \left(\beta|_i\beta|_{i+\frac{n}{2}}\right).[\beta|_i\beta|_{i+\frac{n}{2}}, \tau^k] \\ &= \left(\beta|_i\beta|_{i+\frac{n}{2}}\right)[\beta|_i, \tau^k]^{\beta|_{i+\frac{n}{2}}}[\beta|_{i+\frac{n}{2}}, \tau^k]\end{aligned}$$

$$(\beta|_i\beta|_{i+\frac{n}{2}})^{\beta|_j} = \left(\beta|_j^{-1}\beta|_i\beta|_{i+\frac{n}{2}}\beta|_j\right)\tau^{\Delta(j+\frac{n}{2},i+\frac{n}{2})}\tau^{-\Delta(j+\frac{n}{2},i+\frac{n}{2})} \tag{7.1}$$

$$\overset{(5.3)}{=} \left(\beta|_j^{-1}\beta|_i\right)\tau^{\Delta(j,i)}\left(\beta|_{j+\frac{n}{2}}\beta|_i\right)\tau^{-\Delta(j+\frac{n}{2},i+\frac{n}{2})}$$

$$= \left(\beta|_j^{-1}\beta|_i\beta|_{j+\frac{n}{2}}\right)\tau^{\Delta(j,i)}.[\tau^{\Delta(j,i)}, \beta|_{j+\frac{n}{2}}].\beta|_i\tau^{-\Delta(j+\frac{n}{2},i+\frac{n}{2})}$$

$$\overset{(5.3)}{=} \left(\beta|_j^{-1}\right) \tau^{\Delta(j+\frac{n}{2},i+\frac{n}{2})} \left(\beta|_j \beta|_{i+\frac{n}{2}}\right).$$
$$[\tau^{\Delta(j,i)}, \beta|_{j+\frac{n}{2}}].\beta|_i \tau^{-\Delta(j+\frac{n}{2},i+\frac{n}{2})}$$

$$= \tau^{\Delta(j+\frac{n}{2},i+\frac{n}{2})}.[\tau^{\Delta(j+\frac{n}{2},i+\frac{n}{2})}, \beta|_j].$$
$$\beta|_{i+\frac{n}{2}}[\tau^{\Delta(j,i)}, \beta|_{j+\frac{n}{2}}]\beta|_i \tau^{-\Delta(j+\frac{n}{2},i+\frac{n}{2})}$$

$$\overset{Prop.5.2}{=} \tau^{-\Delta(j,i)}[\tau^{-\Delta(j,i)}, \beta|_j].\beta|_{i+\frac{n}{2}}.$$
$$[\tau^{\Delta(j,i)}, \beta|_{j+\frac{n}{2}}]\beta|_i \tau^{\Delta(j,i)}$$

$$\overset{(5.4)}{=} \tau^{-\Delta(j,i)}\beta|_{i+\frac{n}{2}}.[\tau^{-\Delta(j,i)}, \beta|_{j+\frac{n}{2}}]^{\tau^{\Delta(j,i)}}.[\tau^{\Delta(j,i)}, \beta|_{j+\frac{n}{2}}].\beta|_i \tau^{\Delta(j,i)}$$

$$\overset{(4.1)}{=} \left(\beta|_{i+\frac{n}{2}}\beta|_i\right)^{\tau^{\Delta(j,i)}}.$$

– $\left\langle \beta|_j^2 \tau^{-\Delta(j,j+\frac{n}{2})}\right\rangle^H \leq R$:

$$\begin{aligned} (\beta|_j^2 \tau^{-\Delta(j,j+\frac{n}{2})})^{\tau^k} &= \beta|_j^2 \tau^{-\Delta(j,j+\frac{n}{2})}.[\beta|_j^2 \tau^{-\Delta(j,j+\frac{n}{2})}, \tau^k] \\ = \quad & \beta|_j^2 \tau^{-\Delta(j,j+\frac{n}{2})}.[\beta|_j^2, \tau^k]^{\tau^{-\Delta(j,j+\frac{n}{2})}} \\ = \quad & \beta|_j^2 \tau^{-\Delta(j,j+\frac{n}{2})} \left([\beta|_j, \tau^k]^{\beta|_j}.[\beta|_j, \tau^k]\right)^{\tau^{-\Delta(j,j+\frac{n}{2})}} \end{aligned}$$

By Proposition 5.2 and 5.3, we can show

$$\left(\beta|_j^2 \tau^{-\Delta(j,j+\frac{n}{2})}\right)^{\beta|_i} = \left(\beta|_{j+\frac{n}{2}}^2 \tau^{-\Delta(j+\frac{n}{2},j)}[\tau^{-\Delta(j+\frac{n}{2},j)}, \beta|_{j+\frac{n}{2}}]\right)^{\tau^{\Delta(i,j)}}. \tag{7.2}$$

(II) The subgroup R is abelian:

$$[\beta|_i, \tau^k]^{\beta|_j \tau^t} \overset{Prop.5.3}{=} [\beta|_i, \tau^k]^{\tau^t \beta|_j}; \tag{7.3}$$

$$[\beta|_i, \tau^k]^{\beta|_j \beta|_{j+\frac{n}{2}}} \overset{(5.4)}{=} [\beta|_{i+\frac{n}{2}}, \tau^k]^{\tau^{\Delta(j,i+\frac{n}{2})}\beta|_{j+\frac{n}{2}}} \overset{(7.3)}{=} [\beta|_{i+\frac{n}{2}}, \tau^k]^{\beta|_{j+\frac{n}{2}}\tau^{\Delta(j,i+\frac{n}{2})}} \tag{7.4}$$

$$\overset{(5.4)}{=} [\beta|_i, \tau^k]^{\tau^{\Delta(j+\frac{n}{2},i)+\Delta(j,i+\frac{n}{2})}} \overset{\text{Prop.5.2}}{=} [\beta|_i, \tau^k]$$

$$\begin{aligned} & [\beta|_i, \tau^k]^{\beta|_j^2 \tau^{-\Delta(j,j+\frac{n}{2})}} \overset{(5.4)}{=} [\beta|_{i+\frac{n}{2}}, \tau^k]^{\tau^{\Delta(j,i+\frac{n}{2})}\beta|_j \tau^{-\Delta(j,j+\frac{n}{2})}} \\ \overset{(7.3)}{=} \quad & [\beta|_{i+\frac{n}{2}}, \tau^k]^{\beta|_j \tau^{\Delta(j,i+\frac{n}{2})-\Delta(j,j+\frac{n}{2})}} \overset{(5.4)}{=} [\beta|_i, \tau^k]^{\tau^{\Delta(j,i)+\Delta(j,i+\frac{n}{2})-\Delta(j,j+\frac{n}{2})}} \\ \overset{\text{Prop.5.2}}{=} \quad & [\beta|_i, \tau^k] \end{aligned} \tag{7.5}$$

$$\begin{aligned}\left(\beta|_i\beta|_{i+\frac{n}{2}}\right)^{\beta|_j\beta|_{j+\frac{n}{2}}} &\overset{(7.1)}{=} \left(\beta|_{i+\frac{n}{2}}\beta|_i\right)^{\tau^{\Delta(j,i)}\beta|_{j+\frac{n}{2}}} \\ &= \left(\beta|_{i+\frac{n}{2}}\beta|_i\right)^{\left(\beta|_{j+\frac{n}{2}}\tau^{\Delta(j,i)}[\tau^{\Delta(j,i)},\beta|_{j+\frac{n}{2}}]\right)} \\ &\overset{(7.1)}{=} \left(\beta|_i\beta|_{i+\frac{n}{2}}\right)^{\left(\tau^{\Delta(j+\frac{n}{2},i+\frac{n}{2})+\Delta(j,i)}.[\tau^{\Delta(j,i)},\beta|_{j+\frac{n}{2}}]\right)} \\ &\overset{\text{Prop.5.2}}{=} \left(\beta|_i\beta|_{i+\frac{n}{2}}\right)^{[\tau^{\Delta(j,i)},\beta|_{j+\frac{n}{2}}]} \\ &\overset{(7.4)}{=} \beta|_i\beta|_{i+\frac{n}{2}}\end{aligned}$$

$$\begin{aligned}(\beta|_i\beta|_{i+\frac{n}{2}})^{\beta|_j^2\tau^{-\Delta(j,j+\frac{n}{2})}} &\overset{(7.1)}{=} (\beta|_{i+\frac{n}{2}}\beta|_i)^{\tau^{\Delta(j,i)}\beta|_j\tau^{-\Delta(j,j+\frac{n}{2})}} \\ &= (\beta|_{i+\frac{n}{2}}\beta|_i)^{\beta|_j\tau^{\Delta(j,i)}[\tau^{\Delta(j,i)},\beta|_j]\tau^{-\Delta(j,j+\frac{n}{2})}} \\ &= (\beta|_i\beta|_{i+\frac{n}{2}})^{\tau^{\Delta(j,i+\frac{n}{2})+\Delta(j,i)}[\tau^{\Delta(j,i)},\beta|_j]\tau^{-\Delta(j,j+\frac{n}{2})}} \\ &\overset{\text{Prop.5.2}}{=} (\beta|_i\beta|_{i+\frac{n}{2}})^{[\tau^{\Delta(j,i)},\beta|_j]^{\tau^{\Delta(j+\frac{n}{2},j)}}} \\ &\overset{(4.1),(7.4)}{=} \beta|_i\beta|_{i+\frac{n}{2}}\end{aligned}$$

Let

$$\alpha = \beta|_j^2\tau^{-\Delta(j,j+\frac{n}{2})}[\tau^{-\Delta(j,j+\frac{n}{2})},\beta|_j]. \tag{7.6}$$

Then,

$$\begin{aligned}&\left(\beta|_j^2\tau^{-\Delta(j,j+\frac{n}{2})}\right)^{\beta|_i^2\tau^{-\Delta(i,i+\frac{n}{2})}} \\ &\overset{(7.2)}{=} \left(\beta|_{j+\frac{n}{2}}^2\tau^{-\Delta(j+\frac{n}{2},j)}.[\tau^{-\Delta(j+\frac{n}{2},j)},\beta|_{j+\frac{n}{2}}]\right)^{\tau^{\Delta(i,j)}\beta|_i\tau^{-\Delta(i,i+\frac{n}{2})}} \\ &= \left(\beta|_{j+\frac{n}{2}}^2\tau^{-\Delta(j+\frac{n}{2},j)}.[\tau^{-\Delta(j+\frac{n}{2},j)},\beta|_{j+\frac{n}{2}}]\right)^{\left(\beta|_i\tau^{\Delta(i,j)}.[\tau^{\Delta(i,j)},\beta|_i].\tau^{-\Delta(i,i+\frac{n}{2})}\right)} \\ &= \left(\left(\beta|_{j+\frac{n}{2}}^2\tau^{-\Delta(j+\frac{n}{2},j)}\right)^{\beta|_i}.[\tau^{-\Delta(j+\frac{n}{2},j)},\beta|_{j+\frac{n}{2}}]^{\beta|_i}\right)^{\left(\tau^{\Delta(i,j)}.[\tau^{\Delta(i,j)},\beta|_i].\tau^{-\Delta(i,i+\frac{n}{2})}\right)} \\ &\overset{(5.4)}{=} \left(\left(\beta|_{j+\frac{n}{2}}^2\tau^{-\Delta(j+\frac{n}{2},j)}\right)^{\beta|_i}.[\tau^{-\Delta(j+\frac{n}{2},j)},\beta|_j]^{\tau^{\Delta(i,j)}}\right)^{\left(\tau^{\Delta(i,j)}.[\tau^{\Delta(i,j)},\beta|_i].\tau^{-\Delta(i,i+\frac{n}{2})}\right)} \\ &\overset{(7.2)}{=} \left(\alpha^{\tau^{\Delta(i,j+\frac{n}{2})}}.[\tau^{-\Delta(j+\frac{n}{2},j)},\beta|_j]^{\tau^{\Delta(i,j)}}\right)^{\left(\tau^{\Delta(i,j)}.[\tau^{\Delta(i,j)},\beta|_i].\tau^{-\Delta(i,i+\frac{n}{2})}\right)} \\ &= \left(\alpha.[\tau^{-\Delta(j+\frac{n}{2},j)},\beta|_j]^{\tau^{\Delta(i,j)-\Delta(i,j+\frac{n}{2})}}\right)^{\left(\tau^{\Delta(i,j+\frac{n}{2})+\Delta(i,j)}.[\tau^{\Delta(i,j)},\beta|_i].\tau^{-\Delta(i,i+\frac{n}{2})}\right)} \\ &\overset{\text{Prop.5.2}}{=} \left(\alpha.[\tau^{-\Delta(j+\frac{n}{2},j)},\beta|_j]^{\tau^{\Delta(j+\frac{n}{2},j)}}\right)^{\left(\tau^{\Delta(i,i+\frac{n}{2})}[\tau^{\Delta(i,j)},\beta|_i]\tau^{-\Delta(i,i+\frac{n}{2})}\right)} \\ &\overset{(7.6)}{=} \left(\beta|_j^2\tau^{-\Delta(j,j+\frac{n}{2})}[\tau^{-\Delta(j,j+\frac{n}{2})},\beta|_j][\tau^{\Delta(j+\frac{n}{2},j)},\beta|_j]^{-1}\right)^{[\tau^{\Delta(i,j)},\beta|_i]^{\tau^{-\Delta(i,i+\frac{n}{2})}}} \\ &\overset{\text{Prop.5.2}}{=} \left(\beta|_j^2\tau^{-\Delta(j,j+\frac{n}{2})}\right)^{[\tau^{\Delta(i,j)},\beta|_i]^{\tau^{-\Delta(i,i+\frac{n}{2})}}} \\ &\overset{(4.1),(7.5)}{=} \beta|_j^2\tau^{-\Delta(j,j+\frac{n}{2})}.\end{aligned}$$

Moreover, since

$$\begin{aligned} R\left(\beta|_i\right) R\left(\beta|_j\right) &= R\left(\beta|_i\right)\left(\beta|_j\right) \overset{\text{Prop.5.3}}{=} R\tau^{\Delta(j,i+\frac{n}{2})}\beta|_{j+\frac{n}{2}}\beta|_{i+\frac{n}{2}}\tau^{\Delta(j,i+\frac{n}{2})} \\ &= R\beta|_{j+\frac{n}{2}}\beta|_{i+\frac{n}{2}}\tau^{2\Delta(j,i+\frac{n}{2})} = R\beta|_j^{-1}\beta|_i^{-1}\tau^{2\Delta(j,i+\frac{n}{2})} \\ &= R\beta|_j^{-1}\beta|_j^{2}\tau^{-\Delta(j,j+\frac{n}{2})}\beta|_i^{-1}\beta|_i^{2}\tau^{-\Delta(i,i+\frac{n}{2})}\tau^{2\Delta(j,i+\frac{n}{2})} \\ &= R\beta|_j\beta|_i\tau^{-\Delta(j,j+\frac{n}{2})-\Delta(i,i+\frac{n}{2})+2\Delta(j,i+\frac{n}{2})} \\ &\overset{\text{Prop.5.2}}{=} R\beta|_j\beta|_i = R\beta|_jR\beta|_i \end{aligned}$$

and

$$R\beta|_i = R\beta|_{i+\frac{n}{2}}^{-1}, \quad R\beta|_i^2 = R\tau^{\Delta(i,i+\frac{n}{2})}, \forall i \in Y,$$

we conclude $\dfrac{H}{R}$ is a homomorphic image of

$$\mathbb{Z} \times \underbrace{C_2 \times \cdots \times C_2}_{\frac{n}{2}\ \text{terms}}.$$

□

7.2. The case σ_β transposition.

Theorem 7.2. *Let n be an even number and B an abelian subgroup of $\mathcal{A}_n$ normalized by τ. Suppose $\beta = (\beta|_0, \beta|_1, \ldots, \beta|_{n-1})\sigma_\beta \in B$ where σ_β is a transposition. Then $H = \langle \beta|_i\ \left(0 \le i \le n-1\right), \tau\rangle$ is a metabelian group.*

We prove progressively that

$$\begin{aligned} N &= \left\langle [\beta|_i, \tau^k] \mid k \in \mathbb{Z}, i \in Y \right\rangle, \\ U &= \left\langle N, \quad \beta|_j \mid j \neq 0, \frac{n}{2} \right\rangle, \\ V &= \left\langle U, \quad \beta|_{\frac{n}{2}}\beta|_0, \quad \tau\left(\beta|_0\right)^2 \right\rangle \end{aligned}$$

are normal abelian subgroups of H, from which it follows that $\frac{H}{V}$ is cyclic and therefore H metabelian.

Lemma 7.3. *If the degree of the tree n is even and σ_β is a transposition, then σ_β is $\langle\sigma_\tau\rangle$-conjugate to the transposition $\left(0, \frac{n}{2}\right)$.*

Proof. On conjugating by an appropriate power of σ_τ, we may assume $\sigma_\beta = (0, j)$. The conjugate of σ_β by σ_τ^i is the transposition $(i, j+i)$. In particular, $(j, 2j)$ is a conjugate which is supposed to commute with $(0, j)$. Therefore, $\{0, j\} = \{j, 2j\}$, $2j = 0$ modulo(n), $n = 2n'$ and $j = n'$. □

We go back to part (I) of the Proposition 4.5,

$$\begin{aligned} &\left(\tau^{v}|_{(i)\sigma_\tau^{-v}}\right)^{-1}\left(\beta|_{(i)\sigma_\tau^{-v}}\right)\left(\tau^{v}|_{(i)\sigma_\tau^{-v}\sigma_\beta}\right)\left(\beta|_{(i)\sigma_\tau^{-v}\sigma_\beta\sigma_\tau^{v}}\right) \\ = \;& (\beta|_i)\left(\tau^{v}|_{(i)\sigma_\beta\sigma_\tau^{-v}}\right)^{-1}\left(\beta|_{(i)\sigma_\beta\sigma_\tau^{-v}}\right)\left(\tau^{v}|_{(i)\sigma_\beta\sigma_\tau^{-v}\sigma_\beta}\right)\end{aligned}$$

and set in it $j=(i)\,\sigma_\tau^{-v}$, $v=kn+r$, $r=\overline{v}$ to obtain

$$(\tau^v)|_j^{-1}\beta|_j(\tau^v)|_{(j)\sigma_\beta}\beta|_{(j)\sigma_\beta\sigma_\tau^v} \tag{7.7}$$

$$= \beta|_{(j)\sigma_\tau^v}(\tau^v)|^{-1}_{(j)\sigma_\tau^v\sigma_\beta\sigma_\tau^{-v}}\beta|_{(j)\sigma_\tau^v\sigma_\beta\sigma_\tau^{-v}}(\tau^v)|_{(j)\sigma_\tau^v\sigma_\beta\sigma_\tau^{-v}\sigma_\beta}. \tag{7.8}$$

Proposition 7.4. *The following cases hold for different pairs (j,r).*

- *For $j=0$ there are 3 subcases*
  - *If $r=0$, then*

$$[\beta|_0,\tau^k]^{\beta|_{\frac{n}{2}}}=[\beta|_{\frac{n}{2}},\tau^k],\ \forall k\in\mathbb{Z}; \tag{7.9}$$

  - *If $r=\frac{n}{2}$, then*

$$\beta|_0\tau\beta|_0=\beta|_{\frac{n}{2}}\tau^{-1}\beta|_{\frac{n}{2}}, \tag{7.10}$$

and

$$[\beta|_0,\tau^k]^{\tau(\beta|_0)}=[\beta|_{\frac{n}{2}},\tau^k],\forall k\in\mathbb{Z}. \tag{7.11}$$

  - *If $r\neq 0$ and $r\neq\frac{n}{2}$, then*

$$\tau^{\delta(\frac{n}{2},r)}\beta|_0\beta|_{(\frac{n}{2}+r)}=\beta|_r\tau^{\delta(\frac{n}{2},r)}\beta|_0,\forall r\in Y-\{0,\frac{n}{2}\} \tag{7.12}$$

and

$$[\beta|_0,\tau^k]^{\beta|_r}=[\beta|_0,\tau^k],\forall k\in\mathbb{Z}. \tag{7.13}$$

- *For $j=\frac{n}{2}$ there are 3 subcases*
  - *If $r=0$, then*

$$[\beta|_{\frac{n}{2}},\tau^k]^{\beta|_0}=[\beta|_0,\tau^k],\ \forall k\in\mathbb{Z}; \tag{7.14}$$

  - *If $r=\frac{n}{2}$, then*

$$\tau^{-1}\beta|_{\frac{n}{2}}^2=\beta|_0^2\tau, \tag{7.15}$$

and

$$[\beta|_{\frac{n}{2}},\tau^k]^{\left(\beta|_{\frac{n}{2}}\right)\tau^{-1}}=[\beta|_0,\tau^k],\forall k\in\mathbb{Z}; \tag{7.16}$$

– *If* $r \neq 0$ *and* $r \neq \frac{n}{2}$, *then*

$$\tau^{-\delta(\frac{n}{2},r)}\beta|_{\frac{n}{2}}\beta|_r = \beta|_{\frac{n}{2}+r}\tau^{-\delta(\frac{n}{2},r)}\beta|_{\frac{n}{2}}, \forall r \in Y - \{0, \frac{n}{2}\} \tag{7.17}$$

and

$$[\beta|_{\frac{n}{2}}, \tau^k]^{\beta|_r} = [\beta|_{\frac{n}{2}}, \tau^k], \forall k \in \mathbb{Z}, \forall r \in Y - \{0, \frac{n}{2}\}. \tag{7.18}$$

- *For* $j \neq 0$ *and* $j \neq \frac{n}{2}$, *there are* 5 *subcases:*
  – *If* $j \neq n - r$ *and* $j \neq \frac{n}{2} - r$, *then, by substitution* $t = j + r$, *we have*

$$\beta|_j\beta_t = \beta|_t\beta|_j, \forall j, t \in Y - \{0, \frac{n}{2}\} \tag{7.19}$$

and

$$[\beta|_j, \tau^k]^{\beta|_t} = [\beta|_j, \tau^k], \forall j, t \in Y - \{0, \frac{n}{2}\} \tag{7.20}$$

– *If* $j = n - r$ *and* $0 < r < \frac{n}{2}$, *then, by substitution* $t = j - \frac{n}{2}$, *we have*

$$\tau^{-1}\beta|_{t+\frac{n}{2}}\tau\beta|_0 = \beta|_0\beta|_t, \forall t \in \{1, 2, \cdots, \frac{n}{2} - 1\} \tag{7.21}$$

and

$$[\beta|_{t+\frac{n}{2}}, \tau^k]^{\tau\beta|_0} = [\beta|_t, \tau^k], \forall j \in \{1, 2, \cdots, \frac{n}{2} - 1\} \tag{7.22}$$

– *If* $j = n - r$ *and* $\frac{n}{2} < r \leq n - 1$, *then*

$$\beta|_j\beta|_0 = \beta|_0\beta|_{\frac{n}{2}+j}, \forall j \in \{1, \cdots, \frac{n}{2} - 1\} \tag{7.23}$$

and

$$[\beta|_j, \tau^k]^{\beta|_0} = [\beta|_{\frac{n}{2}+j}, \tau^k], \forall k \in \mathbb{Z}, \forall j \in \{1, \cdots, \frac{n}{2} - 1\} \tag{7.24}$$

– *If* $j = \frac{n}{2} - r$ *and* $0 < r < \frac{n}{2}$, *then*

$$\beta|_j\beta|_{\frac{n}{2}} = \beta|_{\frac{n}{2}}\tau^{-1}\beta|_{j+\frac{n}{2}}\tau, \forall j \in \{1, \ldots, \frac{n}{2} - 1\} \tag{7.25}$$

and

$$[\beta|_j, \tau^k]^{\beta|_{\frac{n}{2}}\tau^{-1}} = [\beta|_{\frac{n}{2}+j}, \tau^k], \forall k \in \mathbb{Z}, \forall j \in \{1, \ldots, \frac{n}{2} - 1\} \tag{7.26}$$

– *If* $j = \frac{n}{2} - r$ *and* $\frac{n}{2} < r \leq n - 1$, *then*

$$\beta|_{\frac{n}{2}}\beta|_j = \beta|_{\frac{n}{2}+j}\beta|_{\frac{n}{2}}, \forall j \in \{1, \cdots, \frac{n}{2} - 1\} \tag{7.27}$$

and

$$[\beta|_j, \tau^k] = [\beta|_{\frac{n}{2}+j}, \tau^k]^{\beta|_{\frac{n}{2}}}, \forall k \in \mathbb{Z}, \forall j \in \{1, \cdots, \frac{n}{2} - 1\}. \tag{7.28}$$

Proof. We will prove (7.19) and (7.20). As $j \notin \{0, \frac{n}{2}, n-r, \frac{n}{2}-r\}$, we have

$$\begin{aligned} (j)\,\sigma_\tau^v &= (j)\,\sigma_\beta\sigma_\tau^v = j+r, \\ (j)\,\sigma_\beta &= (j)\,\sigma_\tau^v\sigma_\beta\sigma_\tau^{-v} = (j)\,\sigma_\tau^v\sigma_\beta\sigma_\tau^{-v}\sigma_\beta = j. \end{aligned}$$

Therefore,

$$\begin{aligned} &\left((\tau^v)|_j^{-1}\beta|_j(\tau^v)|_j\beta|_{j+r} = \beta|_{j+r}(\tau^v)|_j^{-1}\beta|_j(\tau^v)_j, \forall v \in \mathbb{Z}\right) \\ \Leftrightarrow\ &\left(\tau^{-k-\delta(j,r)}\beta|_j\tau^{k+\delta(j,r)}\beta|_{j+r} = \beta|_{j+r}\tau^{-k-\delta(j,r)}\beta|_j\tau^{k+\delta(j,r)}, \forall k \in \mathbb{Z}\right) \\ \Leftrightarrow\ &\left(\beta|_j[\beta|_j,\tau^{k+\delta(j,r)}]\beta|_{j+r} = \beta|_{j+r}\beta|_j[\beta|_j,\tau^{k+\delta(j,r)}], \forall k \in \mathbb{Z}\right), \end{aligned}$$

$$\beta|_j\beta_t = \beta|_t\beta|_j, \forall j,t \in Y - \{0, \frac{n}{2}\} \tag{7.29}$$

and

$$[\beta|_j,\tau^k]^{\beta|_t} = [\beta|_j,\tau^k], \forall j,t \in Y - \{0, \frac{n}{2}\}. \tag{7.30}$$

□

Lemma 7.5. *The group $N = \left\langle [\beta|_i, \ \tau^k] \mid k \in \mathbb{Z}, i \in Y \right\rangle$ is an abelian normal subgroup of H.*

Proof. Define

$$N_i = \left\langle [\beta|_i,\tau^k] \mid k \in \mathbb{Z} \right\rangle$$

for each $i \in Y$. Then, $N = \langle N_i \mid i \in Y \rangle$, each N_i is an abelian subgroup normalized by τ and

$$[\beta|_i,\tau^k]^{\beta|_j^{-1}} = [\beta|_i,\tau^k], \forall k \in \mathbb{Z}, \forall i,j \in Y, j \neq 0, \frac{n}{2} \tag{7.31}$$

We have $[N_i,N_j] = 1, \forall i,j \in Y, j \neq 0, \frac{n}{2}$, because

$$\begin{aligned} [\beta|_i,\tau^k]^{[\beta|_j,\tau^t]} &= [\beta|_i,\tau^k]^{\beta|_j^{-1}\tau^{-t}\beta|_j\tau^t} \overset{(7.31)}{=} [\beta|_i,\tau^k]^{\tau^{-t}\beta|_j\tau^t} \\ &\overset{(4.1)}{=} \left([\beta|_i,\tau^{-t}]^{-1}[\beta|_i,\tau^{k-t}]\right)^{\beta|_j\tau^t} \\ &\overset{(7.31)}{=} \left([\beta|_i,\tau^{-t}]^{-1}[\beta|_i,\tau^{k-t}]\right)^{\tau^t} \\ \overset{(4.1)}{=} [\beta|_i,\tau^k]^{\tau^{-t}\tau^t} &= [\beta|_i,\tau^k], \forall k,t \in \mathbb{Z}, \end{aligned}$$

$\forall i,j \in Y, j \neq 0, \frac{n}{2}$.

Furthermore, $[N_0, N_{\frac{n}{2}}] = 1$, because

$$\begin{aligned} [\beta|_{\frac{n}{2}},\tau^k]^{[\beta|_0,\tau^t]} &= [\beta|_{\frac{n}{2}},\tau^k]^{\beta|_0^{-1}\tau^{-t}\beta|_0\tau^t} \overset{(7.11)}{=} [\beta|_0,\tau^k]^{\tau\tau^{-t}\beta|_0\tau^t} \\ &\overset{(4.1)}{=} \left([\beta|_0,\tau^{-t}]^{-1}[\beta|_0,\tau^{k-t}]\right)^{\tau\beta|_0\tau^t} \\ &\overset{(7.11)}{=} \left([\beta|_{\frac{n}{2}},\tau^{-t}]^{-1}[\beta|_{\frac{n}{2}},\tau^{k-t}]\right)^{\tau^t} \end{aligned}$$

$$\stackrel{(4.1)}{=} [\beta|_{\frac{n}{2}}, \tau^k]^{\tau^{-t}\tau^t} = [\beta|_{\frac{n}{2}}, \tau^k], \forall k, t \in \mathbb{Z}.$$

Therefore N is abelian.

Now, equation (7.31) implies

$$N_i = N_i^{\beta|_j} = N_i^{\beta|_j^{-1}}, \forall i, j \in Y, j \neq 0, \frac{n}{2}; \tag{7.32}$$

equation (7.11) implies

$$\begin{cases} N_{\frac{n}{2}} = N_0^{\beta|_{\frac{n}{2}}},\ N_0 = N_{\frac{n}{2}}^{\beta|_{\frac{n}{2}}^{-1}} \end{cases}; \tag{7.33}$$

equations (4.1), (7.11) imply

$$\begin{cases} N_{\frac{n}{2}} = N_0^{\beta|_0},\ N_0 = N_{\frac{n}{2}}^{\beta|_0^{-1}} \end{cases}; \tag{7.34}$$

equation (7.14) implies

$$\begin{cases} N_0 = N_{\frac{n}{2}}^{\beta|_0},\ N_{\frac{n}{2}} = N_0^{\beta|_0^{-1}} \end{cases}; \tag{7.35}$$

equations (4.1), (7.16) imply

$$\begin{cases} N_0 = N_{\frac{n}{2}}^{\beta|_{\frac{n}{2}}},\ N_{\frac{n}{2}} = N_0^{\beta|_{\frac{n}{2}}^{-1}} \end{cases}; \tag{7.36}$$

equations (4.1), (7.22) imply

$$\begin{cases} N_j = N_{j+\frac{n}{2}}^{\beta|_0},\ N_{j+\frac{n}{2}} = N_j^{\beta|_0^{-1}} \end{cases}, \forall j \in \{1, \ldots, \frac{n}{2} - 1\}; \tag{7.37}$$

equation (7.24) implies

$$\begin{cases} N_{j+\frac{n}{2}} = N_j^{\beta|_0},\ N_j = N_{j+\frac{n}{2}}^{\beta|_0^{-1}} \end{cases}, \forall j \in \{1, \ldots, \frac{n}{2} - 1\}; \tag{7.38}$$

equations (4.1) and (7.26) imply

$$\begin{cases} N_{j+\frac{n}{2}} = N_j^{\beta|_{\frac{n}{2}}},\ N_j = N_{j+\frac{n}{2}}^{\beta|_{\frac{n}{2}}^{-1}} \end{cases}, \forall j \in \{1, \ldots, \frac{n}{2} - 1\}; \tag{7.39}$$

equation (7.28) implies

$$\begin{cases} N_j = N_{j+\frac{n}{2}}^{\beta|_{\frac{n}{2}}},\ N_{j+\frac{n}{2}} = N_j^{\beta|_{\frac{n}{2}}^{-1}} \end{cases}, \forall j \in \{1, \ldots, \frac{n}{2} - 1\}. \tag{7.40}$$

Thus (7.31)-(7.40) prove

$$\begin{aligned} N &= \langle N_i \mid i \in Y \rangle \\ &= \left\langle [\beta|_i, \tau^k] \mid \forall i, k \in \mathbb{Z} \right\rangle \end{aligned}$$

is an abelian normal subgroup of H. □

Lemma 7.6. *The group $U = \left\langle N, \quad \beta|_j \mid j \neq 0, \frac{n}{2} \right\rangle$ is a normal abelian subgroup of H.*

Proof. Lemma 7.5 and equations (7.13), (7.18), (7.19) and (7.20) show that U is abelian.

The fact that N is normal in H, together with the following assertions prove that U is normal in H.

Let $J = \left\langle \beta_0, \beta_{\frac{n}{2}}, \tau \right\rangle$. Then, for $j \in Y - \{0, \frac{n}{2}\}$, we have

(I) $\langle \beta|_j \rangle^J \leq U$:

$$\beta|_j^{\tau^t} = \beta|_j [\beta|_j, \tau^t];$$

$$\beta|_j^{\beta|_0} \overset{(7.23)}{=} \beta|_{j+\frac{n}{2}};$$

$$\beta|_j^{\beta|_0^{-1}} \overset{(7.21)}{=} \tau^{-1} \beta|_{j+\frac{n}{2}} \tau = \beta|_{j+\frac{n}{2}} [\beta|_{j+\frac{n}{2}}, \tau];$$

$$\beta|_j^{\beta|_{\frac{n}{2}}} \overset{(7.25)}{=} \tau^{-1} \beta|_{j+\frac{n}{2}} \tau = \beta|_{j+\frac{n}{2}} [\beta|_{j+\frac{n}{2}}, \tau];$$

$$\beta|_j^{\beta|_{\frac{n}{2}}^{-1}} \overset{(7.27)}{=} \beta|_{j+\frac{n}{2}};$$

(II) $\left\langle \beta|_{j+\frac{n}{2}} \right\rangle^J \leq U$:

$$\beta|_{j+\frac{n}{2}}^{\tau^t} = \beta|_{j+\frac{n}{2}} [\beta|_{j+\frac{n}{2}}, \tau^t];$$

$$\beta|_{j+\frac{n}{2}}^{\beta|_0} \overset{(7.21)}{=} \beta|_0^{-1} \tau \beta|_0 \beta|_j \beta|_0^{-1} \tau^{-1} \beta|_0$$

$$= \left([\beta|_0, \tau]^{-1}\right)^{\tau^{-1}} \beta|_j^{\tau^{-1}} [\beta|_0, \tau]^{\tau^{-1}} \in U;$$

$$\beta|_{j+\frac{n}{2}}^{\beta|_0^{-1}} \overset{(7.23)}{=} \beta|_j \in U;$$

$$\beta|_{j+\frac{n}{2}}^{\beta|_{\frac{n}{2}}} \overset{(7.27)}{=} \beta|_j \in U;$$

$$\beta|_{j+\frac{n}{2}}^{\beta|_{\frac{n}{2}}^{-1}} \overset{(7.25)}{=} \beta|_{\frac{n}{2}} \tau \beta|_{\frac{n}{2}}^{-1} \beta|_j \beta|_{\frac{n}{2}} \tau^{-1} \beta|_{\frac{n}{2}}^{-1}$$

$$= [\beta|_{\frac{n}{2}}, \tau]^{\beta|_{\frac{n}{2}}^{-1} \tau^{-1}} \beta|_j^{\tau^{-1}} \left([\beta|_{\frac{n}{2}}, \tau]^{-1}\right)^{\beta|_{\frac{n}{2}}^{-1} \tau^{-1}}.$$

Hence, U is a normal abelian subgroup of H. □

Lemma 7.7. $V = \left\langle U, \ \beta|_{\frac{n}{2}} \beta|_0, \ \tau \beta|_0^2 \right\rangle$ *is a normal abelian subgroup of* H.

Proof. Lemma 7.6 together with the following assertions proves that V is a normal abelian subgroup of H.

Given $j \in Y - \{0, \frac{n}{2}\}, k \in \mathbb{Z}$, and $J = \left\langle \beta|_0, \beta_{\frac{n}{2}}, \tau \right\rangle$, we prove

(I) $\beta|_{\frac{n}{2}} \beta|_0 \in C_H(U)$:

$$(\beta|_j)^{\beta|_{\frac{n}{2}} \beta|_0} \overset{(7.25)}{=} (\beta|_{j+\frac{n}{2}})^{\tau \beta|_0} \overset{(7.21)}{=} \beta|_j;$$

$$(\beta|_{j+\frac{n}{2}})^{\beta|_{\frac{n}{2}} \beta|_0} \overset{(7.27)}{=} (\beta|_j)^{\beta|_0} \overset{(7.23)}{=} \beta|_{j+\frac{n}{2}};$$

$$[\beta|_j, \tau^k]^{\beta|_{\frac{n}{2}} \beta|_0} = [\beta|_j, \tau^k]^{\beta|_{\frac{n}{2}} \tau^{-1} \tau \beta|_0} \overset{(7.26)}{=} [\beta|_{j+\frac{n}{2}}, \tau^k]^{\tau \beta|_0}$$

$$\overset{(7.22)}{=} [\beta|_j, \tau^k];$$

$$[\beta|_{j+\frac{n}{2}},\tau^k]^{\beta|_{\frac{n}{2}}\beta|_0} \overset{(7.28)}{=} [\beta|_j,\tau^k]^{\beta|_0} \overset{(7.24)}{=} [\beta|_{j+\frac{n}{2}},\tau^k];$$

$$[\beta|_0,\tau^k]^{\beta|_{\frac{n}{2}}\beta|_0} \overset{(7.9)}{=} [\beta|_{\frac{n}{2}},\tau^k]^{\beta|_0} \overset{(7.14)}{=} [\beta|_0,\tau^k];$$

$$\begin{aligned}[\beta|_{\frac{n}{2}},\tau^k]^{\beta|_{\frac{n}{2}}\beta|_0} &= [\beta|_{\frac{n}{2}},\tau^k]^{\beta|_{\frac{n}{2}}\tau^{-1}\tau\beta|_0}\\ &\overset{(7.16)}{=} [\beta|_0,\tau^k]^{\tau\beta|_0} \overset{(7.11)}{=} [\beta|_{\frac{n}{2}},\tau^k];\end{aligned}$$

(II) $\tau\beta|_0^2 \in C_H(U)$:

$$\beta|_j^{\tau\beta|_0^2} = (\beta|_j[\beta|_j,\tau])^{\beta|_0^2} = (\beta|_j^{\beta|_0}[\beta|_j,\tau]^{\beta|_0})^{\beta|_0}$$

$$\overset{(7.23),(7.24)}{=} (\beta|_{j+\frac{n}{2}}[\beta|_{j+\frac{n}{2}},\tau])^{\beta|_0} = \beta|_{j+\frac{n}{2}}^{\tau\beta|_0} \overset{(7.21)}{=} \beta|_j;$$

$$(\beta|_{j+\frac{n}{2}})^{\tau\beta|_0^2} \overset{(7.21)}{=} \beta|_j^{\beta|_0} \overset{(7.23)}{=} \beta|_{j+\frac{n}{2}};$$

$$[\beta|_0,\tau^k]^{\tau\beta|_0^2} \overset{(7.11)}{=} [\beta|_{\frac{n}{2}},\tau^k]^{\beta|_0} \overset{(7.14)}{=} [\beta|_0,\tau^k];$$

$$\begin{aligned}&[\beta|_{\frac{n}{2}},\tau^k]^{\tau\beta|_0^2} \overset{(4.1)}{=} ([\beta|_{\frac{n}{2}},\tau]^{-1}[\beta|_{\frac{n}{2}},\tau^{k+1}])^{\beta|_0^2}\\ &\overset{(7.14)}{=} ([\beta|_0,\tau]^{-1}[\beta|_0,\tau^{k+1}])^{\beta|_0}\\ &\overset{(4.1)}{=} [\beta|_0,\tau^k]^{\tau\beta|_0} \overset{(7.11)}{=} [\beta|_{\frac{n}{2}},\tau^k];\end{aligned}$$

$$\begin{aligned}&[\beta|_j,\tau^k]^{\tau\beta|_0^2} \overset{(4.1)}{=} ([\beta|_j,\tau]^{-1}[\beta|_j,\tau^{k+1}])^{\beta|_0^2}\\ &\overset{(7.24)}{=} ([\beta|_{j+\frac{n}{2}},\tau]^{-1}[\beta|_{j+\frac{n}{2}},\tau^{k+1}])^{\beta|_0}\\ &\overset{(4.1)}{=} [\beta|_{j+\frac{n}{2}},\tau^k]^{\tau\beta|_0} \overset{(7.22)}{=} [\beta|_j,\tau^k];\end{aligned}$$

$$[\beta|_{j+\frac{n}{2}},\tau^k]^{\tau\beta|_0^2} \overset{(7.22)}{=} [\beta|_j,\tau^k]^{\beta|_0} \overset{(7.24)}{=} [\beta|_{j+\frac{n}{2}},\tau^k];$$

(III) $\tau\beta|_0^2 \in C_H(\beta|_{\frac{n}{2}}\beta|_0)$:

$$\begin{aligned}(\beta|_{\frac{n}{2}}\beta|_0)^{\tau\beta|_0^2} &= \beta|_0^{-2}\tau^{-1}\beta|_{\frac{n}{2}}\beta|_0\tau\beta|_0^2\\ &\overset{(7.10)}{=} \beta|_0^{-2}\tau^{-1}\beta|_{\frac{n}{2}}\beta|_{\frac{n}{2}}\tau^{-1}\beta|_{\frac{n}{2}}\beta|_0\end{aligned}$$

$$\begin{aligned}&= \beta|_0^{-2}\tau^{-1}\beta|_{\frac{n}{2}}^2\tau^{-1}\beta|_{\frac{n}{2}}\beta|_0 = (\tau\beta|_0^2)^{-1}\beta|_{\frac{n}{2}}^2\tau^{-1}\beta|_{\frac{n}{2}}\beta|_0\\ &\overset{(7.15)}{=} \beta|_{\frac{n}{2}}\beta|_0;\end{aligned}$$

.

(IV) $\left\langle \tau\beta|_0^2 \right\rangle^J \leq V$:

$$(\tau\beta|_0^2)^{\tau^k} = \tau(\beta|_0^2)^{\tau^k} = \tau\beta|_0^2[\beta|_0^2, \tau^k] = \tau\beta|_0^2[\beta|_0, \tau^k]^{\beta|_0}[\beta|_0, \tau^k];$$

$$(\tau\beta|_0^2)^{\beta|_0} = \beta|_0^{-1}\tau\beta|_0^2\beta|_0 = \tau\tau^{-1}\beta|_0^{-1}\tau\beta|_0\beta|_0^2 = \tau[\tau, \beta|_0]\beta|_0^2$$

$$= \tau[\tau, \beta|_0]\tau^{-1}\tau\beta|_0^2 = ([\beta|_0, \tau]^{-1})^{\tau^{-1}}\tau\beta|_0^2;$$

$$(\tau\beta|_0^2)^{\beta|_0^{-1}} = \beta|_0\tau\beta|_0 = \tau\beta|_0[\beta|_0, \tau]\beta|_0 = \tau\beta|_0^2[\beta|_0, \tau]^{\beta|_0}; \tag{7.41}$$

$$(\tau\beta|_0^2)^{\beta|_{\frac{n}{2}}^{-1}} \overset{(7.41)}{=} \left((\tau\beta|_0^2)^{\beta|_0^{-1}}([\beta|_0, \tau]^{-1})^{\beta|_0}\right)^{\beta|_{\frac{n}{2}}^{-1}}$$

$$= (\tau\beta|_0^2)^{\beta|_0^{-1}\beta|_{\frac{n}{2}}^{-1}}([\beta|_0, \tau]^{-1})^{\beta|_0\beta|_{\frac{n}{2}}^{-1}}$$

$$= (\tau\beta|_0^2)^{(\beta|_{\frac{n}{2}}\beta|_0)^{-1}}([\beta|_0, \tau]^{-1})^{\beta|_0\beta|_{\frac{n}{2}}^{-1}} \overset{(III)}{=} \tau\beta|_0^2([\beta|_0, \tau]^{-1})^{\beta|_0\beta|_{\frac{n}{2}}^{-1}};$$

$$(\tau\beta|_0^2)^{\beta|_{\frac{n}{2}}} = (\tau\beta|_0^2)^{\beta|_{\frac{n}{2}}\beta|_0\beta|_0^{-1}} \overset{(III)}{=} (\tau\beta|_0^2)^{\beta|_0^{-1}} \overset{(7.41)}{=} \tau\beta|_0^2[\beta|_0, \tau]^{\beta|_0}.$$

(V) $\left\langle \beta|_{\frac{n}{2}}\beta|_0 \right\rangle^J \leq V$:

$$(\beta|_{\frac{n}{2}}\beta|_0)^{\tau^k} = \beta|_{\frac{n}{2}}\beta|_0[\beta|_{\frac{n}{2}}\beta|_0, \tau^k] = \beta|_{\frac{n}{2}}\beta|_0[\beta|_{\frac{n}{2}}, \tau^k]^{\beta|_0}[\beta|_0, \tau^k];$$

$$(\beta|_{\frac{n}{2}}\beta|_0)^{\beta|_0} = \beta|_0^{-1}\beta|_{\frac{n}{2}}\beta|_0^2 = \beta|_0^{-1}\beta|_{\frac{n}{2}}\tau^{-1}\tau\beta|_0^2 \tag{7.42}$$

$$= \beta|_0^{-1}\beta|_{\frac{n}{2}}^{-1}\beta|_{\frac{n}{2}}^2\tau^{-1}\tau\beta|_0^2 \overset{(7.15)}{=} (\beta|_{\frac{n}{2}}\beta|_0)^{-1}(\tau\beta|_0^2)^2;$$

$$\beta|_{\frac{n}{2}}\beta|_0 \overset{(7.42)}{=} (\tau\beta|_0^2)^2((\beta|_{\frac{n}{2}}\beta|_0)^{-1})^{\beta|_0}; \tag{7.43}$$

$$(\beta|_{\frac{n}{2}}\beta|_0)^{\beta|_0^{-1}} \overset{(7.43)}{=} ((\tau\beta|_0^2)^2)^{\beta|_0^{-1}}(\beta|_{\frac{n}{2}}\beta|_0)^{-1};$$

$$(\beta|_{\frac{n}{2}}\beta|_0)^{\beta|_{\frac{n}{2}}^{-1}} = \beta|_{\frac{n}{2}}^2\beta|_0\beta|_{\frac{n}{2}}^{-1} = \beta|_{\frac{n}{2}}^2\tau^{-1}\tau\beta|_0\beta|_0\beta|_0^{-1}\beta|_{\frac{n}{2}}^{-1} \tag{7.44}$$

$$\overset{(7.15)}{=} (\tau\beta|_0^2)^2\beta|_0^{-1}\beta|_{\frac{n}{2}}^{-1} = (\tau\beta|_0^2)^2(\beta|_{\frac{n}{2}}\beta|_0)^{-1};$$

$$(\beta|_{\frac{n}{2}}\beta|_0)^{\beta|_{\frac{n}{2}}} \overset{(7.44)}{=} (\beta|_{\frac{n}{2}}\beta|_0)^{-1}((\tau\beta|_0^2)^2)^{\beta|_{\frac{n}{2}}}$$

□

8. **Solvable groups for** $n = 4$

Let B be an abelian subgroup of $\mathcal{A}_4 = Aut(T_4)$ normalized by τ and let $\beta \in B$. Then, by Proposition 4.1 , $\sigma_\beta \in D = \langle (0,1,2,3),(0,2) \rangle$, the unique Sylow 2-subgroup of Σ_4 which contains $\sigma = \sigma_\tau = (0,1,2,3)$.

The normalizer of $\overline{\langle \tau \rangle}$ here is $\Gamma_0 = N_{\mathcal{A}_4}\left(\overline{\langle \tau \rangle}\right) = \langle \Psi, \iota \rangle$ where Ψ is the monic normalizer and where $\iota = \iota^{(1)}(0,3)(1,2)$ inverts τ.

Given a group W, the subgroup generated by squares of its elements is denoted by W^2.

Lemma 8.1. *Let $L = L(D)$ be the layer closure of D above. If $\gamma \in L^2$ then $\gamma\tau$ is conjugate to τ.*

Proof. If $\alpha \in L$ then $\sigma_{\alpha^2} \in \langle \sigma^2 \rangle$ and the product in some order of the states $(\alpha^2)|_i$ $(0 \le i \le 3)$ belongs to $S = L^2$.

Let $\gamma \in S$. Then $\gamma\tau$ is transitive on the 1*st* level of the tree and $(\gamma\tau)^4$ is inactive with conjugate 1st level states, where the first state is

$$(\gamma|_0)(\gamma|_1)(\gamma|_2)(\gamma|_3)\tau \text{ if } \sigma_\gamma = e,$$

and

$$(\gamma|_0)(\gamma|_3)(\gamma|_2)(\gamma|_1)\tau \text{ if } \sigma_\gamma = \sigma^2;$$

in both cases the element is contained in $S\tau$. Therefore, $\gamma\tau$ is transitive on the 2nd level of the tree. Now use induction to prove that $\gamma\tau$ is transitive on all levels of the tree. As $\gamma\tau$ is transitive on all levels of the tree, then $\gamma\tau$ is conjugate to τ. □

8.1. **Cases** $\sigma_\beta \in \{(0,3)(1,2),(0,1)(2,3)\}$.

We will show that these cases cannot occur. We note that σ_τ conjugates $(0,1)(2,3)$ to $(0,3)(1,2)$. Since the argument for β applies to β^τ, it is sufficient to consider the first case.

Suppose $\sigma_\beta = (0,1)(2,3)$. Then,

$$\beta^\tau = \left(\tau^{-1}(\beta|_3), \beta|_0, \beta|_1, \beta|_2\tau\right)(\sigma_\beta)^{\sigma_\tau}.$$

On substituting $\alpha = \beta^\tau$ in $\theta = [\beta, \alpha]$ and in (2.5)

$$\theta|_{(i)\sigma_{\alpha\beta}} = \left(\beta|_{(i)\sigma_\alpha}\right)^{-1}(\alpha|_i)^{-1}(\beta|_i)\left(\alpha|_{(i)\sigma_\beta}\right), \forall i \in Y. \tag{8.1}$$

we get $\theta = e$ and

$$e = \left(\beta|_{(i)\sigma_{\beta^\tau}}\right)^{-1}(\beta^\tau|_i)^{-1}(\beta|_i)\left(\beta^\tau|_{(i)\sigma_\beta}\right), \forall i \in Y \tag{8.2}$$

and so for the index $i = 0$, we obtain

$$\begin{aligned} e &= (\beta|_3)^{-1}\left(\tau^{-1}(\beta|_3)\right)^{-1}(\beta|_0)(\beta|_0), \\ e &= (\beta|_3)^{-2}\tau(\beta|_0)^2 \end{aligned}$$

which is impossible.

8.2. **Cases** $\sigma_\beta \in \{(0,2),(1,3)\}$**.**

Lemma 8.2. *Let* $\alpha, \gamma \in Aut(T_4)$ *be such that*

$$\begin{aligned} \sigma_\alpha, \sigma_\gamma &\in \langle (0,1,2,3),(0,2)\rangle, \\ \tau^{-1}\alpha^2 &= \gamma^2\tau, \\ [\alpha,\tau^k]^\gamma &= [\gamma,\tau^k] \end{aligned}$$

for all $k \in \mathbb{Z}$*. Then,*

$$\sigma_\alpha, \sigma_\gamma \in \langle \sigma \rangle, \qquad \sigma_\alpha\sigma_\gamma = \sigma^{\pm 1}.$$

Proof. From the second and third equations above, we have $\sigma^{-1}\sigma_\alpha^2 = \sigma_\gamma^2\sigma$ and $[\sigma_\alpha,\sigma^k]^{\sigma_\gamma} = [\sigma_\gamma,\sigma^k]$.

(i) Suppose $\sigma_\gamma^2 = e$. Then $\sigma_\alpha^2 = \sigma^2$ and therefore, $\sigma_\alpha = \sigma^{\pm1}$, $[\sigma_\alpha,\sigma^k]^{\sigma_\gamma} = [\sigma_\gamma,\sigma^k] = e$ for all k; thus, $\sigma_\gamma \in \langle\sigma\rangle$ and $\sigma_\gamma \in \langle\sigma^2\rangle$, $\sigma_\alpha\sigma_\gamma = \sigma^{\pm1}$ follows.

(ii) Suppose $o(\sigma_\gamma) = 4$. Then, $\sigma_\gamma = \sigma^{\pm1}$ and $\sigma_\alpha^2 = e$. Since $[\sigma_\alpha,\sigma^k]^{\sigma_\gamma} = e$ for all k, we obtain $\sigma_\alpha \in \langle\sigma\rangle$, $\sigma_\alpha^2 = e$ and $\sigma_\alpha \in \langle\sigma^2\rangle$.Therefore, $\sigma_\alpha\sigma_\gamma = \sigma^{\pm1}$. □

(1) Suppose $\sigma_\beta = (0,2)$. Then by the analysis in Section 7.2, we conclude

$$V = \left\langle [\beta|_i,\tau^k], \beta|_1, \beta|_3, \beta|_2\beta|_0, \tau\beta|_0^2 \mid i \in Y, k \in \mathbb{Z} \right\rangle$$

is an abelian normal subgroup of H.

By Lemma 8.1 , $\tau\beta|_0^2 = \mu$ is a conjugate of τ. As V is abelian, there exist $\xi, t_1, t_2 \in \mathbb{Z}_4$ such that

$$\mu = \tau\beta|_0^2, \beta|_2\beta|_0 = \mu^\xi, \beta|_1 = \mu^{t_1}, \beta|_3 = \mu^{t_2}.$$

Therefore,

$$\beta|_2 = \mu^\xi\beta|_0^{-1}, \tau = \mu\beta|_0^{-2}.$$

On substituting $\gamma = \beta|_0$ and $\alpha = \beta|_2$ in (7.14) and (7.15), by Lemma 8.2, we obtain $\sigma_{\alpha\gamma} = \sigma_{\beta|_2\beta|_0} = \sigma^{\pm1}$. Thus, from $\beta|_2\beta|_0 = \mu^\xi$, we reach $\xi \in U(\mathbb{Z}_4)$.

By (7.15), we have

$$\beta|_2^2\tau^{-1} = \tau\beta|_0^2.$$

It follows then that

$$\begin{aligned} \mu^\xi\beta|_0^{-1}\mu^\xi\beta|_0^{-1}\beta|_0^2\mu^{-1} &= \mu, \\ \left(\mu^\xi\right)^{\beta|_0} &= \mu^{2-\xi}. \end{aligned}$$

Therefore,

$$\mu^{\beta|_0} = \mu^{\frac{2-\xi}{\xi}} \tag{8.3}$$

where $\frac{2-\xi}{\xi} \in \mathbb{Z}_4^1$.

By Equation (7.23) we have

$$\beta|_1^{\beta|_0} = \beta|_3.$$

It follows that

$$(\mu^{t_1})^{\beta|_0} = \mu^{t_2},\ \mu^{t_1\frac{2-\xi}{\xi}} = \mu^{t_2},\ t_2 = t_1\frac{2-\xi}{\xi}.$$

We have reached the form of β,

$$\beta = (\beta|_0, \mu^{t_1}, \mu^{\xi}\beta|_0^{-1}, \mu^{t_1\frac{2-\xi}{\xi}})(0,2)$$

where $\mu = \tau^{\alpha}$ for some $\alpha \in Aut(T_4)$.

Since $\mu^{\beta|_0} = \mu^{\frac{2-\xi}{\xi}}$, we have $\beta|_0 = \left(\lambda_{\frac{2-\xi}{\xi}}\tau^m\right)^{\alpha}$ for some $m \in \mathbb{Z}_4$.

Hence,

$$\begin{aligned} \mu^{t_1} &= (\tau^{t_1})^{\alpha}, \\ \mu^{\xi}\beta|_0^{-1} &= \left(\tau^{\xi}\left(\lambda_{\frac{2-\xi}{\xi}}\tau^m\right)^{-1}\right)^{\alpha} \\ &= \left(\lambda_{\frac{\xi}{2-\xi}}\tau^{(\xi-m)\frac{\xi}{2-\xi}}\right)^{\alpha}. \end{aligned}$$

Thus

$$\beta = (\lambda_{\frac{2-\xi}{\xi}}\tau^m, \tau^{t_1}, \lambda_{\frac{\xi}{2-\xi}}\tau^{(\xi-m)\frac{\xi}{2-\xi}}, \tau^{t_1\frac{2-\xi}{\xi}})^{\alpha^{(1)}}(0,2)$$

and

$$\begin{aligned} \tau &= \mu\beta|_0^{-2} \\ &= \left(\tau\left(\lambda_{\frac{2-\xi}{\xi}}\tau^m\right)^{-2}\right)^{\alpha} \\ &= \left(\lambda_{(\frac{\xi}{2-\xi})^2}\tau^{\left(1-\frac{2m}{\xi}\right)\left(\frac{\xi}{2-\xi}\right)^2}\right)^{\alpha} \end{aligned}$$

We note that in case $\xi = 1, \beta$ has the form

$$\beta = (\tau^m, \tau^{t_1}, \tau^{1-m}, \tau^{t_1})^{\alpha^{(1)}}(0,2)$$

where $\tau = \left(\tau^{1-2m}\right)^{\alpha}$; therefore,

$$\beta = (\tau^{\frac{m}{1-2m}}, \tau^{\frac{t_1}{1-2m}}, \tau^{\frac{1-m}{1-2m}}, \tau^{\frac{t_1}{1-2m}})(0,2).$$

(2) Suppose $\sigma_\beta = (1,3)$. Then, $\gamma = \beta^{\tau}$ satisfies $[\gamma, \gamma^{\tau^k}] = e$. Therefore, the previous case applies and

$$\gamma = (\lambda_{\frac{2-\xi}{\xi}}\tau^m, \tau^{t_1}, \lambda_{\frac{\xi}{2-\xi}}\tau^{(\xi-m)\frac{\xi}{2-\xi}}, \tau^{t_1\frac{2-\xi}{\xi}})^{\alpha^{(1)}}(0,2),$$

where

$$\tau = \left(\lambda_{(\frac{\xi}{2-\xi})^2}\tau^{\left(1-\frac{2m}{\xi}\right)\left(\frac{\xi}{2-\xi}\right)^2}\right)^{\alpha} = (e,e,e,\left(\lambda_{(\frac{\xi}{2-\xi})^2}\tau^{\left(1-\frac{2m}{\xi}\right)\left(\frac{\xi}{2-\xi}\right)^2}\right)^{\alpha})\sigma_\tau.$$

Hence, β has the form

$$\beta = \gamma^{\tau^{-1}} = (\tau^{t_1}, \lambda_{\frac{2-\xi}{\xi}}\tau^{1+m-\xi}, \tau^{t_1\frac{2-\xi}{\xi}}, \lambda_{\frac{\xi}{2-\xi}}\tau^{(1-m)\frac{\xi}{2-\xi}})^{\alpha^{(1)}}(1,3).$$

8.3. **The case $\sigma_\beta = (\sigma_\tau)^2 = (0,2)(1,3)$.** We know that

$$V = \left\langle N, \beta|_i\beta|_{i+2}, \beta|_j^2\tau^{-\Delta(j,j+2)} \mid i,j \in Y \text{ and } k \in \mathbb{Z} \right\rangle$$

is an abelian normal subgroup of H and

$$\tau^{\Delta(i,j)}\beta|_{i+2}\beta|_j\tau^{\Delta(i,j)} = \beta|_{j+2}\beta|_i, \tag{8.4}$$

by analysis of the case 7.1.

From Lemmas 8.1 and 8.2, we have

$$\tau\beta|_0^2 = \mu,\ \beta|_2\beta|_0 = \mu^{\xi_0},\ \beta|_3\beta|_1 = \mu^{\xi_1},\ \tau\beta|_1^2 = \mu^{\xi_2}$$

where $\mu = \tau^\alpha$ and $\xi_0, \xi_1, \xi_2 \in U(\mathbb{Z}_4)$. Therefore,

$$\tau = \mu\beta|_0^{-2} \tag{8.5}$$

$$\beta|_2 = \mu^{\xi_0}\beta|_0^{-1} \tag{8.6}$$

$$\beta|_3 = \mu^{\xi_1}\beta|_1^{-1} \tag{8.7}$$

$$\tau = \mu^{\xi_2}\beta|_1^{-2}. \tag{8.8}$$

Now, we let i,j take their values from Y in (8.4). Note that (i,j) and (j,i) produce equivalent equations and the case where $i=j$ is a tautology. Thus we have to treat the cases $(i,j) = (0,1), (0,2), (1,3), (2,3), (0,3), (1,2)$. Indeed, the last two cases turn out to be superfluous.

(i) Substitute $i=0, j=2$ in (8.4), to obtain

$$\beta|_2^2\tau^{-1} = \tau\beta|_0^2 \tag{8.9}$$

Use (8.5) and (8.6) in (8.9) to get

$$\mu^{\xi_0}\beta|_0^{-1}\mu^{\xi_0}\beta|_0^{-1}\beta|_0^2\mu^{-1} = \mu$$

and so,

$$(\mu^{\xi_0})^{\beta|_0} = \mu^{2-\xi_0}.$$

Therefore,

$$\mu^{\beta|_0} = \mu^{\frac{2-\xi_0}{\xi_0}} \tag{8.10}$$

Since $\frac{2-\xi_0}{\xi_0} \in \mathbb{Z}_4^1$, we find

$$\beta|_0 = \left(\lambda_{\frac{2-\xi_0}{\xi_0}}\tau^{m_0}\right)^\alpha. \tag{8.11}$$

From (8.6),

$$\beta|_2 = \mu^{\xi_0}\beta|_0^{-1} = \left(\tau^{\xi_0}\tau^{-m_0}\lambda_{\frac{\xi_0}{2-\xi_0}}\right)^\alpha = \left(\lambda_{\frac{\xi_0}{2-\xi_0}}\tau^{(\xi_0-m_0)\frac{\xi_0}{2-\xi_0}}\right)^\alpha. \tag{8.12}$$

(ii) Substitute $i = 1, j = 3$ in (8.4) to get

$$\beta|_3^2\tau^{-1} = \tau\beta|_1^2. \tag{8.13}$$

On using (8.7) and (8.8) in (8.13), we obtain

$$\mu^{\xi_1}\beta|_1^{-1}\mu^{\xi_1}\beta|_1^{-1}\beta|_1^2\mu^{-\xi_2} = \mu^{\xi_2}$$

and so,

$$(\mu^{\xi_1})^{\beta|_1} = \mu^{2\xi_2-\xi_1}.$$

Therefore,

$$\mu^{\beta|_1} = \mu^{\frac{2\xi_2-\xi_1}{\xi_1}}. \tag{8.14}$$

Since $\frac{2\xi_2-\xi_1}{\xi_1} \in \mathbb{Z}_4^1$, we have

$$\beta|_1 = \left(\lambda_{\frac{2\xi_2-\xi_1}{\xi_1}}\tau^{m_1}\right)^\alpha. \tag{8.15}$$

By (8.7), we find

$$\beta|_3 = \mu^{\xi_1}\beta|_1^{-1} = \left(\tau^{\xi_1}\tau^{-m_1}\lambda_{\frac{\xi_1}{2\xi_2-\xi_1}}\right)^\alpha = \left(\lambda_{\frac{\xi_1}{2\xi_2-\xi_1}}\tau^{(\xi_1-m_1)\frac{\xi_1}{2\xi_2-\xi_1}}\right)^\alpha. \tag{8.16}$$

(iii) Substitute $i = 0, j = 1$ in (8.4) to get

$$\beta|_2\beta|_1 = \beta|_3\beta|_0. \tag{8.17}$$

Use (8.11), (8.12), (8.15) and (8.16) in (8.17), to obtain

$$\lambda_{\frac{\xi_0}{2-\xi_0}}\tau^{(\xi_0-m_0)\frac{\xi_0}{2-\xi_0}}\lambda_{\frac{2\xi_2-\xi_1}{\xi_1}}\tau^{m_1} = \lambda_{\frac{\xi_1}{2\xi_2-\xi_1}}\tau^{(\xi_1-m_1)\frac{\xi_1}{2\xi_2-\xi_1}}\lambda_{\frac{2-\xi_0}{\xi_0}}\tau^{m_0}$$

and so,

$$\lambda_{\frac{\xi_0}{2-\xi_0}\frac{2\xi_2-\xi_1}{\xi_1}}\tau^{(\xi_0-m_0)\frac{\xi_0}{2-\xi_0}\frac{2\xi_2-\xi_1}{\xi_1}+m_1} = \lambda_{\frac{\xi_1}{2\xi_2-\xi_1}\frac{2-\xi_0}{\xi_0}}\tau^{(\xi_1-m_1)\frac{\xi_1}{2\xi_2-\xi_1}\frac{2-\xi_0}{\xi_0}+m_0}.$$

Therefore,

$$\left(\frac{\xi_1}{2\xi_2-\xi_1}\right)^2 = \left(\frac{\xi_0}{2-\xi_0}\right)^2 \tag{8.18}$$

and

$$(\xi_0-m_0)\frac{\xi_0}{2-\xi_0}\frac{2\xi_2-\xi_1}{\xi_1} + m_1 = (\xi_1-m_1)\frac{\xi_1}{2\xi_2-\xi_1}\frac{2-\xi_0}{\xi_0} + m_0. \tag{8.19}$$

(iv) Substitute $i = 2, j = 3$ in (8.4) to get

$$\beta|_0\beta|_3 = \beta|_1\beta|_2. \tag{8.20}$$

Use (8.11), (8.12), (8.15) and (8.16) in (8.20), to obtain

$$\lambda_{\frac{2-\xi_0}{\xi_0}}\tau^{m_0}\lambda_{\frac{\xi_1}{2\xi_2-\xi_1}}\tau^{(\xi_1-m_1)\frac{\xi_1}{2\xi_2-\xi_1}} = \lambda_{\frac{2\xi_2-\xi_1}{\xi_1}}\tau^{m_1}\lambda_{\frac{\xi_0}{2-\xi_0}}\tau^{(\xi_0-m_0)\frac{\xi_0}{2-\xi_0}}$$

and so,

$$\lambda_{\frac{\xi_0}{2-\xi_0}\frac{\xi_1}{2\xi_2-\xi_1}}\tau^{m_0\frac{\xi_1}{2\xi_2-\xi_1}+(\xi_1-m_1)\frac{\xi_1}{2\xi_2-\xi_1}} = \lambda_{\frac{2\xi_2-\xi_1}{\xi_1}\frac{\xi_0}{2-\xi_0}}\tau^{m_1\frac{\xi_0}{2-\xi_0}+(\xi_0-m_0)\frac{\xi_0}{2-\xi_0}}.$$

Therefore,

$$\left(\frac{\xi_1}{2\xi_2-\xi_1}\right)^2 = \left(\frac{\xi_0}{2-\xi_0}\right)^2$$

and

$$m_0\frac{\xi_1}{2\xi_2-\xi_1} + (\xi_1-m_1)\frac{\xi_1}{2\xi_2-\xi_1} = m_1\frac{\xi_0}{2-\xi_0} + (\xi_0-m_0)\frac{\xi_0}{2-\xi_0}. \tag{8.21}$$

We have from (8.18)

$$\frac{\xi_0}{2-\xi_0} = \pm\frac{\xi_1}{2\xi_2-\xi_1}. \tag{8.22}$$

(a) If

$$\frac{\xi_0}{2-\xi_0} = \frac{\xi_1}{2\xi_2-\xi_1},$$

then

$$2\xi_2\xi_0 - \xi_1\xi_0 = 2\xi_1 - \xi_1\xi_0,$$

and so,

$$\xi_2 = \frac{\xi_1}{\xi_0}. \tag{8.23}$$

From (8.19), we get

$$m_1 = \frac{\xi_1-\xi_0}{2} + m_0. \tag{8.24}$$

(b) If

$$\frac{\xi_0}{2-\xi_0} = -\frac{\xi_1}{2\xi_2-\xi_1}$$

then by (8.19) and (8.21),

$$m_0 - \xi_0 + m_1 = m_1 - \xi_1 + m_0$$

$$m_0 + \xi_1 - m_1 = -m_1 - \xi_0 + m_0,$$

which implies $\xi_1 = \xi_0 = 0$,which is impossible.

Now by (8.23) and (8.24), we have

$$\beta|_1 = \left(\lambda_{\frac{2-\xi_0}{\xi_0}}\tau^{\frac{\xi_1-\xi_0}{2}+m_0}\right)^\alpha \tag{8.25}$$

and

$$\beta|_3 = \left(\lambda_{\frac{\xi_0}{2-\xi_0}}\tau^{\left(\frac{\xi_1+\xi_0}{2}-m_0\right)\frac{\xi_0}{2-\xi_0}}\right)^\alpha. \tag{8.26}$$

Therefore,

$$\beta = (\beta|_0, \beta|_1, \beta|_2, \beta|_3)(0,2)(1,3)$$

where $\beta|_0, \beta|_1, \beta|_2$ and $\beta|_3$ are described in (8.11),(8.25), (8.12) and (8.26), respectively, and

$$\begin{aligned} \tau &= \mu\beta|_0^{-2} \\ &= \left(\tau\left(\lambda_{\frac{2-\xi_0}{\xi_0}}\tau^{m_0}\right)^{-2}\right)^{\alpha} \\ &= \left(\lambda_{(\frac{\xi_0}{2-\xi_0})^2}\tau^{\left(1-\frac{2m_0}{\xi_0}\right)\left(\frac{\xi_0}{2-\xi_0}\right)^2}\right)^{\alpha}. \end{aligned}$$

(v) The cases $(i,j) = (1,2), (0,3)$ in (8.4) do not add any more information about β.

Summarizing, we have found

$$\beta|_0 = \left(\lambda_{\frac{2-\xi_0}{\xi_0}}\tau^{m_0}\right)^{\alpha},\ \beta|_1 = \left(\lambda_{\frac{2-\xi_0}{\xi_0}}\tau^{\frac{\xi_1-\xi_0}{2}+m_0}\right)^{\alpha}, \tag{8.27}$$

$$\beta|_2 = \left(\lambda_{\frac{\xi_0}{2-\xi_0}}\tau^{(\xi_0-m_0)\frac{\xi_0}{2-\xi_0}}\right)^{\alpha}, \beta|_3 = \left(\lambda_{\frac{\xi_0}{2-\xi_0}}\tau^{\left(\frac{\xi_1+\xi_0}{2}-m_0\right)\frac{\xi_0}{2-\xi_0}}\right)^{\alpha}, \tag{8.28}$$

$$\tau = \left(\lambda_{(\frac{\xi_0}{2-\xi_0})^2}\tau^{\left(1-\frac{2m_0}{\xi_0}\right)\left(\frac{\xi_0}{2-\xi_0}\right)^2}\right)^{\alpha}. \tag{8.29}$$

In the particular case where $\xi_0 = 1$, β has the form

$$\beta = (\tau^{\frac{m_0}{1-2m_0}}, \tau^{\frac{\frac{\xi_1-1}{2}+m_0}{1-2m_0}}, \tau^{\frac{1-m_0}{1-2m_0}}, \tau^{\frac{\frac{\xi_1+1}{2}-m_0}{1-2m_0}})(0,2)(1,3)$$

where $\tau = \left(\tau^{1-2m_0}\right)^{\alpha}$.

8.4. **Cases** $\sigma_\beta \in \{e, \sigma_\tau, \sigma_\tau^{-1}\}$**.** (1) Suppose $\sigma_\beta = e$ and let β stabilize the kth level of the tree. Then by Proposition 4.3, we have

$$[\beta|_u, \beta|_v^{\tau^{\xi}}] = e, \text{ for all } u, v \in \mathcal{M} \text{ with } |u| = |v| = k.$$

Therefore, $\dot{N} = \langle \beta|_w \mid |w| = k, w \in \mathcal{M}\rangle$ is abelian and so is its normal closure $\dot{M}$ under $\left\langle \dot{N}, \tau \right\rangle$. Also, active elements in $\dot{M}$ are characterized in 8.1, 8.2, 8.3 and 8.4. In particular, there exists $\kappa \in \dot{M}$ such that $\sigma_\kappa = (0,2)(1,3)$ and $\beta \in \times_{p^k} C(\kappa)$.

(2) Suppose $\sigma_\beta = \sigma_\tau = (0,1,2,3)$. Then, clearly the element

$$\beta^2 = (\beta|_0\beta|_1,\ \beta|_1\beta|_2,\ \beta|_2\beta|_3,\ \beta|_3\beta|_0)(0,2)(1,3)$$

satisfies $[\beta^2, \left(\beta^2\right)^{\tau^k}] = e$ for all $k \in \mathbb{Z}_4$. Therefore, by the previous analysis, we have

$$\beta|_0\beta|_1 = \left(\lambda_{\frac{2-\xi_0}{\xi_0}}\tau^{m_0}\right)^{\alpha}, \tag{8.30}$$

$$\beta|_1\beta|_2 = \left(\lambda_{\frac{2-\xi_0}{\xi_0}}\tau^{\frac{\xi_1-\xi_0}{2}+m_0}\right)^{\alpha}, \tag{8.31}$$

$$\beta|_2\beta|_3 = \left(\lambda_{\frac{\xi_0}{2-\xi_0}}\tau^{(\xi_0-m_0)\frac{\xi_0}{2-\xi_0}}\right)^{\alpha}, \tag{8.32}$$

$$\beta|_3\beta|_0 = \left(\lambda_{\frac{\xi_0}{2-\xi_0}}\tau^{\left(\frac{\xi_1+\xi_0}{2}-m_0\right)\frac{\xi_0}{2-\xi_0}}\right)^{\alpha}, \tag{8.33}$$

$$\tau = \left(\lambda_{(\frac{\xi_0}{2-\xi_0})^2}\tau^{\left(1-\frac{2m_0}{\xi_0}\right)\left(\frac{\xi_0}{2-\xi_0}\right)^2}\right)^{\alpha}. \tag{8.34}$$

Hence, multiplying (8.30) by (8.32), we obtain

$$\beta|_0\beta|_1\beta|_2\beta|_3 = \left(\lambda_{\frac{2-\xi_0}{\xi_0}}\tau^{m_0}\lambda_{\frac{\xi_0}{2-\xi_0}}\tau^{(\xi_0-m_0)\frac{\xi_0}{2-\xi_0}}\right)^{\alpha} = \left(\tau^{\frac{\xi_0^2}{2-\xi_0}}\right)^{\alpha}. \tag{8.35}$$

We define

$$\psi_\eta = \begin{cases} \lambda_\eta, & \text{if } \eta \in \mathbb{Z}_4^1 \\ \theta\lambda_{-\eta}, & \text{if } -\eta \in \mathbb{Z}_4^1 \end{cases}, \tag{8.36}$$

$$\theta = \theta^{(1)}(e,\tau^{-1},\tau^{-1},\tau^{-1})(1,3)$$

(an invertor of τ) and $\gamma = (e, (\beta|_0)^{-1}, (\beta|_0\beta|_1)^{-1}, (\beta|_0\beta|_1\beta|_2)^{-1})\left(\alpha^{-1}\psi_{\frac{2-\xi_0}{\xi_0^2}}\right)^{(1)}$.

We verify, by (8.35), that γ conjugates β to

$$(e,e,e,\beta|_0\beta|_1\beta|_2\beta|_3)^{\left(\alpha^{-1}\psi_{\frac{2-\xi_0}{\xi_0^2}}\right)^{(1)}}\sigma$$

which is equal to τ.

(3) Suppose $\sigma_\beta = \sigma_\tau^{-1} = (0,3,2,1)$. Then, β^{-1} satisfies the previous case and $\beta^{-1} = \tau^\gamma$ for some $\gamma \in \mathcal{A}_4$. Therefore, as θ inverts τ, we have

$$\beta = \left(\beta^{-1}\right)^{-1} = (\tau^\gamma)^{-1} = (\tau)^{\theta\gamma} \tag{8.37}$$

8.5. **Final Step.** We finish the proof of the second part of Theorem A. For the case where the activity of β is a 4-cycle, we use the fact that $\beta^2 \in B$, which we have already described. Next, from the description of the centralizer of β^2, we are able to pin down the form of β.

Proposition 8.3. *Let $\beta = (\beta|_0, \beta|_1, \beta|_2, \beta|_3)(0,2)(1,3)$ be such that $(\beta|_0)(\beta|_2) = \tau^{\theta_1}$ and $(\beta|_1)(\beta|_3) = \tau^{\theta_2}$, for some $\theta_1, \theta_2 \in Aut(T_4)$. Then, β is conjugate to τ^2.*

Proof. Let $\alpha = (e, e, \beta|_0^{-1}, \beta|_1^{-1})$. Then,

$$\beta^{\alpha} = (e, e, \ \beta|_0\beta|_2 \ , \beta|_1\beta|_3)(0,2)(1,3). \tag{8.38}$$

Therefore, substituting $\beta|_0\beta|_2 = \tau^{\theta_1}$ and $\beta|_1\beta|_3 = \tau^{\theta_2}$ in the above equation, we have

$$\beta^{\alpha} = (e, e, \tau^{\theta_1}, \tau^{\theta_2})(0,2)(1,3).$$

Conjugating β^{α} by $\gamma = (\theta_1^{-1}, \theta_2^{-1}, \theta_1^{-1}, \theta_2^{-1})$ we produce

$$\beta^{\alpha\gamma} = \tau^2.$$

□

We show below that active elements of B produce within B elements conjugate to τ^2.

Proposition 8.4. *Let $\beta \in B$ with nontrivial σ_β. Then*

(i) *If $\sigma_\beta = \sigma_\tau^2$, then β is a conjugate of τ^2.*
(ii) *If $\sigma_\beta \in \{(0,2),(1,3)\}$, then $\beta\beta^\tau$ is a conjugate τ^2.*
(iii) *If $\sigma_\beta \in \{\sigma_\tau, \sigma_\tau^{-1}\}$, then β^2 is a conjugate of τ^2.*

Proof. It is enough to prove (i), since (ii), (iii) are just special cases.

If $\sigma_\beta = \sigma_\tau^2$, then

$$\beta|_0 = \left(\lambda_{\frac{2-\xi_0}{\xi_0}}\tau^{m_0}\right)^{\alpha}, \ \beta|_1 = \left(\lambda_{\frac{2-\xi_0}{\xi_0}}\tau^{\frac{\xi_1-\xi_0}{2}+m_0}\right)^{\alpha}, \tag{8.39}$$

$$\beta|_2 = \left(\lambda_{\frac{\xi_0}{2-\xi_0}}\tau^{(\xi_0-m_0)\frac{\xi_0}{2-\xi_0}}\right)^{\alpha}, \beta|_3 = \left(\lambda_{\frac{\xi_0}{2-\xi_0}}\tau^{\left(\frac{\xi_1+\xi_0}{2}-m_0\right)\frac{\xi_0}{2-\xi_0}}\right)^{\alpha}, \tag{8.40}$$

$$\tau = \left(\lambda_{(\frac{\xi_0}{2-\xi_0})^2}\tau^{\left(1-\frac{2m_0}{\xi_0}\right)\left(\frac{\xi_0}{2-\xi_0}\right)^2}\right)^{\alpha}, \tag{8.41}$$

where $\xi_0, \xi_1 \in U(\mathbb{Z}_4), \ m_0 \in \mathbb{Z}_4$.

Therefore,

$$\beta|_0\beta|_2 = \left(\lambda_{\frac{2-\xi_0}{\xi_0}}\tau^{m_0}\lambda_{\frac{\xi_0}{2-\xi_0}}\tau^{(\xi_0-m_0)\frac{\xi_0}{2-\xi_0}}\right)^{\alpha} = \left(\tau^{\frac{\xi_0^2}{2-\xi_0}}\right)^{\alpha} = \tau^{\left(\psi\frac{\xi_0^2}{2-\xi_0}\right)\alpha}$$

$$\beta|_1\beta|_3 = \left(\lambda_{\frac{2-\xi_0}{\xi_0}}\tau^{\frac{\xi_1-\xi_0}{2}+m_0}\lambda_{\frac{\xi_0}{2-\xi_0}}\tau^{\left(\frac{\xi_1+\xi_0}{2}-m_0\right)\frac{\xi_0}{2-\xi_0}}\right)^{\alpha} = \left(\tau^{\frac{\xi_1\xi_0}{2-\xi_0}}\right)^{\alpha} = \tau^{\left(\psi\frac{\xi_1\xi_0}{2-\xi_0}\right)\alpha}$$

It follows from Proposition 8.3, that β is a conjugate of τ^2. □

Corollary 8.5. *Suppose $\beta \in B$ is an active element. Then, B is conjugate to a subgroup of the centralizer $C(\tau^2)$.*

Proposition 8.6. *Let $\gamma \in C(\tau^2)$. Then,*

$$\gamma = (\tau^{m_0}, \tau^{m_1}, \tau^{m_0+\delta((0)\sigma_\gamma,\ 2)}, \tau^{m_1+\delta((1)\sigma_\gamma,\ 2)})\sigma_\gamma, \tag{8.42}$$

where $m_0, m_1 \in \mathbb{Z}_4, \sigma_\gamma \in C_{\Sigma_4}(\sigma^2)$.

Proof. Write $\gamma = (\gamma|_0, \gamma|_1, \gamma|_2, \gamma|_3)\sigma_\gamma$. Then $\tau^2\gamma = \gamma\tau^2$ translates to

$$\begin{aligned} & (e,e,\tau,\tau)(0,2)(1,3)(\gamma|_0, \gamma|_1, \gamma|_2, \gamma|_3)\sigma_\gamma \\ = \ & (\gamma|_0, \gamma|_1, \gamma|_2, \gamma|_3)\sigma_\gamma(e,e,\tau,\tau)(0,2)(1,3), \end{aligned}$$

and this in turn translates to

$$\begin{aligned} & (\gamma|_2, \gamma|_3, \tau\gamma|_0, \tau\gamma|_1)(0,2)(1,3)\sigma_\gamma \\ = \ & (\gamma|_0, \gamma|_1, \gamma|_2, \gamma|_3)\sigma_\gamma(\tau^{\delta(0,2)}, \tau^{\delta(1,2)}, \tau^{\delta(2,2)}, \tau^{\delta(3,2)})(0,2)(1,3) \\ = \ & (\gamma|_0, \gamma|_1, \gamma|_2, \gamma|_3)(\tau^{\delta((0)\sigma_\gamma,2)}, \tau^{\delta((1)\sigma_\gamma,2)}, \tau^{\delta((2)\sigma_\gamma,2)}, \tau^{\delta((3)\sigma_\gamma,2)})\sigma_\gamma(0,2)(1,3) \\ = \ & (\gamma|_0\tau^{\delta((0)\sigma_\gamma,2)}, \gamma|_1\tau^{\delta((1)\sigma_\gamma,2)}, \gamma|_2\tau^{\delta((2)\sigma_\gamma,2)}, \gamma|_3\tau^{\delta((3)\sigma_\gamma,2)})\sigma_\gamma(0,2)(1,3) \end{aligned}$$

Thus, we have

$$\begin{cases} \gamma|_2 = \gamma|_0\tau^{\delta((0)\sigma_\gamma,2)}, \\ \gamma|_3 = \gamma|_1\tau^{\delta((1)\sigma_\gamma,2)}, \\ \tau\gamma|_0 = \gamma|_2\tau^{\delta((2)\sigma_\gamma,2)}, \\ \tau\gamma|_1 = \gamma|_3\tau^{\delta((3)\sigma_\gamma,2)}. \end{cases}$$

Hence,

$$\begin{cases} \gamma|_2 = \gamma|_0\tau^{\delta((0)\sigma_\gamma,2)},\ \gamma|_3 = \gamma|_1\tau^{\delta((1)\sigma_\gamma,2)}, \\ \tau^{\gamma|_0} = \tau^{\delta((0)\sigma_\gamma,2)+\delta((2)\sigma_\gamma,2)} = \tau,\ \tau^{\gamma|_1} = \tau^{\delta((1)\sigma_\gamma,2)+\delta((3)\sigma_\gamma,2)} = \tau \end{cases}.$$

Therefore, there exist $m_0, m_1 \in \mathbb{Z}_4$ such that

$$\begin{cases} \gamma|_0 = \tau^{m_0},\ \gamma|_1 = \tau^{m_1}, \\ \gamma|_2 = \tau^{m_0+\delta((0)\sigma_\gamma,2)},\ \gamma|_3 = \tau^{m_1+\delta((1)\sigma_\gamma,2)} \end{cases}.$$

Hence, γ has the form

$$\gamma = (\tau^{m_0}, \tau^{m_1}, \tau^{m_0+\delta((0)\sigma_\gamma,2)}, \tau^{m_1+\delta((1)\sigma_\gamma,2)})\sigma_\gamma, \tag{8.43}$$

where $\sigma_\gamma \in C_{\Sigma_4}(\sigma^2)$. □

Corollary 8.7. *The centralizer of τ^2 in $\mathcal{A}_4$ is*

$$C(\tau^2) = \langle (e,e,\tau,e)(0,2),\ \tau,\ (\tau^{m_0}, \tau^{m_1}, \tau^{m_0}, \tau^{m_1}) \mid m_0, m_1 \in \mathbb{Z}_4 \rangle.$$

Corollary 8.8. *Let $\gamma \in C(\tau^2)$ be such that $\sigma_\gamma \in \langle (0,2)(1,3) \rangle$. Then*

$$\gamma \in \left\langle (\tau^{m_0}, \tau^{m_1}, \tau^{m_0}, \tau^{m_1}),\ \tau^2 \mid m_0, m_1 \in \mathbb{Z}_4 \right\rangle.$$

Proposition 8.9. *Let $\dot{H} = \langle (\tau^{m_0}, \tau^{m_1}, \tau^{m_0}, \tau^{m_1}), \tau^2 \mid m_0, m_1 \in \mathbb{Z}_4 \rangle$. Then the normalizer $N_{\mathcal{A}_4}(\dot{H})$ is the group*

$$\langle C(\tau^2), (\psi_{2m_0+1}, \psi_{2m_1+1}, \psi_{2m_0+1}\tau^{m_0}, \psi_{2m_1+1}\tau^{m_1}) \mid m_0, m_1 \in \mathbb{Z}_4 \rangle,$$

where, for each $\eta \in U(\mathbb{Z}_4)$, ψ_η is defined by (8.36) and

$$\tau^{\psi_\eta} = \tau^\eta.$$

Proof. As

$$\alpha = (\psi_{2m_0+1}\psi_{2m_1+1}, \psi_{2m_0+1}\tau^{m_0}, \psi_{2m_1+1}\tau^{m_1}), \tag{8.44}$$

conjugates τ^2 to

$$(\tau^{m_0}, \tau^{m_1}, \tau^{m_0+1}, \tau^{m_1+1})(0,2)(1,3),$$

where $m_0, m_1 \in \mathbb{Z}_4$, and any other element in $N_{\mathcal{A}_4}(\dot{H})$ is equal to an element in $C(\tau^2)$ times an element of the form (8.44), then $N_{\mathcal{A}_4}(\dot{H})$ is the desired subgroup. □

Theorem 8.10. *Let G be a solvable subgroup of $Aut(T_4)$ which contains τ. Then, G is a subgroup of*

$$\times_4 (\cdots (\times_4 (\times_4 T^\alpha \rtimes S_4) \rtimes S_4) \cdots) \rtimes S_4 \tag{8.45}$$

for some $\alpha \in \mathcal{A}_4$, where T is the normalizer in $\mathcal{A}_4$ of $C(\tau^2)$.

Proof. As in the case $n = p$, we assume G has derived length $d \geq 2$ and let B be the $(d-1)$th term of the derived series of G. Then, B is an abelian group normalized by τ. On analyzing the case 8.4 and the final step, there exists a level t such that B is a subgroup of $\dot{V} = \times_{4^k} C(\mu^2)$,where $\mu = \tau^\alpha$ for some $\alpha \in \mathcal{A}_4$ and where $\sigma_{\mu^2} = (0,2)(1,3)$. There also exists $\beta \in B$ such that $\beta|_u = \mu^2$ for some index $u \in \mathcal{M}$.

Moreover, if T is the normalizer of $C(\tau^2)$, then clearly, T^α is the normalizer of $C(\mu^2)$.

We will show now that G is a subgroup of

$$\dot{J} = \times_4 (\cdots (\times_4 (\times_4 T^\alpha \rtimes S_4) \rtimes S_4) \cdots) \rtimes S_4$$

where the cartesian product $\times_4$appears t times..

Let $\gamma \notin \dot{J}$. Since $\gamma \notin \dot{J}$, there exists $w \in \mathcal{M}$ having $|w| = t$ and $\gamma|_w \notin T^\alpha$. Since τ is transitive on all levels of the tree, by Corollary 8.8 we can conjugate β by an appropriate power of τ to get $\theta \in B$ such that

$$\theta|_w = \mu^2 \text{ or } \theta|_w = (\mu^2)^\tau = ((\tau^{m_0}, \tau^{m_1}, \tau^{m_0+1}, \tau^{m_1+1})(0,2)(1,3))^\alpha,$$

where $m_0, m_1 \in \mathbb{Z}_4$. Thus, for $v = w^\gamma$ we have

$$(\theta^\gamma)|_v \overset{(2.7)}{=} \theta|_{v^{\gamma^{-1}}}^{\gamma_{v^{\gamma^{-1}}}} = \theta|_w^{\gamma|_w} \notin C(\mu^2)$$

which implies $\theta^\gamma \notin B \leq \dot{V}$ and $\gamma \notin G$. Hence, G is a subgroup of $\dot{J}$. □

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