THE $n$-ARY ADDING MACHINE AND SOLVABLE GROUPS

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Abstract. We describe under various conditions abelian subgroups of the automorphism group $\text{Aut}(T_n)$ of the regular $n$-ary tree $T_n$, which are normalized by the $n$-ary adding machine $\tau = (e, \ldots, e, \tau)\sigma_\tau$ where $\sigma_\tau$ is the $n$-cycle $(0, 1, \ldots, n-1)$. As an application, for $n = p$ a prime number, and for $n = 4$, we prove that every soluble subgroup of $\text{Aut}(T_n)$, containing $\tau$ is an extension of a torsion-free metabelian group by a finite group.

1. Introduction

Adding machines have played an important role in dynamical systems, and in the theory of groups acting on trees: see [1, 6, 7, 3, 5].

An element $\alpha$ in the automorphism group $A_n = \text{Aut}(T_n)$ of the $n$-ary tree $T_n$, is represented as $\alpha = \alpha|_\phi = (\alpha|_0, \ldots, \alpha|_{n-1})\sigma_\alpha$ where $\phi$ is the empty sequence from the free monoid $\mathcal{M}$ generated by $Y = \{0, 1, \ldots, n-1\}$, where $\alpha|_i \in A_n$, for $i \in Y$, are called 1st level states of $\alpha$ and where $\sigma_\alpha$ (the activity of $\alpha$) is a permutation in the symmetric group $\Sigma_n$ on $Y$ extended ‘rigidly’ to act on the tree; if $\sigma_\alpha = e$, we say that $\alpha$ is inactive.

In applying the same representation to $\alpha|_0$ we produce $\alpha|_0$ for all $i \in Y$ and we produce in general $\{\alpha|_u \mid u \in \mathcal{M}\}$ the set of states of $\alpha$. Following this notation, the $n$-ary adding machine is represented as $\tau = (e, \ldots, e, \tau)\sigma_\tau$ where $e$ is the identity automorphism and $\sigma_\tau$ is the regular permutation $\sigma = (0, 1, \ldots, n-1)$. In this sense, the adding machine is an infinite variant of the regular permutation which appears often in geometric and combinatorial contexts.

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A characteristic feature of $\tau$ is that its $n$-th power $\tau^n$ is the diagonal automorphism of the tree $(\tau, \ldots, \tau)$. This fact implies that the centralizer of the cyclic group $\langle \tau \rangle$ in $A_n$ is equal to its topological closure $\overline{\langle \tau \rangle}$ in the group $A_n$ when considered as a topological group with respect to the the natural topology induced by the tree. The pro-cyclic group $\langle \tau \rangle$ is isomorphic to $\mathbb{Z}/n\mathbb{Z}$, the ring of $n$-adic integers

$$\xi = \sum_{i \geq 0} a_i n^i \ (0 \leq a_i \leq n-1 \text{ for all } i).$$

A large variety of subgroups of $A_n$ which contain $\tau$ have been constructed, including groups which are torsion-free and just non-solvable without free subgroups of rank 2 (see, [2, 8] and generalizations thereof [10]). Furthermore, the free group of rank 2 has been represented on the binary tree as a group generated by two conjugates of the adding machine $\tau$ each having a finite number of states [11]. On the other hand, the restricted structure of its centralizer indicate that solvable groups which contain $\tau$ have restricted structure. For nilpotent groups we show

**Proposition.** Let $G$ be a nilpotent subgroup of $A_n$ which contains the $n$-adic adding machine $\tau$. Then $G$ is a subgroup of $\langle \tau \rangle$.

The most visible examples of solvable groups containing $\tau$ are conjugate to subgroups of those belonging to the infinite sequence of groups

$$\Gamma_0 = N_{A_n}(\overline{\langle \tau \rangle}),$$

$$\Gamma_{i+1} = (\times_n \Gamma_i) \rtimes G_{i+1} \ (i \geq 0)$$

where $\times_n \Gamma_i$ is a direct product of $n$ copies of $\Gamma_i$ (seen as a subgroup of the 1st level stabilizer of the tree) and where $G_i$ is a solvable subgroup of the symmetric group $\Sigma_n$ in its canonical action on the tree and containing the cycle $\sigma_\tau$. We observe that for all $i$, the groups $\Gamma_i$ are metabelian by ‘finite solvable subgroups of $\Sigma_n$’. It was shown by the second author that for $n = 2$, solvable groups which contain the binary adding machine are conjugate to some subgroups of $\Gamma_i$ acting on the binary tree [9]. This appears to be the general pattern. However, the description for degrees $n > 2$ requires a classification of solvable subgroups of $\Sigma_n$ which contain the cycle $\sigma = (0, 1, \ldots, n-1)$[4]. This in itself is an open problem, even for metabelian groups. On the other hand, the answer for primitive solvable subgroups of $\Sigma_n$ is simple and classical. For then, $n$ is a prime number $p$ or $n = 4$. In case $n = p$, the solvable subgroups $G_i$ can all be taken to be the normalizer $F = N_{\Sigma_n}(\langle \sigma \rangle)$ of order $p(p-1)$ and in case $n = 4$, the $G_i$’s can all be taken to be the symmetric group $\Sigma_4$.

Given this background, the main theorem of this paper is

**Theorem A.** Let $n = p$, a prime number, or $n = 4$. Then any solvable subgroup of $A_n$ which contains the $n$-ary machine $\tau$ is conjugate to a subgroup of $\Gamma_i$ for some $i$.

The result follows first from general analysis of the conditions $[\beta, \beta^{x^n}] = e$ (for some $\beta \in A_n$ and all $x \in \mathbb{Z}$), then their impact on the 1st level states of the subgroup $\langle \beta, \tau \rangle$ and on how these in turn translate successively to conditions on states at lower levels. It is somewhat surprising that the process converges to a clear global description for trees of degrees $p$ and 4.

The first step of this analysis leads to the following description of the normal closure of $\langle \beta \rangle$ under the action of $\tau$. 


Theorem B. Let $B$ be an abelian subgroup of $A_n$ normalized by $\tau$, let $\beta = (\beta|_0, \beta|_1, \ldots, \beta|_{n-1}) \sigma_\beta \in B$ and define the subgroup $H = \langle \beta|_i (i \in Y), \tau \rangle$ generated by the first level states of $\beta$ and $\tau$.

(I) Suppose $\sigma_\beta = (\sigma_\tau)^s$ for some integer $s$. Then $H$ is metabelian-by-finite. More precisely, let $m = \frac{n}{\gcd(n,s)}$, define the product $\pi_i = \beta|_i \beta|_{i+s} \beta|_{i+2s} \cdots \beta|_{i+(m-1)s}$ (the notation $\beta|_j$ means $\beta|_{\bar{j}}$ where $\bar{j}$ is the representative of $j$ in $Y$ modulo $n$) and define the subgroup

$$K = \left\langle [\beta|_i, \tau^k], \pi_i \mid k \in \mathbb{Z}, i \in Y \right\rangle$$

Then $K$ is an abelian group and $H$ affords the normal series

$$H \triangleright K \langle \tau \rangle (= O) \triangleright K$$

where the quotient group $\frac{H}{O}$ is a homomorphic image of a subgroup of the wreath product $C_m \wr C_n$ of the cyclic groups $C_m, C_n$.

(II) Let $n$ be an even number. Then $H$ is a metabelian group if $s = \frac{n}{2}$ or if $\sigma_\beta$ is a transposition.

Part (I) of Theorem B will be proven in Sections 4 and 5 and part (II) in Section 7.

Let $P$ be a subgroup of $\Sigma_n$. The layer closure of $P$ in $A_n$ is the group $L(P)$ formed by elements of $A_n$ whose states have activities in $P$. The following result is yet another characterization of the adding machine.

Theorem C. Let $n$ be an odd number and let $L = L(\langle \sigma \rangle)$, the layer closure of $\langle \sigma \rangle$ in $A_n$. Let $s$ be an integer which is relatively prime to $n$ and let $\beta = (\beta|_0, \beta|_1, \ldots, \beta|_{n-1}) \sigma^s \in L$ be such that $[\beta, \beta^{x^s}] = e$ for all $x \in \mathbb{Z}$. Then $\beta$ is a conjugate of $\tau$ in $L$.

2. Preliminaries

We start by introducing definitions and notation. The $n$-ary tree $T_n$ can be identified with the free monoid $\mathcal{M} = \langle 0, 1, \ldots, n-1 \rangle^*$ of finite sequences from $Y = \{0, 1, \ldots, n-1\}$, ordered by $v \leq u$ provided $u$ is an initial subword of $v$.

The identity element of $\mathcal{M}$ is the empty sequence $\phi$. The level function for $T_n$, denoted by $|m|$ is the length of $m \in \mathcal{M}$; the root vertex $\phi$ has level 0.

![Figure 1. The Binary Tree](image)

The action $\rho : i \to j$ of a permutation $\rho \in \Sigma_n$ will be from the right and written as $(i) \rho = j$ or as $i^\rho = j$. If $i, j$ are integers then the action of $\rho$ on $i$ is to be identified with its action on its
representatives \( \overline{j} \) in \( Y \), modulo \( n \). Permutations \( \sigma \) in \( \Sigma_n \) are extended ‘rigidly’ to automorphisms of \( A_n \) by

\[
(y,u)\rho = (y)\sigma, \forall y \in Y, \forall u \in M.
\]

An automorphism \( \alpha \in A_n \) induces a permutation \( \sigma_\alpha \) on the set \( Y \). Consequently, \( \alpha \) affords the representation \( \alpha = \alpha' \sigma_\alpha \) where \( \alpha' \) fixes \( Y \) point-wise and for each \( i \in Y \), \( \alpha' \) induces \( \alpha|_i \) on the subtree whose vertices form the set \( i \cdot M \). If \( j \) is an integer the \( \alpha|_j \) will be understood as \( \alpha|_{\overline{j}} \) where \( \overline{j} \) is the representative of \( j \) in \( Y \) modulo \( n \).

Given \( i \) in \( Y \), we use the canonical isomorphism \( i \cdot u \mapsto u \) between \( i \cdot M \) and the tree \( T_n \), and thus identify \( \alpha|_i \) with an automorphism of \( T_n \); therefore, \( \alpha' \in \mathcal{F}(Y, A_n) \), the set of functions from \( Y \) into \( A_n \), or what is the same, the 1st level stabilizer \( \text{Stab}(1) \) of the tree. This provides us with the factorization \( A_n = \mathcal{F}(Y, A_n) \cdot \Sigma_n \).

Let \( \alpha, \beta, \gamma \in A_n \). Then the following formulas hold

\[
\begin{align*}
\sigma_{\alpha^{-1}} &= (\sigma_\alpha)^{-1}, \quad \sigma_\alpha \sigma_\beta = \sigma_{\alpha \beta}, \\
(\alpha^{-1})|_u &= \left(\alpha|_u\right)^{\alpha^{-1}}, \\
(\alpha \beta)|_u &= (\alpha|_u)(\gamma|_u) \quad \text{where} \quad \gamma|_u = \beta|_u \alpha, \\
\gamma &= \alpha^{-1}\beta \alpha \iff \left( \sigma_{\gamma} = \sigma_{\alpha^{-1}} \sigma_\beta \sigma_\alpha \text{ and } \gamma|_{(i)\sigma_\alpha} = \alpha|_{(i)\sigma_\alpha}^{-1} \beta |_{(i)\sigma_\alpha} \alpha|_{(i)\sigma_\beta}, \forall i \in Y \right), \\
\theta &= [\beta, \alpha] = \beta^{-1} \beta^\alpha \iff \\
\theta|_{(i)\sigma_\alpha} &= \left( \beta|_{(i)\sigma_\alpha} \right)^{-1} \left( \alpha|_{(i)\sigma_\alpha} \right)^{-1} \left( \beta|_{(i)\sigma_\alpha} \right) \left( \alpha|_{(i)\sigma_\alpha} \right), \forall i \in Y. \\
(\alpha^m)|_i &= (\alpha|_i) \left( \alpha|_{(i)\sigma_\alpha} \right) \left( \alpha|_{(i)\sigma_\alpha^2} \right) \cdots \left( \alpha|_{(i)\sigma_\alpha^{m-1}} \right) \\
(\beta^\alpha)|_u &= \left( \beta|_u \alpha^{-1} \right)^{(\alpha|_u)^{-1}}, \quad \text{where} \quad \beta \in \text{Stab}(k) \text{ and } |u| \leq k.
\end{align*}
\]

An automorphism \( \alpha \in A_n \) corresponds to an input-output automaton with alphabet \( Y \) and with set of states \( Q(\alpha) = \{ \alpha|_u \mid u \in M \} \). The automaton \( \alpha \) transforms the letters as follows: if the automaton is in state \( \alpha|_u \) and reads a letter \( i \in Y \) then it outputs the letter \( j = (i) \alpha|_u \) and its state changes to \( \alpha|_{ui} \); these operations can be best described by the labeled edge \( \alpha|_u \xrightarrow{i} j \Rightarrow \alpha|_{ui} \). Following terminology of automata theory, every automorphism \( \alpha|_u \) is called the state of \( \alpha \) at \( u \).

The tree \( T_n \) is a topological space which is the direct limit of its truncations at the \( n \)-th levels. Thus the group \( A_n \) is the inverse limit of the permutation groups it induces on the \( n \)-th level vertices. This transforms \( A_n \) into a topological group. An infinite product of elements \( A_n \) is a well-defined element of \( A_n \) provided that for any given level \( l \), only finitely many of the elements in the product have non-trivial
action on vertices at level \( l \). Thus, if \( \alpha \in A_n \) and \( \xi = \sum_{i \geq 0} a_i n^i \in \mathbb{Z}_n \) then \( \alpha^\xi = \alpha^{a_0} \cdot \alpha^{n a_1} \cdots \alpha^{n^i a_i} \cdots \) is a well defined element of \( A_n \). The notation \( \alpha|_{\xi} \) is to be understood as \( \alpha|_i \) where \( i = a_0 \).

The topological closure of a subgroup \( H \) in \( A_n \) will be indicated by \( \overline{H} \). We note that if \( H \) is abelian then

\[
\overline{H} = \{ h^\xi \mid h \in H, \xi \in \mathbb{Z}_n \}.
\]

One of the characterizing aspects of the \( n \)-ary adding machine is that the centralizer of \( \tau \) is a pro-cyclic group; namely,

\[
C_{A_n}(\tau) = \langle \tau \rangle = \{ \tau^\xi \mid \xi \in \mathbb{Z}_n \}.
\]

Let \( v = yu \) where \( y \in Y, u \in \mathcal{M} \). The image of \( v \) under the action of \( \alpha \) is

\[
(v)\alpha = (yu)\alpha = (y)\sigma_\alpha (u)\alpha|_y.
\]

The action extends to infinite sequences (or boundary points of the tree) in the same manner. A boundary point of the tree \( c = c_0c_1c_2 \ldots \), where \( c_i \in Y \) for all \( i \), corresponds also to the \( n \)-adic integer \( \xi = \sum \{ c_i n^i | i \geq 0 \} \in \mathbb{Z}_n \). Thus the action of the tree automorphism \( \alpha \) can thus be translated to an action on the ring of \( n \)-adic integers. We will indicate \( c_0 \) by \( \bar{\xi} \) which is \( \xi \) modulo \( n \). In the case of the automorphism \( \tau = (e, e, \ldots, e, \tau, \sigma) \), the action of \( \tau \) on \( c \) is

\[
(c)\tau = \begin{cases} 
(c_0 + 1) c_1 c_2 \ldots & \text{if } 0 \leq c_0 \leq n - 2, \\
0(c_1 c_2 \cdots)\tau, & \text{if } c_0 = n - 1,
\end{cases}
\]

which translates to the \( n \)-ary addition

\[
\xi^\tau = \xi + 1.
\]

![Figure 2. The binary adding machine](image)

3. Normalizer of the topological closure \( \overline{\langle \tau \rangle} \)

An element \( \xi = \sum_{i \geq 0} a_i n^i \in \mathbb{Z}_n \) is a unit in \( \mathbb{Z}_n \) if and only if \( \bar{\xi} (= a_0) \) is a unit in \( \mathbb{Z} \) modulo \( n \). The group of automorphisms of \( \mathbb{Z}_n \) is isomorphic to the multiplicative group of units \( U(\mathbb{Z}_n) \). The subgroup of \( U(\mathbb{Z}_n) \) consisting of elements \( \xi \) with \( \bar{\xi} = 1 \) is denoted by \( \mathbb{Z}_n^1 \). This subgroup has the transversal \( \{ j \mid 1 \leq j \leq n - 1, \gcd(j, n) = 1 \} \) in \( U(\mathbb{Z}_n) \) and therefore has index \( [U(\mathbb{Z}_n) : \mathbb{Z}_n^1] = \varphi(n) \) where \( \varphi \) is the Euler function.

Given \( \alpha \in A_n \) we denote the diagonal automorphism \( (\alpha, ..., \alpha) \) by \( \alpha^{(1)} \) and define inductively \( \alpha^{(i+1)} = (\alpha^{(i)})^{(1)} \) for all \( i \geq 1 \).
3.1. **Powers of** $\tau$. Let $\xi = \sum_{i \geq 0} a_i n^i \in \mathbb{Z}_n$. Then $\sum_{i \geq 1} a_i n^{i-1} = \frac{\xi - \xi}{n}$.

**Lemma 3.1.** Let $\xi \in \mathbb{Z}_n$. Then

$$
\tau^\xi = (\tau^{\frac{\xi-a_0}{n}}, \ldots, \tau^{\frac{\xi-a_0+1}{n}}, \ldots, \tau^{\frac{\xi-a_0+1}{n}})_{a_0 \text{ terms}}^{a_0}.
$$

**Proof.** For $j$ an integer with $1 \leq j \leq n - 1$, we have

$$
\tau^j = \left( e, \ldots, e, \tau, \ldots, \tau \right)_{j \text{ terms}}^{\sigma^j},
$$

and $\tau^n = (\tau, \ldots, \tau) = \tau^{(1)}$.

Given $\xi = \sum_{i \geq 0} a_i n^i$, then

\begin{align}
\tau^{a_0} &= (e, \ldots, e, \tau, \ldots, \tau)_{a_0 \text{ terms}}^{\sigma^{a_0}}, \\
\tau^{a_j n^j} &= \tau^{(a_j n^j-1)n} = \left( \tau^{a_j n^j-1} \right)^{(1)}, \\
\tau^\xi &= (\frac{\xi-a_0}{n}, \ldots, \frac{\xi-a_0}{n}, \frac{\xi-a_0+1}{n}, \ldots, \frac{\xi-a_0+1}{n})_{a_0 \text{ terms}}^{\sigma^{a_0}} \\
&= (\frac{\xi-a_0}{n}, \ldots, \frac{\xi-a_0}{n}, \frac{\xi-a_0+1}{n}, \ldots, \frac{\xi-a_0+1}{n})_{\xi \text{ terms}}^{\sigma^{\xi}}.
\end{align}

As we have seen, the description of $\tau^\xi$ involves the partition of the interval $[0, \ldots, n - 1]$ into two subintervals. It is convenient to use here the carry 2-cocycle $\delta : \mathbb{Z}_n \times \mathbb{Z}_n \rightarrow \{0, 1\}$ defined by

$$
\delta(\eta, \kappa) = \frac{\eta + \kappa - \eta + \kappa}{n} = \begin{cases} 
0, & \text{if } \eta + \kappa < n \\
1, & \text{otherwise}
\end{cases}.
$$

We call this 2-valued function by *Delta-2* (*later on we will introduce a 3-valued function *Delta-3*). Using *Delta-2*, the notation for the power of $\tau$ becomes

\begin{equation}
\tau^\xi = \left( \frac{\xi-a_0}{n} + \delta(i, \xi) \right)_{0 \leq i \leq n-1}^{\sigma^{\xi}}.
\end{equation}
3.2. Centralizer of \( \tau \).

**Lemma 3.2.** \( C_{A_n} (\tau) = \langle \tau \rangle \).

**Proof.** Let \( \alpha \in A_n \) commute with \( \tau \). Then, \([\sigma_\alpha, \sigma_\tau] = e\) and therefore \( \sigma_\alpha = (\sigma_\tau)^{s_0} \) for some integer \( 0 \leq s_0 \leq n - 1 \). Therefore, \( \beta = \alpha \tau^{-s_0} = (\beta|_0, ... , \beta|_{n-1}) \) commutes with \( \tau \) and \( \sigma_\beta = e \). Now,

\[
\beta^* = ((\beta|_{n-1})^*, \beta|_0, ..., \beta|_{n-2}) = \beta
\]

implies \( \beta|_i = \beta|_0 \) for all \( 0 \leq i \leq n - 1 \) and \( \beta|_0 \) commutes with \( \tau \). Therefore \( \beta = (\beta|_0)^{(1)} \) and \( \beta|_0 \) replaces \( \alpha \) in the previous argument. Hence, there exists an integer \( s_1 \) such that \( 0 \leq s_1 \leq n - 1 \) and \( \gamma = \beta|_0 \tau^{-s_1} = (\gamma|_0)^{(1)} \). From this we conclude

\[
\alpha = \beta \tau^{s_0} = (\beta|_0)^{(1)} \tau^{s_0}
\]

\[
= \left( (\gamma|_0)^{(1)} \tau^{s_1}, ..., (\gamma|_0)^{(1)} \tau^{s_1} \right) \tau^{s_0}
\]

\[
= (\gamma|_0)^{(2)} \tau^{ns_1} \tau^{s_0} = (\gamma|_0)^{(2)} \tau^{ns_1+s_0}.
\]

We obtain the desired form inductively, \( \alpha = \tau^\xi \) where

\[
\xi = s_0 + s_1 n + s_2 n^2 + ...
\]

The characterization of nilpotent groups which contain \( \tau \), announced in the introduction, follows.

**Proposition 3.3.** Let \( G \) be a nilpotent subgroup of \( A_n \) which contains the \( n \)-adic adding machine \( \tau \). Then \( G \) is a subgroup of \( \langle \tau \rangle \).

**Proof.** Suppose \( G \) is a nilpotent group of class \( k > 1 \) which contains \( \tau \). Then, the center \( Z(G) \) is contained in \( \langle \tau \rangle \). Let \( j \) be the maximum index such that \( Z_j(G) \leq \langle \tau \rangle \) and let \( \alpha \in Z_{j+1}(G) \setminus \langle \tau \rangle \).

Then \([\tau, \alpha] \in Z_j(G)\) and therefore \([\tau, \alpha] = \tau^\xi \) for some \( \xi \in \mathbb{Z}_n \setminus \{0\} \). Therefore

\[
[\tau, 2\alpha] = [\tau, \alpha, \alpha] = [\tau^\xi, \alpha]
\]

\[
= [\tau, \alpha]^\xi = \tau^{\xi^2} \in Z_{j-1}(G)
\]

and more generally, for \( l \geq 1 \), we have \([\tau, l\alpha] = \tau^{\xi^l} \in Z_{j-l+1}(G) \). It follows that \( \tau^{\xi^j} \in Z_0(G) = \{e\} \). Thus, \( \xi^{j-1} = 0 \) and \( \xi = 0 \); a contradiction. \( \square \)
3.3. Normalizer of \( \langle \tau \rangle \).

**Lemma 3.4.** The group \( \Gamma_0 = N_{A_n} \langle \tau \rangle \) is metabelian. Indeed, the derived subgroup \( \Gamma'_0 \) is contained in \( \langle \tau \rangle \).

**Proof.** Let \( \alpha, \beta \in \Gamma_0 \), then \( \tau^\alpha = \tau^\xi \) and \( \tau^\beta = \tau^n \) for some \( \eta, \xi \in U(\mathbb{Z}_n) \). Therefore,

\[
\tau^\alpha = \tau^\xi, \tau = (\tau^\xi)^{\alpha^{-1}} = (\tau^{\alpha^{-1}})^\xi, \quad \tau^{\alpha^{-1}} = \tau^{\xi^{-1}}.
\]

Likewise, \( \tau^{\beta^{-1}} = \tau^{n^{-1}} \). Thus, \( \tau^{[\alpha, \beta]} = \tau \) and \( \Gamma'_0 \leq C_{A_n}(\tau) = \langle \tau \rangle \) follows. \( \square \)

We present properties of the Delta-2 function which we will use in the sequel.

**Lemma 3.5.** For all \( 0 \leq i, j < n \) and \( \xi \in \mathbb{Z}_n \) we have

\[
\sum_{i=0}^{n-1} \delta(i, j) = j,
\]

\[
\delta(i, j\xi) = j \left( \frac{\xi - \xi}{n} \right) - \frac{j\xi - j\xi}{n} + \sum_{k=0}^{j-1} \delta(i + k\xi, \xi).
\]

**Proof.** The first assertion is easy to verify.

The second is obtained from

\[
(\tau^\xi)^{|i} = (\tau^\xi)|_{i} (\tau^\xi)|_{i+\xi} \cdots (\tau^\xi)|_{i+(j-1)\xi},
\]

by substituting

\[
(\tau^\xi)|_{i} = \tau^{\xi - \xi n + \delta(i, \xi)}
\]

in its right hand side and

\[
\tau^{\xi j}|_{i} = \tau^{\frac{j\xi - j\xi}{n} + \delta(i, j\xi)}
\]

in its left. \( \square \)

**Proposition 3.6.** Given \( \alpha \in A_n \) and \( \xi \in U(\mathbb{Z}_n) \). Then the condition \( \tau^\alpha = \tau^\xi \) is equivalent to conditions (i), (ii) and (iii) below.

(i)

\[
\alpha|_i = (\alpha|_0) \tau^{\mu_i} \quad (1 \leq i \leq n - 1)
\]

where

\[
\mu_i = i \frac{(\xi - \xi)}{n} + \sum_{k=0}^{i-1} \delta((v(\alpha) + k)\xi, \xi)
\]

and where \( v(\alpha) \) is defined by

\[
0 \leq v(\alpha) \leq n - 1,
\]

(0) \( \sigma_\alpha = \frac{v(\alpha)\xi}{\xi}; \)
Therefore, there exists
\[ \begin{aligned}
( j ) & = ( v( \alpha ) + j ) \xi \\
& ( 0 \leq j \leq n - 1 ).
\end{aligned} \]

Furthermore, if \( \xi \in \mathbb{Z}_n^1 \), then \( v( \alpha ) = 0 , \)
\( ( j ) = j \xi \) and \( \mu_i = i^{\xi - 1}_n . \)

\textbf{Proof.} Since \( \sigma_\xi^\alpha = \sigma_\xi^\tau \), we have an equality between the permutations
\[ ((0) \sigma_\alpha , (1) \sigma_\alpha , \ldots , (n - 1) \sigma_\alpha ) = (0, \xi, 2\xi, \ldots , (n - 1) \xi) . \]

Therefore, there exists \( v( \alpha ) \in Y \) such that
\[ ( j ) \sigma_\alpha = \bar{v(\alpha)} \xi \] and so,
\[ \begin{aligned}
( j ) & = \bar{v(\alpha)} \xi , \forall j \in Y .
\end{aligned} \]

Now, \( \tau^\alpha = \tau^\xi \) is equivalent to \( \alpha = \tau^{-s} \alpha \tau^s \xi \) for every \( s \in \mathbb{Z} \), which in turn is equivalent to
\[ \alpha|_{(i)\sigma_\tau^\xi} = ((\tau^s)|_i)^{-1}(\alpha|_i) (\tau^\xi)|_i \sigma_\alpha , \forall i \in Y , \forall s \in \mathbb{Z} . \]

The latter conditions are equivalent to
\[ \begin{aligned}
\alpha|_0 & = \alpha|_{(0)\sigma_\tau^\xi} = ((\tau^n)|_0)^{-1}(\alpha|_0) (\tau^n)|_0 \sigma_\alpha , \\
\alpha|_i & = \alpha|_{(i)\sigma_\tau^\xi} = ((\tau^i)|_0)^{-1}(\alpha|_0 (\tau^i)|_0 \sigma_\alpha , \forall i \in Y \setminus \{0\}
\end{aligned} \]

and these in turn are equivalent to
\[ \begin{aligned}
\alpha|_i & = \alpha|_0 \tau^{\xi - n \tau^{-1} + \delta(v(\alpha) \xi, \xi)} = \alpha|_0 \tau^{\mu_i} , \\
\mu_i & = i \left( \frac{\xi - \xi}{n} \right) + \sum_{k=0}^{i-1} \delta((v(\alpha) + k) \xi, \xi) \forall i \in Y \setminus \{0\} .
\end{aligned} \]

If \( \xi \in \mathbb{Z}_n^1 \), then \( \sum_{k=0}^{i-1} \delta(k \xi, \xi) = \sum_{k=0}^{i-1} \delta(k, 1) = 0 . \) The rest of the assertion follows directly. \( \square \)

\textbf{Corollary 3.7.} Let \( \xi \in U(\mathbb{Z}_n) \), \( \sigma_\alpha \) and \( \mu_i \) be as above. Then \( \alpha = (\alpha)^{(1)}_e (e, \tau^{\mu_0}, \ldots , \tau^{\mu_{n-1}}) \sigma_\alpha \) conjugates \( \tau \) to \( \tau^\xi \). In particular, if \( \xi \in \mathbb{Z}_n^1 \), then \( \alpha = (\alpha)^{(1)}_e (e, \tau^{\frac{\xi}{n}} , \tau^{\frac{\xi}{n} + 1} , \ldots , \tau^{(n-1)\frac{\xi}{n} + 1} ) \) (denoted by \( \lambda_\xi \)) conjugates \( \tau \) to \( \tau^\xi \).

Although we have computed above an automorphism which inverts \( \tau \), we give another with a simpler description. Define the permutation
\[ \varepsilon = (0, n - 1) (1, n - 2) \ldots \left( \left[ \frac{n - 2}{2} \right] , \left[ \frac{n + 1}{2} \right] \right) . \]

Then \( \varepsilon \) inverts \( \sigma_\tau = (0, 1, \ldots , n - 1) \) and
\[ \begin{aligned}
\iota & = \iota^{(1)}_\varepsilon \\
\end{aligned} \]

inverts \( \tau \).

Define
\[ \begin{aligned}
\Lambda & = \{ \lambda_\xi | \xi \in \mathbb{Z}_n^1 \} , \\
\Psi & = \{ \lambda_\xi \tau^t | \xi \in \mathbb{Z}_n^1 , t \in \mathbb{Z}_n \} .
\end{aligned} \]
and call $\Psi$ the monic normalizer of $(\tau)$.

**Proposition 3.8.** (i) $\Lambda$ is an abelian group isomorphic to $\mathbb{Z}_n^1$;
(ii) $\Psi = \Lambda \ltimes (\tau) \cong \mathbb{Z}_n^1 \ltimes \mathbb{Z}_n$;
(iii) on letting $\Psi'$ denote the derived subgroup of $\Psi$, we have $\Psi' = (\tau^n)$.

**Proof.** (i) Let $\xi, \theta \in \mathbb{Z}_n^1$. Then, as $\lambda_\xi, \lambda_\theta$ and $\lambda_{\xi\theta}$ are inactive, it follows that

$$
(\lambda_\xi \lambda_\theta \lambda_{\xi\theta}^{-1})|_i = (\lambda_\xi)|_i (\lambda_\theta)|_i ((\lambda_{\xi\theta})|_i)^{-1}
$$

$$
= \lambda_\xi \tau^{\xi \frac{1}{n}} \lambda_\theta \tau^{\theta \frac{1}{n}} \left( \lambda_{\xi\theta} \tau^{\xi \theta \frac{1}{n}} \right)^{-1} = \lambda_\xi \lambda_\theta \lambda_{\xi\theta}^{-1} \tau^{\xi \frac{1}{n}} \lambda_\theta \tau^{\theta \frac{1}{n}} \tau^{-\xi \theta \frac{1}{n}} \lambda_{\xi\theta}^{-1}
$$

$$
= \lambda_\xi \lambda_\theta \left( \tau^{i \theta \xi \frac{1}{n}} \tau^{i \theta \frac{1}{n}} \tau^{-i \theta \frac{1}{n}} \right) \lambda_{\xi\theta}^{-1} = \lambda_\xi \lambda_\theta \lambda_{\xi\theta}^{-1}, \forall i \in \{0, \ldots, n-1\}.
$$

Therefore, $\lambda_\xi \lambda_\theta = \lambda_{\xi\theta}$. In addition, $\lambda_\xi = e$ if and only if $\xi = 1$.

(ii) This factorization is clear.

(iii) Let $\theta = 1 + \theta'n, \eta \in \mathbb{Z}_n$. Then

$$
[\tau, \lambda_\theta] = \tau^{-\eta} \lambda_\theta \tau^\eta = 1.
$$

$$
\tau^{-\eta} \tau^\eta = \tau^{(\eta-1)} = (\tau^n)^{\eta'}. \quad \square
$$

We prove below the existence of conjugates $\tau^\alpha$ of $\tau$ in $N_{A_n}(\overline{(\tau)})$, which lie outside $(\tau)$. This fact allows us to construct the first important type of metabelian groups $(\tau) (\tau^\alpha)$ containing $\tau$.

**Proposition 3.9.** Given $\xi, \rho \in \mathbb{Z}_n^1$ with $\xi \neq 1$. Then for all $n$ odd and for all $n$ even such that $2n \mid (\xi - 1)$, an element $\alpha = (\alpha|_0, \ldots, \alpha|_{n-1})$ in $A_n$ satisfies $\tau^\alpha = \lambda_\xi \tau^\rho$ if and only if

$$
\begin{cases}
\alpha|_{i+1} = (\alpha|_0) \lambda_{\xi^{i+1}} \tau^{\xi \frac{1}{n}} (\rho \xi^{i+1} \xi^{-1} - (i+1)) \quad (0 \leq i \leq n-2), \\
\tau^\alpha|_0 = \lambda_{\xi^n} \tau^{\frac{1}{n}} (\rho \xi^{n-1} \xi^{-1}).
\end{cases}
$$

**Proof.** From $\tau^\alpha = \lambda_\xi \tau^{1+\kappa}$, we obtain using (2.4),

$$
\begin{cases}
\lambda_\xi \tau^{i \xi \frac{1}{n} + \kappa} = (\alpha|_i)^{-1} \alpha|_{i+1}, \text{ if } i \in Y - \{n-1\} \\
\lambda_\xi \tau^{(n-1) \xi \frac{1}{n} + \kappa + 1} = (\alpha|_{n-1})^{-1} \tau \alpha|_0.
\end{cases}
$$

Therefore,

$$
\alpha|_{i+1} = (\alpha|_0) \lambda_\xi \tau^\kappa \lambda_\xi \tau^{i \xi \frac{1}{n} + \kappa} \ldots \lambda_\xi \tau^{i \xi \frac{1}{n} + \kappa}, \text{ for } i = 0, 1, \ldots, n-2,
$$

$$
\alpha|_0 = \tau^{-1} (\alpha|_{n-1}) \lambda_\xi \tau^{(n-1) \xi \frac{1}{n} + \kappa + 1}.
$$

The first equations can be expressed as

$$
\alpha|_{i+1} = (\alpha|_0) \lambda_\xi \tau^\kappa \left( \sum_{j=0}^i \xi^j \left( \frac{\xi^{i+1} \xi^{-1}}{\xi^{-1} - (i+1)} \right) \right)
$$

and the last as

$$
\alpha|_0 = \left( \sum_{j=0}^{n-1} \xi^j \xi \right) \frac{\lambda_\xi \xi^n \tau^{(1+\kappa) \xi^{-1} - (n-1)}}{\xi^{-1} - (i+1)}
$$
\begin{align*}
\alpha|_0 & = \tau^{-1}(\alpha|_0) \lambda \xi^n \tau \frac{1}{n} \left[ (1 + \kappa n) \xi^{n-1} - (n-1) \right] \tau \left( n-1 \right) \xi^{-1} + \kappa + 1 \\
& = \tau^{-1}(\alpha|_0) \lambda \xi^n \tau \frac{1}{n} \left[ (1 + \kappa n) \xi^{n-1} \right].
\end{align*}

Now, we need to show that \( \tau^{\alpha|_0} = \lambda \xi^n \tau \frac{1}{n} \left[ (1 + \kappa n) \xi^{n-1} \right] \) satisfies the same conditions as those for \( \alpha \); that is, both \( \xi^n, \rho' = \frac{1}{n} \left[ (1 + \kappa n) \xi^{n-1} \right] \in \mathbb{Z}_n^1 \).

Of course, \( \xi^n \in \mathbb{Z}_n^1 \), so let us consider \( \rho \left( \xi^n - 1 \right) / n(\xi - 1) \). Since \( \xi \in \mathbb{Z}_n^1 \), we can write \( \xi = 1 + \ell n \), and then

\[
\frac{\xi^n - 1}{\xi - 1} \equiv n + \left( \frac{n}{2} \right) \ell n \pmod{n^2},
\]

by using the Binomial Theorem. Since \( \rho \equiv 1 \pmod{n} \), it follows that

\[
\frac{\rho(\xi^n - 1)}{n(\xi - 1)} \equiv 1 + \left( \frac{n}{2} \right) \ell \pmod{n},
\]

and consequently, \( \rho(\xi^n - 1)/n(\xi - 1) \in \mathbb{Z}_n^1 \) if and only if \( n \mid \left( \frac{n}{2} \right) \ell \) (that is, if and only if \( (n - 1) \ell \) is even). So \( \rho(\xi^n - 1)/n(\xi - 1) \in \mathbb{Z}_n^1 \) holds for odd \( n \), and for even \( n \) provided that \( 2n \mid (\xi - 1) \). \( \square \)

4. Abelian groups \( B \) normalized by \( \tau \)

Let \( B \) be an abelian subgroup of \( A_n \) normalized by \( \tau \). For a fixed \( \beta \in B \), we define the ‘1st level state closure’ of \( \langle \beta, \tau \rangle \) as the group

\[
H = \langle \beta|_i \mid i \in Y \rangle, \tau \rangle.
\]

We will be dealing frequently with the following subgroups of \( H \),

\[
N = \left\langle [\beta|_i, \tau^{k_i}] \mid k_i \in \mathbb{Z}, i \in Y \right\rangle \\
M = N \langle \tau \rangle.
\]

When \( \sigma_\beta = (\sigma_\tau)^s \) for some integer \( s \), \( m = \frac{n}{\gcd(n,s)} \) and

\[
\pi_i = \beta|_{i+s} \beta|_{i+2s} \cdots \beta|_{i+(m-1)s}
\]

we will also be dealing with the subgroups

\[
K = \langle N, \pi_i \mid i \in Y \rangle, \\
O = K \langle \tau \rangle.
\]

We show below that when \( n \) is a power of a prime number \( p^k \), the activity range of \( \beta \) narrows down to a Sylow \( p \)-subgroup of \( \Sigma_n \). This is used to restrict the location of an abelian group \( B \) normalized by \( \tau \), within \( A_n \).
Proposition 4.1. Let \( n = p^k \), \( \sigma = (0,1,\ldots,n-1) \) and \( P \) be a Sylow \( p \)-subgroup \( P \) of \( \Sigma_n \) which contains \( \sigma \). Then

(i) \( P \) is isomorphic to \(((\ldots(\ldots C_p)\ldots) C_p)\ldots C_p\) a wreath product of the cyclic group \( C_p \) of order \( p \) iterated \( k-1 \) times; the normalizer of \( P \) in \( \Sigma_n \) is \( N_{\Sigma_n}(P) = P \langle c \rangle \) where \( c \) is cyclic of order \( p-1 \);

(ii) \( P \) is the unique Sylow \( p \)-subgroup \( P \) of \( \Sigma_n \) which contains \( \sigma \);

(iii) if \( W \) is an abelian subgroup of \( \Sigma_n \) normalized by \( \sigma \) then \( W \) is contained in \( P \).

\[ \text{Proof.} \]
(i) The structure of \( P \) as an iterated wreath product is well-known. The center of \( P \) is \( Z = \langle z = \sigma^{p^k-1} \rangle \) and \( C_{\Sigma_n}(z) = P \). Therefore, \( N_{\Sigma_n}(P) = N_{\Sigma_n}(Z) = P \langle c \rangle \) where \( c \) is cyclic of order \( p-1 \).

(ii) If \( \sigma \in P^g \) for some \( g \in \Sigma_n \) then \( z^g \in C_{\Sigma_n}(\sigma) = \langle \sigma \rangle \) and therefore \( \langle z^g \rangle = \langle z \rangle \), \( P^g = P \). Thus, \( P \) is the unique Sylow \( p \)-subgroup of \( \Sigma_n \) to contain \( \sigma \).

(iii) Let \( W \) be an abelian subgroup of \( \Sigma_n \) normalized by \( \sigma \). Let \( V = W\langle \sigma \rangle \) and \( V_0 \) be the stabilizer of 0 in \( V \). Then, since \( \sigma \) is a regular cycle, it follows that \( V = V_0 \langle \sigma \rangle \), \( V_0 \cap \langle \sigma \rangle = \{e\} \). Suppose that there exists a prime \( q \) different from \( p \) which divides the order of \( W \) and let \( Q \) be the unique Sylow \( q \)-subgroup of \( W \). Then \( Q \) is the unique Sylow \( q \)-subgroup of \( V \) and \( Q \leq V_0 \). Therefore, \( Q = \{e\} \) and \( W \) is a \( p \)-group. As \( \sigma \in V \), we conclude \( W \leq P \). \( \square \)

Lemma 4.2. (a) Let \( \gamma \in A_n \). Conditions (i), (ii) below are equivalent:

(i) \( [\gamma, \gamma^k] = e \) for all \( k \in \mathbb{Z} \);
(ii) \( [\tau^k, \gamma, \gamma] = e \) for all \( k \in \mathbb{Z} \).

Condition (i) implies

(iii) \( \langle [\gamma, \tau^k] \mid k \in \mathbb{Z} \rangle \) is a commutative group.

Condition (iii) implies

\( \langle [\gamma^u, \tau^k] \mid k \in \mathbb{Z} \rangle \) is a commutative group for all indices \( u \).

(b) Let \( n = p^k \). Then any abelian subgroup \( B \) normalized by \( \tau \) is contained in the layer closure \( L = L(N_{\Sigma_n}(P)) \).

\[ \text{Proof.} \]
(a) First,

\[ [\gamma, \gamma^k] = \gamma^{-1} (\tau^{-k} \gamma^{-1} \tau^k) \gamma (\tau^{-k} \gamma \tau^k) = \gamma^{-1} (\tau^{-k} \gamma^{-1} \tau^k) \gamma (\gamma^{-1} \tau^{-k} \gamma \tau^k) = [\tau^k, \gamma][\gamma, \tau^k] \]

and so,

\[ [\gamma, \gamma^k] = e \iff [\gamma, \tau^k]^{-1} = [\gamma, \tau^k]. \]

Furthermore, since

\[ [\gamma, \tau^{k_1} \tau^{k_2}] = [\gamma, \tau^{k_2}]^{-1} [\gamma, \tau^{k_1+k_2}] \]

(4.1)
for all integers $k_1, k_2$, condition (ii) implies
\[
[\gamma, \tau^{k_1}][\gamma, \tau^{k_2}] = [\gamma, \tau^{k_1}]^{\gamma^{-1}\tau^{-k_2}\gamma\tau^{k_2}} = [\gamma, \tau^{k_1} ]^{\tau^{-k_2}\gamma\tau^{k_2}} \\
= \left( [\gamma, \tau^{-k_2}]^{-1} [\gamma, \tau^{k_1-k_2}] \right)^{\gamma\tau^{k_2}} = \left( [\gamma, \tau^{-k_2}]^{-1} [\gamma, \tau^{k_1-k_2}] \right)^{\tau^{k_2}} \\
= [\gamma, \tau^{k_1}] .
\]

Finally, we note that by (2.5),
\[
([\gamma, \tau^{nk}])_{|i}\sigma_\gamma = (\gamma^{-1})_{|i}\sigma_\gamma (\tau^{-nk})_{|i} (\gamma|_i) (\tau^{nk})_{|i}\sigma_\gamma = (\gamma|_i)^{-1} \tau^{-k} (\gamma|_i) \tau^k \\
= [\gamma|_i, \tau^k].
\]

Since $[\gamma, \tau^{kn}]$ is inactive for all $k \in \mathbb{Z}$, we obtain $\{[\gamma|_i, \tau^k] | k \in \mathbb{Z}\}$ is a commutative set for all $i$. The rest of the assertion follows by induction on the tree level.

(b) Let $\beta \in B$. Since the normal closure of $\langle \sigma_\beta \rangle$ under the action of $\sigma_\gamma$ is an abelian subgroup, it follows that $\sigma_\beta \in P$. Furthermore, as $\langle [\beta|_u, \tau^k] | k \in \mathbb{Z}\rangle$ is an abelian group normalized by $\tau$, it follows that $[\sigma_\beta|_u, \sigma] \in P$ and therefore $\sigma^{\sigma_\beta|_u} \in P$. Thus, we conclude $\sigma_{\beta|_u} \in N_{\Sigma^*_n}(P)$ and $\beta \in L$. \hfill \Box

**Proposition 4.3.** Let $l \geq 1$ and suppose $\alpha, \gamma \in \text{Stab}(l)$ satisfy $[\alpha, \gamma^\tau] = e$ for all $x \in \mathbb{Z}$. Then
\[
[\alpha|_u, (\gamma|_v)^\tau] = e, \forall u, v \in \mathcal{M} \text{ having } |u| = |v| \leq l \text{ and } \forall x \in \mathbb{Z}.
\]

**Proof.** We start with the case $l = 1$. Write $x = r + kn$ where $r = \overline{x}$.

By (2.4),
\[
(\gamma^\tau)|_{|i}^{rx} = ((\tau^x)|_{|i})^{-1} (\gamma|_i) (\tau^x)|_{|i}, \\
(\gamma^\tau_i)|_{|i} = \tau^{-k-\delta(i-r,r)} (\gamma|_{i-r}) \tau^{k+\delta(i-r,r)}.
\]

As $[\alpha, \gamma^\tau] = e$ and $\alpha, \gamma^\tau \in \text{Stab}(1)$, we have, for all $i, j, r \in Y$ and all $k, x \in \mathbb{Z},$
\[
[\alpha|_i, (\gamma|_j)^\tau] = e, \quad [\alpha|_i, (\gamma|_{i-r})^{-k+\delta(i-r,r)}] = e, \\
[\alpha|_i, (\gamma|_{i-r})^\tau] = e.
\]

The general case $l \geq 1$ follows by induction. \hfill \Box

We apply the above proposition to $\beta \in B$.

**Corollary 4.4.** Let $\sigma_\beta = e$. Then for all $i, j \in Y$ and for all $x \in \mathbb{Z}$
\[
([\beta|_i], (\beta|_j)^\tau) = e.
\]

We derive further relations in $H = \langle \beta|_i \ (i \in Y) \rangle$. \hfill \Box

**Proposition 4.5.** Let $\beta \in B$. Then the following relations hold in $H$ for all $v \in \mathbb{Z}$ and for all $i \in Y$:
Proof. (I) Clearly \( [\beta, \beta^v] = e \) implies \( [\sigma_\beta, \sigma_\beta^v] = e \). It also implies

\[
\left( \beta (i) \sigma_{\beta^v} \right)^{-1} (\beta^v (i))^{-1} (\beta (i) \beta^v) = e,
\]

\[
(\beta^v (i)) \left( \beta (i) \sigma_{\beta^v} \right) = (\beta (i) \beta^v).
\]

\[
\left( \tau^v (i) \sigma_{\beta^v} \right)^{-1} (\beta (i) \sigma_{\beta^v}) (\tau^v (i) (\sigma_{\beta^v})) (\beta (i) \sigma_{\beta^v}) = (\beta (i) \tau^v) (i) \sigma_{\beta^v} (\tau^v).
\]

(II) On changing \( v \) to \( nv \) in (I), we obtain:

\[
\tau^{-v} (\beta (i) \tau^v) (\beta (i) \sigma_{\beta^v}) = (\beta (i) \tau^{-v} (\beta (i) \sigma_{\beta^v}) \tau^v,
\]

\[
\left( \beta (i) \sigma_{\beta^v} \right)^{-1} ((\beta (i) \tau^{-v} (\beta (i) \tau^v) \beta (i) \sigma_{\beta^v}) \right) = (\beta (i) \tau^{-v} (\beta (i) \tau^v) \beta (i) \sigma_{\beta^v}) \tau^v.
\]

(III) From (II), we derive

\[
[\beta (i) \tau^v] (\beta (i) \sigma_{\beta^v}) = [\beta (i) \sigma_{\beta^v}, \tau^v] (\beta (i) \sigma_{\beta^v}) = ... = [\beta (i) \tau^v].
\]

\[\square\]
5. The case $\beta \in B$ with $\sigma_\beta \in \langle \sigma_r \rangle$

This section is devoted to the proof of the second part of Theorem B. For this purpose, we introduce the following 3-variable combination of Delta-2 functions

$$\Delta_s(i, t) = \delta(i, t - i) - \delta(i - s, t - i)$$

which we call the Delta-3 function.

**Lemma 5.1.** Let $\beta \in A_n$ such that $[\beta, \beta^{x^2}] = e$ for any $x \in \mathbb{Z}$ and let $\sigma_\beta = \sigma^x_r$ for some $s \in Y$. Then,

$$\tau^{\Delta_s(i, t)} (\beta|_{i-s}) [\beta|_{i-s}, \tau^z] (\beta|_t) = (\beta|_{t-s}) (\beta|_i) [\beta|_i, \tau^z] \tau^{\Delta_s(i+s, t+s)}$$

for all $i, t \in \{0, 1, \ldots, n - 1\}$, $z \in \mathbb{Z}$.

**Proof.** Since $\sigma_\beta = \sigma^x_r$, we have $\sigma^{x^2}_\beta = \sigma^x_r = \sigma^x_r$.

From (2.4) and (2.5), we obtain

$$\tau^{-\frac{z-x}{n}} \delta(j-x, x) (\beta|_{j-x}) \tau^{-\frac{z-x}{n}} \delta(j-x+s, x) (\beta|_{j+s}) = (\beta|_j) \tau^{-\frac{z-x}{n}} \delta(j+s-x, x) (\beta|_{j+s-x}) \tau^{-\frac{z-x}{n}} \delta(j+2s-x, x)$$

Setting $k = \frac{x-y}{n}$ and $r = \frac{z}{n}$ and using (5.1), we have

$$\tau^{-k} \delta(j-r, r) (\beta|_{j-r}) \tau^k \delta(j+s-r, r) (\beta|_{j+s}) = (\beta|_j) \tau^{-k} \delta(j+s-r, r) (\beta|_{j+s-r}) \tau^k \delta(j+2s-r, r),$$

for all $r, j \in Y$ and all $k \in \mathbb{Z}$.

Also on setting $t = j + s, i = j + s - r$ and $z = k + \delta(j + s - r, r) (i = k + \delta(i, t - i))$ and using (5.2), we obtain

$$\tau^{-z} \delta(i, t - i) - \delta(i-s, t-i) (\beta|_{i-s}) \tau^z \beta|_{t} = (\beta|_{t-s}) \tau^{-z} \beta|_{i+1} \tau^z - \delta(i, t - i) + \delta(i+s, t-i),$$

for all $t, i \in \{0, 1, \ldots, n - 1\}$ and all $z \in \mathbb{Z}$.

It follows that

$$\tau^\delta(i, t - i) - \delta(i-s, t-i) (\beta|_{i-s}) \tau^z \beta|_{t} = (\beta|_{t-s}) \tau^\delta(i, t - i) + \delta(i+s, t-i) \tau^{-\delta(i, t - i) - \delta(i+s, t-i)}$$

for all $t, i \in \{0, 1, \ldots, n - 1\}$ and all $z \in \mathbb{Z}$. $\square$

We develop below some properties of the $\Delta_s$ function to be used in the sequel.

**Proposition 5.2.** The Delta-3 function satisfies

(i) $\Delta_s(i, t) = \delta(i, -s) - \delta(t, -s) = \begin{cases} 
0, & \text{if } \bar{i}, \bar{t} \geq \bar{s} \text{ or } \bar{i}, \bar{t} < \bar{s} \\
1, & \text{if } \bar{i} < \bar{s} \leq \bar{t} \\
-1, & \text{if } \bar{i} < \bar{s} \leq \bar{t} \end{cases}$,
(ii) \( \Delta_s(i, t) = -\Delta_s(t, i) \),

(iii) \( \Delta_s(i + s, t + s) = -\Delta_s(i, t) \),

(iv) \( \Delta_s(i, t) = \Delta_s(i, z) + \Delta_s(z, t) \),

(v) \( \sum_{k=0}^{n-1} \Delta_s(i + ks, t + ks) = 0 \),

(vi) \( \sum_{k=0}^{n-1} \Delta_s(k, t) = \begin{cases} n - \bar{s}, & \text{if } \bar{t} < \bar{s} \\ -\bar{s}, & \text{if } \bar{t} \geq \bar{s} \end{cases} \), for all \( i, t, z \in \mathbb{Z} \).

**Proof.**

(i) Using the definition \( \delta(i, j) = \frac{i + j - i + j}{n} \) we have

\[
\Delta_s(i, t) = \frac{i + t - i - t}{n} - \frac{i - s + t - i - t - s}{n} = \frac{\bar{i} + \bar{s} - \bar{i} - \bar{s}}{n} - \frac{\bar{t} + \bar{s} - \bar{t} - \bar{s}}{n}
\]

\[
= \delta(i, -s) - \delta(t, -s) = \begin{cases} 0, & \text{if } \bar{t}, \bar{i} \geq \bar{s} \text{ or } \bar{t}, \bar{i} < \bar{s} \\ 1, & \text{if } \bar{t} < \bar{s} \leq \bar{i} \\ -1, & \text{if } \bar{i} < \bar{s} \leq \bar{t} \end{cases}
\]

(ii) Follows from (i).

(iii) Calculate

\[
\Delta_s(i + s, t + s) = \delta(i + s, t - i) - \delta(i, t - i)
\]

\[
= - (\delta(i, t - i) - \delta(i + s, t - i)) = -\Delta_s(i, t).
\]

(iv) This part follows from (i).

(v) From the definition of the Delta-2 function

\[
\sum_{k=0}^{n-1} \delta(i + ks, t - i) = \sum_{k=0}^{n-1} \delta(i + (k - 1)s, t - i).
\]

(vi) Finally, we have

\[
\sum_{k=0}^{n-1} \Delta_s(k, t) = \sum_{k=0}^{\bar{s}-1} \Delta_s(k, t) + \sum_{k=\bar{s}}^{n-1} \Delta_s(k, t)
\]

\[
= \begin{cases} n - \bar{s}, & \text{if } \bar{t} < \bar{s} \\ -\bar{s}, & \text{if } \bar{t} \geq \bar{s} \end{cases}.
\]

□

With the use of the Delta-3 function we obtain
Proposition 5.3. The following relations are verified in $H = \langle \beta_i \mid (i \in Y), \tau \rangle$, for all $x, z \in \mathbb{Z}$ and all $i, t \in Y$:

(I) $\tau^{\Delta_s(i, t)} (\beta|_{i-s}) (\beta|_t) = (\beta|_{t-s}) (\beta|_i) \tau^{\Delta_s(i+s, t+s)}$;
(II) $[\beta|_{i-s}, \tau^{\alpha}_{\beta|_i}] \tau^{\Delta_s(i+s, t+s)} = [\beta|_i, \tau^{\alpha}]$;
(III) $[[\beta|_i, \tau^{\alpha}], [\beta|_t, \tau^{\alpha}]] = e$.

Proof. Returning to Lemma 5.1, we have

$$\tau^{\Delta_s(i, t)} (\beta|_{i-s}) (\beta|_t) = (\beta|_{t-s}) (\beta|_i) \tau^{\Delta_s(i+s, t+s)},$$

Consequently,

$$\tau^{\Delta_s(i, t)} (\beta|_{i-s}) (\beta|_t) = (\beta|_{t-s}) (\beta|_i) \tau^{\Delta_s(i+s, t+s)}$$

and

$$[\beta|_{i-s}, \tau^{\alpha}] (\beta|_t) = [\beta|_{i}, \tau^{\alpha}] \tau^{\Delta_s(i+s, t+s)} = [\beta|_i, \tau^{\alpha}],$$

for all $t, i \in Y$ and all $z \in \mathbb{Z}$.

From (5.4) and (4.1), $N = \langle [\beta|_i, \tau^{k}] \mid k \in \mathbb{Z}, i \in Y \rangle$ is a normal subgroup of $H$. Moreover, by applying alternately the above equations, we obtain

$$[\beta|_i, \tau^{\alpha}] (\beta|_t) = [\beta|_{i}, \tau^{\alpha}] \tau^{\Delta_s(i+s, t+s)},$$

for all $t, i \in Y$ and all $z \in \mathbb{Z}$.

By applying alternately the above equations, we obtain

$$[\beta|_i, \tau^{\alpha}] = [\beta|_{i}, \tau^{\alpha}] \tau^{\Delta_s(i+s, t+s)}.$$
Corollary 5.4. Let $\beta \in A_n$ such that $[\beta, \beta^x] = e$ for every $x \in \mathbb{Z}$ with $\sigma_\beta = \sigma_\beta^s$ for some $s \in \{0, 1, \ldots, n - 1\}$. Then

$$M = \left\langle [\beta_i, \tau^{k_i}] \mid k_i \in \mathbb{Z}, 0 \leq i \leq n - 1 \right\rangle$$

is a normal metabelian subgroup of $H$.

Proof. By Proposition 5.3, $N = \langle [\beta_i, \tau^{k_i}] \mid k_i \in \mathbb{Z}, 0 \leq i \leq n - 1 \rangle$ is abelian and normal in $H$. Since $N\tau \in Z(H/N)$, it follows that $M = N\langle \tau \rangle$ is a normal subgroup of $H$ and is clearly metabelian. □

We are ready to prove part (I) of Theorem B.

Theorem 5.5. Let $\beta \in A_n$ be such that $[\beta, \beta^x] = e, \forall x \in \mathbb{Z}$ and $\sigma_\beta = \sigma_\beta^s$ for some $s \in Y$ and $H = \langle \beta_0, \ldots, \beta_n, \tau \rangle$. Recall $\pi_j = \beta_j|\beta|\beta|_{j+2} \cdots \beta|_{j+(m-1)s}$. Then,

(i) the group $K = \langle [\beta_i, \tau^2], \pi_j \mid i,j \in Y, x \in \mathbb{Z} \rangle$ is an abelian normal subgroup of $H$ and the group $O = K\langle \tau \rangle$ is a metabelian normal subgroup of $H$;

(ii) the quotient group $H/O$ is a homomorphic image of a subgroup of $C_m \wr C_n$.

In particular, $H$ is metabelian-by-finite.

Proof. (i) Recall

$$N = \left\langle [\beta_i, \tau^{k_i}] \mid k_i \in \mathbb{Z}, i \in Y \right\rangle,$$

$$K = N\langle \pi_j \mid j \in Y \rangle$$

where $m = \frac{n}{\gcd(n,s)}$. Then, by Proposition 5.3, $N$ is an abelian normal subgroup of $H$.

By (5.4), we have

$$[\beta_i, \tau^x]^{\pi_j} = [\beta_i, \tau^x]^{\tau^{\Delta_k(i+k+1)s} \beta|_{j+k} \cdots \beta|_{j+(m-1)s}}$$

$$= [\beta_i, \tau^x]^{\tau^{\Delta_k(i+k+1)s} \beta|_{j+k} \cdots \beta|_{j+(m-1)s}}$$

$$= [\beta_i, \tau^x]^{\tau^{\Sigma_{k=0}^{m-1} \Delta_k(i+k+1)s, j+k}}$$

Thus,

(5.5) $[[\beta_i, \tau^z], (\beta^m)|_j] = e, \forall i,j \in Y, \forall z \in \mathbb{Z}$

Since $\sigma_\beta^m = e$, we have by Corollary 4.4

(5.6) $[(\beta^m)|_i, (\beta^m)|_j] = e, \forall i,j \in Y$.

Moreover,

(5.7) $(\beta^m)|_i^\tau = ((\beta^m)|_i) [(\beta^m)|_i, \tau]$.

Since $[\beta, \beta^x] = e, \forall x \in \mathbb{Z}$, it follows that $[\beta^m, \beta^x] = e, \forall x \in \mathbb{Z}$. 

Therefore, by (2.5),
\[ e = (\beta^m)_{i}^{-1}(\beta^{\tau^x})_{i}^{-1}(\beta^m)_{i}|(\beta^{\tau^x})_{i}, \forall x \in \mathbb{Z}, \forall i \in Y. \]

Now, as \( \sigma_{\beta} = \sigma_{\beta}^i \) and \( \sigma_{\beta}^m = e \), we reach

(5.8) \[ (\beta^m)_{i+s} = (\beta^m)_{i}|(\beta^{\tau^x})_{i}, \forall x \in \mathbb{Z}, \forall i \in Y. \]

By (2.4), the following
\[ (\beta^{\tau^x})_{i} = (\tau^x)_{i}^{-1}(\beta^{\tau^x})_{i}^{-1}(\beta^x)_{i}^{-1}(\beta^m)_{i}(\beta^{\tau^x})_{i}, \forall x \in \mathbb{Z}, \forall i \in Y. \]
holds for all \( i \in Y \) and all \( x \in \mathbb{Z} \).

From which we derive
(5.9) \[ (\beta^{\tau^x})_{i} = \tau^{-\frac{x - \tau - \delta(i-x,x)}{m}} \beta_{i-x}^{\tau} \tau^{\frac{x - \tau - \delta(i-x+s,x)}{m}} \]
for all \( i \in Y \) and all \( x \in \mathbb{Z} \).

Therefore, by (5.8) and (5.9),
\[ (\beta^m)_{i+s} = (\beta^m)_{i}|(\tau^{-\frac{x - \tau - \delta(i-x,x)}{m}} \beta_{i-x}^{\tau} \tau^{\frac{x - \tau - \delta(i-x+s,x)}{m}}), \]
for all \( i \in Y \) and all \( x \in \mathbb{Z} \).

On writing \( x = kn + \tau = kn + r, r \in \mathbb{Z} \) in the above equation, we obtain
\[ (\beta^m)_{i+s} = (\beta^m)_{i}|(\tau^{-k-\delta(i-r,r)} \beta_{i-r}^{\tau} \tau^{k+\delta(i-r+s,r)} \]
\[ \Rightarrow (\beta^m)_{i+s} = (\beta^m)_{i}|(\beta_{i-r})^{\tau^{-k-\delta(i-r,r)} \tau^{k+\delta(i-r+s,r)}} \]
\[ \Rightarrow (\beta^m)_{i+s} = (\beta^m)_{i}|(\beta_{i-r})^{\tau^{-k-\delta(i-r+s,r)} \tau^{k+\delta(i-r,r)}} \]
for all \( i, r \in Y \) and all \( k \in \mathbb{Z} \).

By (5.5), (5.7) and using the fact that \( N \) is abelian and normal in \( H \), we find
\[ (\beta^m)_{i+s}^{\delta(i-r,r)-\delta(i-r+s,r)} = (\beta^m)_{i}|(\beta_{i-r})^{\tau^{-k-\delta(i-r,r)} \tau^{k+\delta(i-r+s,r)}} \]
for all \( i, r \in Y \).

On setting \( j = i - r \), we get
(5.10) \[ (\beta^m)_{i+s}^{\Delta_{-s}(i-r,i)} = (\beta^m)_{i}|(\beta_{i-r})^{\tau^{\Delta_{-s}(i-r,i)}} \]
for all \( i, j \in Y \).

Further, by using equations (5.5), (5.6), (5.7), (5.10) and
(5.11) \[ (\beta^m)_{i} = \pi_{i}, \]
we conclude that also \( K \) is an abelian normal subgroup of \( H \).
Now, \( O = K \langle \tau \rangle \) is metabelian. Moreover it is normal in \( H \), because
\[
\tau^{\beta i} = \tau \tau^{-1} \tau^{\beta i} = \tau [\tau, \beta]_i \in O
\]
for all \( i \in Y \).

(ii) Consider the following Fibonacci type group
\[
X = \langle b_0, \ldots, b_{n-1} \mid b_i b_{j+s} = b_j b_{i+s}, b_i b_{n-1+s} \cdots b_{i+(n-1)s} = e, \forall i, j \in Y \rangle
\]
where the bar notation indicates ‘modulo \( n \)’.

The Equation (5.3) shows that \( \frac{H}{O} \) is a homomorphic image of \( X \). We will prove that \( X \) is isomorphic to a subgroup of the wreath product \( C_m \wr C_n \).

As a matter of fact the group \( C_m \wr C_n \) has the presentation
\[
\langle a, b \mid a^n = e, b^a = a^b, b^{a+s} = b^{a+s}, \ldots b^{a+(m-1)s} = e, \forall i, j \in Y \rangle.
\]

On defining \( b = a^{s}u^{-1} \), we have
\[
u^{m} = e
\]
\[
\Rightarrow (a^{-s}b)^m = e
\]
\[
\Rightarrow (a^{-s}b \cdots a^{-s}b)^{a^{-s}} = e
\]
\[
\Rightarrow b^{a+s}b^{a+s} \cdots b^{a+s} = e.
\]

Also, the commutation relation
\[
u^a u^a = u^a v^a
\]
implies
\[
(b^{-1}a^s)^a (b^{-1}a^s)^a = (b^{-1}a^s)^a (b^{-1}a^s)^a
\]
\[
\Rightarrow (a^{-s}b)^a (a^{-s}b)^a = (a^{-s}b)^a (a^{-s}b)^a
\]
\[
\Rightarrow b^{a+s}a^{-s}b^{a+s} = b^{a+s}a^{-s}b^{a+s}
\]
\[
\Rightarrow b^{a+s}b^{a+s} = b^{a+s}b^{a+s}.
\]

By using Tietze transformations we conclude that \( C_m \wr C_n \) has the presentation
\[
\langle a, b \mid a^n = e, b^{a+s}b^{a+s} = b^{a+s}b^{a+s}, b^a b^a \cdots b^{a+(m-1)s} = e, \forall i, j \in Y \rangle.
\]

Then, on introducing \( b_i = b^i, i = 0, \ldots, n-1 \), the above presentation is expressed as
\[
\langle a, b_0, \ldots, b_{n-1} \mid a^n = e, b_i = b_0^i, b_j b_{j+s} = b_j b_{j+s}, b_i b_{n-1+s} \cdots b_{i+(m-1)s} = e, \forall i, j \in Y \rangle.
\]

\[\square\]

The next results leads to a proof of Theorem C.
Lemma 5.6. Let $\sigma = (0, 1, \ldots, n-1) \in \Sigma_n$ and let $L$ be the layer closure of $\langle \sigma \rangle$ in $A_n$. Suppose $\beta = (\beta_0, \beta_1, \ldots, \beta_{n-1}) \sigma \beta \in L$ satisfies $[\beta, \beta^x] = e$ for all $x \in \mathbb{Z}$. Write $\sigma_\beta = \sigma^x$ and $\sigma_{\beta|_i} = \sigma^{m_i}$ for all $i \in Y$. Then for all $i, t \in Y$, the following congruence holds

$$\Delta_s(i, t) + m_{-t-s} + m_t \equiv m_{-t-s} + m_i + \Delta_s(i + s, t + s) \mod n.$$  

Proof. Since $\sigma_{\beta|_i} = \sigma^{m_i}$, we conclude by (5.3),

$$\sigma^n \equiv \sigma^{m_i + \Delta(i, t)} \equiv \sigma^{m_{-t-s} + m_i + \Delta_s(i + s, t + s)}$$

and therefore, $\Delta_s(i, t) + m_{-t-s} + m_t \equiv m_{-t-s} + m_i + \Delta_s(i + s, t + s) \mod n.$ \hfill \Box

Lemma 5.7. Maintain the notation of the previous lemma and let $s = 1$. Then,

$$\sigma_{(\beta^s)_0} = \sigma_{(\beta_0)(\beta_1)\cdots(\beta_{n-1})} = \sigma.$$  

Proof. The case $n = 2$ is covered by Proposition 6 of [9].

Now let $n$ be an odd prime. From

$$\Delta_1(i, t) + m_{-t-1} + m_t \equiv m_{-t-1} + m_i + \Delta_1(i + 1, t + 1) \mod n$$

we conclude

$$\sum_{i=0}^{n-2} \sum_{t=i+1}^{n-1} (\Delta_1(i, t) + m_{-t-1} + m_t) \equiv \sum_{i=0}^{n-2} \sum_{t=i+1}^{n-1} (m_{-t-1} + m_i + \Delta_1(i + 1, t + 1)) \mod n.$$  

Now,

$$\sum_{i=0}^{n-2} \sum_{t=i+1}^{n-1} \Delta_1(i, t) \quad \text{Prop 5.2(i)} \quad \sum_{t=0}^{n-1} \Delta_1(0, t) \quad \text{Prop 5.2(ii)} \quad \sum_{t=0}^{n-1} \Delta_1(0, t)$$

$$\sum_{t=0}^{n-1} \Delta_1(0, t) \quad \text{Prop 5.2(ii)} \quad \sum_{t=0}^{n-1} -\Delta_1(t, 0) \quad \text{Prop 5.2(ii)} \quad \sum_{t=0}^{n-1} -\Delta_1(t, 0) \quad \text{Prop 5.2(ii)} \quad (n-1),$$

$$\sum_{i=0}^{n-2} \sum_{t=i+1}^{n-1} \Delta_1(i + 1, t + 1) \quad \text{Prop 5.2(i)} \quad \sum_{i=0}^{n-2} \sum_{i=0}^{n-2} \Delta_1(i + 1, 0) \quad \text{Prop 5.2(ii)} \quad \sum_{i=0}^{n-1} \Delta_1(i, 0) \quad \text{Prop 5.2(ii)} \quad (n-1),$$

$$\sum_{i=0}^{n-2} \sum_{t=i+1}^{n-1} (m_{-t-1} + m_t) = 2(n-1)m_{n-1} + (n-2)\sum_{k=0}^{n-1} m_k$$

and

$$\sum_{i=0}^{n-2} \sum_{t=i+1}^{n-1} (m_{-t-1} + m_t) = n \sum_{k=0}^{n-1} m_k.$$  

Since $n$ is odd, we have

$$\sum_{k=0}^{n-1} m_k \equiv 1 \mod n.$$
and therefore, $\sigma_{(\beta|_0)\cdots(\beta|_{n-1})} = \sigma^{m_0 + \cdots + m_{n-1}} = \sigma$. \hfill \Box$

Now we prove Theorem C.

**Theorem 5.8.** Let $n$ be an odd number, $\sigma = (0, \ldots, n-1) \in \Sigma_n$ and let $L$ be the layer closure of $\langle \sigma \rangle$ in $A_n$. Let $s$ be an integer relatively prime to $n$ and $\beta = (\beta|_0, \beta|_1, \ldots, \beta|_{n-1}) \sigma^s \in L$ be such that $[\beta, \beta^s] = e$ for all $x \in \mathbb{Z}$. Then $\beta$ is a conjugate of $\tau$ in $A_n$.

**Proof.** We start with the case $s = 1$. The element

$$\alpha(1) = (e, \beta|_0^{-1}, (\beta|_0(\beta|_1))^{-1}, \ldots, ((\beta|_0) \cdots (\beta|_{n-2}))^{-1}) \in \text{Stab}(1)$$

conjugates $\beta$ to

$$\beta^{\alpha(1)} = (e, \ldots, e, (\beta|_0) \cdots (\beta|_{n-1})).$$

By Lemma 5.7 we find $\sigma_{(\beta|_0)\cdots(\beta|_{n-1})} = \sigma$. Moreover by Proposition 4.3

$$[(\beta^n)|_0, ((\beta^n)|_0)^s] = [(\beta|_0) \cdots (\beta|_{n-1}), ((\beta|_0) \cdots (\beta|_{n-1}))^s] = e,$$

for all integers $x$. Therefore $(\beta|_0) \cdots (\beta|_{n-1})$ satisfies the hypothesis of the theorem. The process can be repeated until we obtain a sequence $(\alpha(k))_{k \in \mathbb{N}}$ such that $\beta^{\alpha(1)\alpha(2)\cdots\alpha(k-1)} = \tau$, where $\alpha(k) \in \text{Stab}(k)$ satisfies $\alpha(k)|_u = \alpha(k)|_v$ for all $u, v \in M$ with $|u| = |v| = k - 1$.

Now, suppose more generally $s$ is such that $\gcd(s, n) = 1$ and let $k$ be the minimum positive integer for which $sk \equiv 1 \mod(n)$. Then $\beta^k$ satisfies the hypothesis of the first part and so, there exists $\alpha \in L$ such that $(\beta^k)^\alpha = \tau$. Since $k$ is invertible in $\mathbb{Z}_n$, there exists $\gamma \in A_n$ such that $\tau^{-1} = \tau^k$. Thus, $\beta^{\alpha \gamma^{-1}} = \tau$. \hfill \Box

6. *Solvable groups for $n = p$, a prime number*

We will prove in this section the case $n = p$ of Theorem A.

Let $B$ be an abelian subgroup of $Aut(T_p)$ normalized by $\tau$ and let $\beta \in B$. By Proposition 4.1 $\sigma_\beta \in \langle \sigma_\tau \rangle$ and therefore we have in effect two cases, namely, $\sigma_\beta = e, \sigma_\tau$.

**Proposition 6.1.** Suppose $\sigma_\beta \in \langle \sigma_\tau \rangle$. Then, $\sigma_{(\beta|_i)} \in \langle \sigma_\tau \rangle$ for all $i \in Y$.

**Proof.** By Theorem 5.5 $K$ is an abelian normal subgroup of $H$ and $\overline{H}$ is homomorphic to a subgroup of $C_p \cap C_p$ for $O = K(\tau)$.

By Proposition 4.1 $K$ is a subgroup of $\langle \sigma_\tau \rangle$ modulo $\text{Stab}(1)$. So the same is true for $O = K(\tau)$.

Therefore, $H$ is a $p$-group modulo $\text{Stab}(1)$. Since $H$ is a $p$-group modulo $\text{Stab}(1)$ and since $\tau \in H$, it follows that $H$ coincides with $\langle \sigma_\tau \rangle$ modulo $\text{Stab}(1)$, by Proposition 4.1. Hence, necessarily we have $\sigma_{(\beta|_i)} \in \langle \sigma_\tau \rangle$. \hfill \Box

**Theorem 6.2.** Let $p$ be a prime number and $\beta \in Aut(T_p)$ such that $\sigma_\beta = \sigma_\tau^s$ for some integer $s$ relatively prime to $p$. Suppose $[\beta, \beta^s] = e$ for all $x \in \mathbb{Z}$. Then $\beta$ is conjugate to $\tau$ in $Aut(T_p)$. 

Proof. As the second author showed the case $p=2$ in [9], we will show the case $p$ odd.

Suppose $s = 1$. Recall that

$$\alpha(1) = (e, \beta|_0^{-1}, (\beta|_0^2)^{-1}, \ldots, \beta|_{p-2})^{-1}) \in \text{Stab}(1)$$

conjugates $\beta$ to its normal form

$$\beta^{\alpha(1)} = (e, \ldots, e, \beta|_0 \cdots \beta|_{p-1})\sigma.$$ 

By Lemma 5.7 we have $\sigma\beta|_0 \cdots \beta|_{p-1} = \sigma$. Moreover by Proposition 4.3

$$[\beta|_0^p, (\beta|_0^p)_\tau^x] = [\beta|_0 \cdots \beta|_{p-1}, (\beta|_0 \cdots \beta|_{p-1})^\tau] = e,$$

for all integers $x$. Therefore $\beta|_0 \cdots \beta|_{p-1}$ satisfies the condition of the theorem. This process can be repeated to produce a sequence $(\alpha(k))_{k \in \mathbb{N}}$ such that $\beta^{\alpha(1)}\alpha(2) \cdots \alpha(k) \cdots = \tau$, where $\alpha(k) \in \text{Stab}(k)$ satisfies $\alpha(k)|_u = \alpha(k)|_v$ for all $u, v \in M$ where $|u| = |v| = k - 1$.

In the general case, $s$ is such that $\gcd(p, s) = 1$. Let $k$ be the minimum positive integer which is the inverse of $s$ modulo $p$. Then, $\sigma|_{\beta^k} = \sigma_\tau$ and $\beta^k$ satisfies the hypotheses. Thus there exists $\alpha \in A_p$ such that $(\beta^k)^\alpha = \tau$. Let $k$ be the inverse of $k$ in $U(\mathbb{Z}_n)$; then $\beta^\alpha = \tau^k$. There exists $\gamma \in N_{A_p}(<\tau>)$ which conjugates $\tau$ to $\tau^k$ and so, $(\beta^\alpha)^{\gamma^{-1}} = \tau$.

Lemma 6.3. Let $p$ be a prime number and $\beta \in \text{Aut}(T_p)$ such that $[\beta, \beta^x] = e$ for all $x \in \mathbb{Z}$. Then, there exists a tree level $m$ and a conjugate $\mu$ of $\tau$ such that $\beta \in \times_{p^m} <\mu>$ and there exists an index $u$ of length $m$ such that $\beta_i|_u = \mu$.

Proof. Let $m$ be the minimum tree level such that $\sigma_{\beta|_u} \neq e$ for some $|u| = m$. Therefore, $\beta \in \text{Stab}(m)$ and $\sigma_{\beta|_u} = \sigma_{\beta^s}$ for some integer $s$ such that $\gcd(p, s) = 1$. By Proposition 4.3 $[\beta|_u, \beta|_u^s] = e$ for all indices $v$ such that $|v| = m$ and for all $k \in \mathbb{Z}$, So, by Theorem 6.2 $\mu = \beta|_u$ is conjugate to $\tau$ in $\text{Aut}(T_p)$ and $\beta|_v \in <\mu>$ for all $v$ such that $|v| = m$, by Lemma 3.2. \hfill $\square$

Theorem 6.4. Let $p$ be a prime number, $\sigma = (0, 1, \ldots, p - 1) \in \Sigma_p$, $F = N_{\Sigma_p}(<\sigma>)$, $\Gamma_0 = N_A(<\tau>)$ . Let $G$ be a solvable subgroup of $\text{Aut}(T_p)$ which contains the $p$-adic adding machine $\tau$. Then, there exists an integer $t \geq 1$ such that $G$ is conjugate to a subgroup of

$$\times_p (\cdots (\times_p (\times_p \Gamma_0 \times F) \times) \cdots) \times F,$$

where $\times_p$ appears $t$ times.

Proof. We may suppose $G$ has derived length $d \geq 2$. Let $B$ be the $(d - 1)$-th term of the derived series of $G$. By Lemma 6.3 there exists a level $t$ such that $B$ is a subgroup of $V = \times_{p^t} <\mu>$ where $\mu = \tau^\alpha$ for some $\alpha \in \text{Aut}(T_p)$.

We will show that $G$ is a subgroup of

$$\hat{J} = \times_p (\cdots (\times_p (\times_p (\Gamma_0^\alpha \times \Sigma_p) \times \Sigma_p) \cdots) \times \Sigma_p),$$

where $\times_p$ appears $t$ times.
Let $\gamma \in G \setminus \hat{J}$. Then there exists an index $w$ of length $t$ such that $\gamma|_w \not\in (\Gamma_0)^\alpha$. Since $B$ is an abelian subgroup normalized by $\tau$ and $\tau$ is transitive on all levels of the tree, by Lemma 6.3 there exists $\beta \in B$ such that $\beta|_w = \mu^\eta$ for some $\eta \in U(\mathbb{Z}_p)$.

Write $v = w^\gamma$. Then,

$$
(\beta^\gamma)|_v \overset{227}{=} (\beta|_{v^{\gamma^{-1}}}) \left(\gamma|_v^{\gamma^{-1}}\right) = (\beta|_w)^{\gamma|_w} \not\in \langle \mu \rangle,
$$

and this implies $\beta^\gamma \not\in B \leq \times_p \langle \mu \rangle$ and $\gamma \not\in G$. Hence, $G$ is a subgroup of $\hat{J}$.

Now, since $G$ is a solvable group containing $\tau$, there exist $G_i (0 \leq i \leq t)$ solvable subgroups of $\Sigma_p$ containing $\sigma = (0, 1, \ldots, p - 1)$ such that $G$ is a subgroup of

$$
R_t (\alpha) = \times_p (\times_p (\Gamma_0) \rtimes G_1 \rtimes G_2) \cdots \rtimes G_t.
$$

Since for all $i$, we have $G_i \leq F$ we may substitute every the $G_i$ by $F$. Finally, $R_t (\alpha)$ is a conjugate of $R_t (1)$ by the diagonal automorphism $\alpha^t$.

\[\square\]

7. Two cases for $n$ even

We prove in this section part (II) of Theorem B.

7.1. The case $\sigma_\beta = (\sigma_\tau)^{\frac{n}{2}}$.

**Theorem 7.1.** Let $n$ be an even number, $\beta \in A_n$ such that $\sigma_\beta = \sigma_\tau^{\frac{n}{2}}$ and $[\beta, \beta^{\tau_x}] = e$ for all $x \in \mathbb{Z}$. Then $H = \langle \beta \rangle_i (0 \leq i \leq n - 1) , \tau \rangle$ is a metabelian subgroup of $A_n$.

**Proof.** Denote $\Delta_{\frac{n}{2}}(i, j)$ by $\Delta(i, j)$.

Define the subgroup

$$
R = \left\langle [\beta|_i, \tau^k], \beta|_i \beta|_{i+\frac{n}{2}}, \beta|_i^{\frac{n}{2}} \tau^{-\Delta(j, j + \frac{n}{2})} \mid k \in \mathbb{Z} \text{ and } i, j, t \in Y \right\rangle.
$$

We will prove that $R$ is an abelian normal subgroup of $H$.

(1) $R$ is normal in $H$:

- $\langle [\beta|_i, \tau^k] \rangle^H \leq R$:

$$
[\beta|_i^{\frac{n}{2}}, \tau^k] \beta|_i^{\frac{n}{2}} \beta|_i, \tau^k \overset{5.3}{=} [\beta|_i, \tau^k]^{\Delta(i, 0)};
$$

- $\langle \beta|_i \beta|_{i+\frac{n}{2}} \rangle^H \leq R$:

$$
(\beta|_i \beta|_{i+\frac{n}{2}})^{\tau^k} = \left(\beta|_i \beta|_{i+\frac{n}{2}}\right)^{[\beta|_i \beta|_{i+\frac{n}{2}}, \tau^k]} = \left(\beta|_i \beta|_{i+\frac{n}{2}}\right)^{[\beta|_i, \tau^k]^{\beta|_i^{\frac{n}{2}} \beta|_i}} \beta|_i^{\frac{n}{2}}, \tau^k
$$

(7.1)

$$
(\beta|_i \beta|_{j+\frac{n}{2}})^{\beta|_j} = \left(\beta|_j^{-1} \beta|_i \beta|_{j+\frac{n}{2}} \beta|_j\right)^{\tau^{\Delta(j, j + \frac{n}{2})} \tau^{-\Delta(j, j + \frac{n}{2})}} \overset{5.3}{=} \left(\beta|_j^{-1} \beta|_i\right)^{\tau^{\Delta(j, j)}} \left(\beta|_j \beta|_{j+\frac{n}{2}} \beta|_i\right)^{\tau^{-\Delta(j, j + \frac{n}{2})}} = \left(\beta|_j^{-1} \beta|_i \beta|_{j+\frac{n}{2}}\right)^{\tau^{\Delta(j, j), \beta|_j \beta|_{j+\frac{n}{2}}, \beta|_i^{\tau^{-\Delta(j, j + \frac{n}{2})}}}}.
$$
\[ (5.3) \quad \left( \beta_{\bar{i}}^{-1} \right) \tau^{\Delta(j,\frac{\bar{i}}{2} + \frac{\bar{i}}{2})} \left( \beta_{\bar{j}} \beta_{\bar{i}} + \frac{\bar{i}}{2} \right) \phi_{\bar{i}j} + \phi_{\bar{j}i} \beta_{\bar{i}} \mid \tau - \Delta(j,\frac{\bar{i}}{2} + \frac{\bar{i}}{2}) \]

\[ = \tau^{\Delta(j,\frac{\bar{i}}{2} + \frac{\bar{i}}{2})} \left[ \tau^{\Delta(j,\frac{\bar{i}}{2} + \frac{\bar{i}}{2})} \beta_{\bar{j}} \right] \beta_{\bar{i}} \mid \tau - \Delta(j,\frac{\bar{i}}{2} + \frac{\bar{i}}{2}) \]

\[ \text{Prop} \Rightarrow 5.2 \quad \tau^{\Delta(j,\bar{i})} \left[ \tau^{\Delta(j,\bar{i})} \beta_{\bar{j}} \right] \beta_{\bar{i}} \mid \tau^{\Delta(j,\bar{i})} \]

\[ \left( \beta_{\bar{i}} \right)^{\beta_{\bar{j}}} \tau^{-\Delta(j,\bar{i} + \frac{\bar{i}}{2})} \leq R : \]

\[ \left( \beta_{\bar{j}} \right)^{\beta_{\bar{i}}} \tau^{-\Delta(j,\bar{i} + \frac{\bar{i}}{2})} = \left( \beta_{\bar{j}} \right)^{\beta_{\bar{i}}} \tau^{-\Delta(j,\bar{i} + \frac{\bar{i}}{2})} \]

By Proposition 5.2 and 5.3, we can show

\[ \left( \beta_{\bar{i}} \right)^{\beta_{\bar{j}}} \tau^{-\Delta(j,\bar{i} + \frac{\bar{i}}{2})} \]

(II) The subgroup \( R \) is abelian:

\[ \left( \beta_{\bar{i}} \right)^{\beta_{\bar{j}}} \tau^{-\Delta(j,\bar{i} + \frac{\bar{i}}{2})} \mid \tau^{\Delta(j,\bar{i} + \frac{\bar{i}}{2})} \]

\[ \left( \beta_{\bar{i}} \right)^{\beta_{\bar{j}}} \tau^{-\Delta(j,\bar{i} + \frac{\bar{i}}{2})} \mid \tau^{\Delta(j,\bar{i} + \frac{\bar{i}}{2})} \]

\[ \left( \beta_{\bar{i}} \right)^{\beta_{\bar{j}}} \tau^{-\Delta(j,\bar{i} + \frac{\bar{i}}{2})} \mid \tau^{\Delta(j,\bar{i} + \frac{\bar{i}}{2})} \]

\[ \left( \beta_{\bar{i}} \right)^{\beta_{\bar{j}}} \tau^{-\Delta(j,\bar{i} + \frac{\bar{i}}{2})} \mid \tau^{\Delta(j,\bar{i} + \frac{\bar{i}}{2})} \]

\[ \left( \beta_{\bar{i}} \right)^{\beta_{\bar{j}}} \tau^{-\Delta(j,\bar{i} + \frac{\bar{i}}{2})} \mid \tau^{\Delta(j,\bar{i} + \frac{\bar{i}}{2})} \]

\[ \left( \beta_{\bar{i}} \right)^{\beta_{\bar{j}}} \tau^{-\Delta(j,\bar{i} + \frac{\bar{i}}{2})} \mid \tau^{\Delta(j,\bar{i} + \frac{\bar{i}}{2})} \]

\[ \left( \beta_{\bar{i}} \right)^{\beta_{\bar{j}}} \tau^{-\Delta(j,\bar{i} + \frac{\bar{i}}{2})} \mid \tau^{\Delta(j,\bar{i} + \frac{\bar{i}}{2})} \]
\[
(\beta_i | \beta_i | i+\frac{\beta}{2})^2 = \beta_i | \beta_i | i+\frac{\beta}{2}
\]

Let

\[
(7.6) \quad \alpha = \beta_j | \beta_j | i+\frac{\beta}{2} | \tau - (j+\frac{\beta}{2}), \beta_j | i+\frac{\beta}{2}.
\]

Then,

\[
\left( \beta_j | \beta_j | i+\frac{\beta}{2} \right)^2 \beta_j | \beta_j | i+\frac{\beta}{2} | \tau - (j+\frac{\beta}{2}), \beta_j | i+\frac{\beta}{2}
\]

\[
= \left( \beta_j | \beta_j | i+\frac{\beta}{2} \right)^2 \beta_j | \beta_j | i+\frac{\beta}{2} | \tau - (j+\frac{\beta}{2}), \beta_j | i+\frac{\beta}{2}
\]

\[
= \left( \beta_j | \beta_j | i+\frac{\beta}{2} \right)^2 \beta_j | \beta_j | i+\frac{\beta}{2} | \tau - (j+\frac{\beta}{2}), \beta_j | i+\frac{\beta}{2}
\]

\[
\alpha = \beta_j | \beta_j | i+\frac{\beta}{2} | \tau - (j+\frac{\beta}{2}), \beta_j | i+\frac{\beta}{2}
\]
Moreover, since
\[ R(\beta_i) R(\beta_j) = R(\beta_i) (\beta_j) \overset{\text{Prop. 5.3}}{=} R\tau^{\Delta(j,i+\frac{n}{2})}\beta_{j+i+\frac{n}{2}} \tau^{\Delta(j,i+\frac{n}{2})} \]
\[ = R\beta_{j+i+\frac{n}{2}} \tau^{2\Delta(j,i+\frac{n}{2})} = R\beta_i^{-1} \beta_j^{-1} \tau^{2\Delta(j,i+\frac{n}{2})} \]
\[ = R\beta_j \beta_i \tau^{\Delta(j,i+\frac{n}{2})} \tau^{\Delta(i,j+\frac{n}{2})} + 2\Delta(j,i+\frac{n}{2}) \]
\[ \overset{\text{Prop. 5.2}}{=} R\beta_i R\beta_j \]
and
\[ R\beta_i = R\beta_i^{-1}, \quad R\beta_i^2 = R\tau^{\Delta(i,j+\frac{n}{2})}, \quad \forall i \in Y, \]
we conclude \( \frac{H}{R} \) is a homomorphic image of
\[ \mathbb{Z} \times C_2 \times \cdots \times C_2, \]
\[ \frac{n}{2} \text{ terms} \]

7.2. The case \( \sigma_\beta \) transposition.

**Theorem 7.2.** Let \( n \) be an even number and \( B \) an abelian subgroup of \( A_n \) normalized by \( \tau \). Suppose \( \beta = (\beta_0, \beta_1, \ldots, \beta_{n-1}) \in B \) where \( \sigma_\beta \) is a transposition. Then \( H = \langle \beta_i \ (0 \leq i \leq n-1), \tau \rangle \) is a metabelian group.

We prove progressively that
\[ N = \left\langle [\beta_i, \tau^k] \mid k \in \mathbb{Z}, i \in Y \right\rangle, \]
\[ U = \left\langle N, \beta_j \mid j \neq 0, \frac{n}{2} \right\rangle, \]
\[ V = \left\langle U, \beta_{\frac{n}{2}}, \tau (\beta_0)^2 \right\rangle \]
are normal abelian subgroups of \( H \), from which it follows that \( \frac{H}{V} \) is cyclic and therefore \( H \) metabelian.

**Lemma 7.3.** If the degree of the tree \( n \) is even and \( \sigma_\beta \) is a transposition, then \( \sigma_\beta \) is \( \langle \tau \rangle \)-conjugate to the transposition \( (0, \frac{n}{2}) \).

**Proof.** On conjugating by an appropriate power of \( \sigma_\tau \), we may assume \( \sigma_\beta = (0, j) \). The conjugate of \( \sigma_\beta \) by \( \sigma_i^j \) is the transposition \( (i, j+i) \). In particular, \( (j, 2j) \) is a conjugate which is supposed to commute with \( (0, j) \). Therefore, \( \{0, j\} = \{j, 2j\}, \ 2j = 0 \mod(n), \ n = 2n' \) and \( j = n' \). \( \square \)
We go back to part (I) of the Proposition 4.5.

\[
(\tau^v|_{(i)\sigma^{-v}})^{-1}\left(\beta|_{(i)\sigma^{-v}}\right) \left(\tau^v|_{(i)\sigma^{-v}\sigma_{\beta}}\right) \left(\beta|_{(i)\sigma^{-v}\sigma_{\beta}\sigma_{\beta}}\right)
= (\beta|_{i}) \left(\tau^v|_{(i)\sigma_{\beta}\sigma^{-v}}\right)^{-1}\left(\beta|_{(i)\sigma_{\beta}\sigma^{-v}}\right) \left(\tau^v|_{(i)\sigma_{\beta}\sigma^{-v}\sigma_{\beta}}\right)
\]

and set in it \(j = (i)\sigma^{-v}, v = kn + r, r = v\) to obtain

\[
(\tau^v)|_{j}^{-1}\beta|_{j}(\tau^v)|_{(j)\sigma_{\beta}\beta|_{(j)\sigma_{\beta}\sigma^{-v}}}
= \beta|_{(j)\sigma_{v}(\tau^v)|_{(j)\sigma_{v}\sigma_{\beta}\sigma_{\beta\sigma_{\beta}}}^{-1}}\beta|_{(j)\sigma_{v}\sigma_{\beta}\sigma_{\beta}(\tau^v)|_{(j)\sigma_{v}\sigma_{\beta}\sigma_{\beta}}}.
\]

**Proposition 7.4.** The following cases hold for different pairs \((j, r)\).

- For \(j = 0\) there are 3 subcases
  - If \(r = 0\), then
    \[
    [\beta|_{0}, \tau^{k}]^{\beta|_{0}} = [\beta|_{0}, \tau^{k}], \forall k \in \mathbb{Z};
    \]
  - If \(r = \frac{n}{2}\), then
    \[
    \beta|_{0} \tau \beta|_{0} = \beta|_{\frac{n}{2}} \tau^{-1} \beta|_{\frac{n}{2}},
    \]
    and
    \[
    [\beta|_{0}, \tau^{k}]^{\tau(0)} = [\beta|_{0}, \tau^{k}], \forall k \in \mathbb{Z}.
    \]
  - If \(r \neq 0\) and \(r \neq \frac{n}{2}\), then
    \[
    \tau^{\delta(\frac{n}{2}, r)} \beta|_{0} \beta|_{(\frac{n}{2} + r)} = \beta|_{r} \tau^{\delta(\frac{n}{2}, r)} \beta|_{0}, \forall r \in Y - \{0, \frac{n}{2}\}
    \]
    and
    \[
    [\beta|_{0}, \tau^{k}]^{\beta|_{0}} = [\beta|_{0}, \tau^{k}], \forall k \in \mathbb{Z}.
    \]

- For \(j = \frac{n}{2}\) there are 3 subcases
  - If \(r = 0\), then
    \[
    [\beta|_{\frac{n}{2}}, \tau^{k}]^{\beta|_{0}} = [\beta|_{0}, \tau^{k}], \forall k \in \mathbb{Z};
    \]
  - If \(r = \frac{n}{2}\), then
    \[
    \tau^{-1} \beta|_{\frac{n}{2}} \tau = \beta|_{\frac{n}{2}} \tau,
    \]
    and
    \[
    [\beta|_{\frac{n}{2}}, \tau^{k}]^{\beta|_{0}} = [\beta|_{0}, \tau^{k}], \forall k \in \mathbb{Z};
    \]
– If \( r \neq 0 \) and \( r \neq \frac{n}{2} \), then
\[
\tau^{-\delta(\frac{n}{2}, r)\beta\beta}|_r = \beta|_{\frac{n}{2} + r}, \forall r \in Y - \{0, \frac{n}{2}\}
\]
and
\[
[\beta|_{\frac{n}{2}}, \tau^k]^{\beta\beta}|_r = [\beta|_{\frac{n}{2}}, \tau^k], \forall k \in \mathbb{Z}, \forall r \in Y - \{0, \frac{n}{2}\}. \tag{7.18}
\]

• For \( j \neq 0 \) and \( j \neq \frac{n}{2} \), there are 5 subcases:

  – If \( j \neq n - r \) and \( j \neq \frac{n}{2} - r \), then, by substitution \( t = j + r \), we have
\[
\beta|_j \beta_t = \beta|_t \beta_j, \forall j, t \in Y - \{0, \frac{n}{2}\} \tag{7.19}
\]
and
\[
[\beta|_j, \tau^k]^{\beta\beta}|_t = [\beta|_j, \tau^k], \forall j, t \in Y - \{0, \frac{n}{2}\} \tag{7.20}
\]

  – If \( j = n - r \) and \( 0 < r < \frac{n}{2} \), then, by substitution \( t = j - \frac{n}{2} \), we have
\[
\tau^{-1}\beta|_{t + \frac{n}{2}} \tau \beta|_0 = \beta|_0 \beta|_{t + \frac{n}{2}}, \forall t \in \{1, 2, \ldots, \frac{n}{2} - 1\} \tag{7.21}
\]
and
\[
[\beta|_{t + \frac{n}{2}}, \tau^k]^{\tau \beta|_o} = [\beta|_t, \tau^k], \forall j \in \{1, 2, \ldots, \frac{n}{2} - 1\} \tag{7.22}
\]

  – If \( j = n - r \) and \( \frac{n}{2} < r \leq n - 1 \), then
\[
\beta|_j \beta|_0 = \beta|_0 \beta|_{\frac{n}{2} + j}, \forall j \in \{1, \ldots, \frac{n}{2} - 1\} \tag{7.23}
\]
and
\[
[\beta|_j, \tau^k]^{\beta\beta}|_0 = [\beta|_{\frac{n}{2} + j}, \tau^k], \forall k \in \mathbb{Z}, \forall j \in \{1, \ldots, \frac{n}{2} - 1\} \tag{7.24}
\]

  – If \( j = \frac{n}{2} - r \) and \( 0 < r < \frac{n}{2} \), then
\[
\beta|_{\frac{n}{2}} \beta|_j = \beta|_{\frac{n}{2} + j} \tau^{-1}\beta|_{j + \frac{n}{2} \tau}, \forall j \in \{1, \ldots, \frac{n}{2} - 1\} \tag{7.25}
\]
and
\[
[\beta|_{\frac{n}{2}}, \tau^k]^{\beta\beta}|_{\frac{n}{2}} \tau^{-1} = [\beta|_{\frac{n}{2} + j}, \tau^k], \forall k \in \mathbb{Z}, \forall j \in \{1, \ldots, \frac{n}{2} - 1\} \tag{7.26}
\]

  – If \( j = \frac{n}{2} - r \) and \( \frac{n}{2} < r \leq n - 1 \), then
\[
\beta|_{\frac{n}{2}} \beta|_j = \beta|_{\frac{n}{2} + j} \beta|_{\frac{n}{2}}, \forall j \in \{1, \ldots, \frac{n}{2} - 1\} \tag{7.27}
\]
and
\[
[\beta|_{\frac{n}{2}}, \tau^k] = [\beta|_{\frac{n}{2} + j}, \tau^k]^{\beta\beta}|_{\frac{n}{2}}, \forall k \in \mathbb{Z}, \forall j \in \{1, \ldots, \frac{n}{2} - 1\} \tag{7.28}
\]
Proof. We will prove (7.19) and (7.20). As $j \notin \{0, \frac{n}{2}, n - r, \frac{n}{2} - r\}$, we have

\[
(j) \sigma^v \tau = (j) \sigma \sigma^v \tau = j + r,
\]

\[
(j) \beta = (j) \sigma^v \sigma^{-v} \beta = (j) \sigma^v \sigma^v \sigma^{-v} \beta = j.
\]

Therefore,

\[
\left( (\tau^v)^{-1} \beta_{j}(\tau^v) \right)^{-1} \beta_{j+r} = \beta_{j+r}(\tau^v)^{-1} \beta_{j}(\tau^v), \forall \nu \in \mathbb{Z}
\]

\[
\iff \left( \tau^{-k-\delta(j,r)} \beta_{j+r} = \beta_{j+r} \tau^{-k-\delta(j,r)} \beta_{j+r}, \forall k \in \mathbb{Z} \right)
\]

\[
\iff \left( \beta_{j}[\beta_{j}, \tau^{k+\delta(j,r)}] \beta_{j+r} = \beta_{j+r} \beta_{j}[\beta_{j}, \tau^{k+\delta(j,r)}], \forall k \in \mathbb{Z} \right)
\]

(7.29)

\[
\beta_{j} \beta_{t} = \beta_{t} \beta_{j}, \forall j, t \in Y - \left\{0, \frac{n}{2} \right\}
\]

and

(7.30)

\[
[\beta_{j}, \tau^{k}]^{\beta_{t}} = [\beta_{j}, \tau^{k}], \forall j, t \in Y - \left\{0, \frac{n}{2} \right\}.
\]

\[
\square
\]

Lemma 7.5. The group $N = \langle [\beta_{i}, \tau^{k}] | k \in \mathbb{Z}, i \in Y \rangle$ is an abelian normal subgroup of $H$.

Proof. Define

\[
N_{i} = \langle [\beta_{i}, \tau^{k}] | k \in \mathbb{Z} \rangle
\]

for each $i \in Y$. Then, $N = \langle N_{i} | i \in Y \rangle$, each $N_{i}$ is an abelian subgroup normalized by $\tau$ and

(7.31)

\[
[\beta_{i}, \tau^{k}]^{\beta_{j}} = [\beta_{i}, \tau^{k}], \forall k \in \mathbb{Z}, \forall i, j \in Y, j \neq 0, \frac{n}{2}
\]

We have $[N_{i}, N_{j}] = 1$, $\forall i, j \in Y, j \neq 0, \frac{n}{2}$, because

\[
[\beta_{i}, \tau^{k}][\beta_{j}, \tau^{t}] = [\beta_{i}, \tau^{k}][\beta_{j}^{-1} \tau^{-t} \beta_{j} \tau^{t}] \overset{(7.31)}{=} [\beta_{i}, \tau^{k}]^{\tau^{-t} \beta_{j} \tau^{t}} \overset{(1.1)}{=} ([\beta_{i}, \tau^{-t} \beta_{j} \tau^{t}] )^{\beta_{j} \tau^{t}} \overset{(7.31)}{=} \left( [\beta_{i}, \tau^{-t}]^{-1} [\beta_{i}, \tau^{k-t}] \right)^{\tau^{t}}
\]

\[
[\beta_{i}, \tau^{k}]^{\tau^{-t} \beta_{j} \tau^{t}} = [\beta_{i}, \tau^{k}], \forall k, t \in \mathbb{Z}, \forall i, j \in Y, j \neq 0, \frac{n}{2}
\]

Furthermore, $[N_{0}, N_{\frac{n}{2}}] = 1$, because

\[
[\beta_{\frac{n}{2}}, \tau^{k}]^{\beta_{0} \tau^{t}} = [\beta_{\frac{n}{2}}, \tau^{k}][\beta_{0}^{-1} \tau^{-t} \beta_{0} \tau^{t}] \overset{(1.1)}{=} [\beta_{\frac{n}{2}}, \tau^{k}]^{\tau^{-t} \beta_{0} \tau^{t}} \overset{(7.31)}{=} \left( [\beta_{0}, \tau^{-t}]^{-1} [\beta_{0}, \tau^{k-t}] \right)^{\tau \beta_{0} \tau^{t}} \overset{(7.31)}{=} \left( [\beta_{\frac{n}{2}}, \tau^{-t}]^{-1} [\beta_{\frac{n}{2}}, \tau^{k-t}] \right)^{\tau^{t}}
\]
\[ (1.1) [\beta|_{\frac{n}{2}}, \tau^k]^{\tau^{-t}\tau^t} = [\beta|_{\frac{n}{2}}, \tau^k], \forall k, t \in \mathbb{Z}. \]

Therefore \( N \) is abelian.

Now, equation (7.31) implies
\[
(7.32) \quad N_i = N_i^{\beta|_j} = N_i^{\beta|^{-1}_j}, \forall i, j \in Y, j \neq 0, \frac{n}{2};
\]
equation (7.11) implies
\[
(7.33) \quad \begin{cases} N_{\frac{n}{2}} = N_0^{\beta|_{\frac{n}{2}}}, N_0 = N_0^{\beta|^{-1}_{\frac{n}{2}}}; \end{cases}
\]
equations (4.1), (7.11) imply
\[
(7.34) \quad \begin{cases} N_{\frac{n}{2}} = N_0^{\beta|_0}, N_0 = N_0^{\beta|^{-1}_0}; \end{cases}
\]
equation (7.14) implies
\[
(7.35) \quad \begin{cases} N_0 = N_0^{\beta|_0}, N_{\frac{n}{2}} = N_0^{\beta|^{-1}_{\frac{n}{2}}}; \end{cases}
\]
equations (4.1), (7.16) imply
\[
(7.36) \quad \begin{cases} N_0 = N_0^{\beta|_{\frac{n}{2}}}, N_{\frac{n}{2}} = N_0^{\beta|^{-1}_{\frac{n}{2}}}; \end{cases}
\]
equations (4.1), (7.22) imply
\[
(7.37) \quad \begin{cases} N_j = N_j^{\beta|_0}, N_j + \frac{n}{2} = N_j^{\beta|^{-1}_0}, \forall j \in \{1, \ldots, \frac{n}{2} - 1\}; \end{cases}
\]
equation (7.24) implies
\[
(7.38) \quad \begin{cases} N_{j + \frac{n}{2}} = N_j^{\beta|_0}, N_j = N_j^{\beta|^{-1}_0}, \forall j \in \{1, \ldots, \frac{n}{2} - 1\}; \end{cases}
\]
equations (4.1) and (7.26) imply
\[
(7.39) \quad \begin{cases} N_{j + \frac{n}{2}} = N_j^{\beta|_{\frac{n}{2}}}, N_j = N_j^{\beta|^{-1}_{\frac{n}{2}}}, \forall j \in \{1, \ldots, \frac{n}{2} - 1\}; \end{cases}
\]
equation (7.28) implies
\[
(7.40) \quad \begin{cases} N_j = N_j^{\beta|_{\frac{n}{2}}}, N_j + \frac{n}{2} = N_j^{\beta|^{-1}_{\frac{n}{2}}}, \forall j \in \{1, \ldots, \frac{n}{2} - 1\}. \end{cases}
\]
Thus (7.31)-(7.40) prove
\[
N = \langle N_i \mid i \in Y \rangle = \langle [\beta|_i, \tau^k] \mid \forall i, k \in \mathbb{Z} \rangle
\]
is an abelian normal subgroup of \( H \).

\[ \square \]

**Lemma 7.6.** The group \( U = \langle N, \beta|_j \mid j \neq 0, \frac{n}{2} \rangle \) is a normal abelian subgroup of \( H \).
Proof. Lemma 7.5 and equations (7.13), (7.18), (7.19) and (7.20) show that $U$ is abelian.

The fact that $N$ is normal in $H$, together with the following assertions proves that $U$ is normal in $H$.

Let $J = \langle \beta_0, \beta_{\frac{n}{2}}, \tau \rangle$. Then, for $j \in Y - \{0, \frac{n}{2}\}$, we have

(I) $\langle \beta|_j \rangle^J \leq U$:

\[
\begin{align*}
\beta|_j^{\tau^I} &= \beta|_j[\beta|_j, \tau^I]; \\
\beta|_j^{\beta|_0} &= \beta|_{j+\frac{n}{2}}; \\
\beta|_j^{\beta|_0^{-1}} &= \beta|_{j+\frac{n}{2}} \tau^{-1} \beta|_{j+\frac{n}{2}} \tau = \beta|_{j+\frac{n}{2}}[\beta|_{j+\frac{n}{2}}, \tau]; \\
\beta|_j^{\beta|_j} &= \beta|_{j+\frac{n}{2}} \tau^{-1} \beta|_{j+\frac{n}{2}} \tau = \beta|_{j+\frac{n}{2}}[\beta|_{j+\frac{n}{2}}, \tau]; \\
\beta|_j^{\beta|_j^{-1}} &= \beta|_{j+\frac{n}{2}}.
\end{align*}
\]

(II) $\langle \beta|_{j+\frac{n}{2}} \rangle^J \leq U$:

\[
\begin{align*}
\beta|_{j+\frac{n}{2}}^{\tau^I} &= \beta|_{j+\frac{n}{2}}[\beta|_{j+\frac{n}{2}}, \tau^I]; \\
\beta|_{j+\frac{n}{2}}^{\beta|_0} &= \beta|_{j+\frac{n}{2}}[\beta|_{j+\frac{n}{2}}, \tau^I]; \\
\beta|_{j+\frac{n}{2}}^{\beta|_0^{-1}} &= \beta|_{j+\frac{n}{2}} \tau^{-1} \beta|_{j+\frac{n}{2}} \tau = \beta|_{j+\frac{n}{2}}[\beta|_{j+\frac{n}{2}}, \tau]; \\
\beta|_{j+\frac{n}{2}}^{\beta|_j} &= \beta|_{j+\frac{n}{2}} \tau^{-1} \beta|_{j+\frac{n}{2}} \tau = \beta|_{j+\frac{n}{2}}[\beta|_{j+\frac{n}{2}}, \tau]; \\
\beta|_{j+\frac{n}{2}}^{\beta|_j^{-1}} &= \beta|_{j+\frac{n}{2}}.
\end{align*}
\]

Hence, $U$ is a normal abelian subgroup of $H$. \hfill $\Box$

Lemma 7.7. $V = \langle U, \beta|_{\frac{n}{2}} \beta|_0, \tau \beta|_0^2 \rangle$ is a normal abelian subgroup of $H$.

Proof. Lemma 7.6 together with the following assertions proves that $V$ is a normal abelian subgroup of $H$.

Given $j \in Y - \{0, \frac{n}{2}\}$, $k \in \mathbb{Z}$, and $J = \langle \beta|_0, \beta_{\frac{n}{2}}, \tau \rangle$, we prove

(1) $\beta|_{\frac{n}{2}} \beta|_0 \in C_H(U)$:

\[
\begin{align*}
(\beta|_j)^{\beta|_j \beta|_0} &= (\beta|_j + \frac{n}{2})^{\tau \beta|_0} \beta|_j; \\
(\beta|_{j+\frac{n}{2}})^{\beta|_{j+\frac{n}{2}} \beta|_0} &= (\beta|_{j+\frac{n}{2}})^{\tau \beta|_0} \beta|_{j+\frac{n}{2}}; \\
[\beta|_j, \tau^k]^{\beta|_j \beta|_0} &= [\beta|_j, \tau^k]^{\tau^{-1} \tau \beta|_0} \beta|_0 \beta|_{j+\frac{n}{2}, \tau^k} \tau \beta|_0 \\
&= \beta|_{j+\frac{n}{2}} \beta|_0.
\end{align*}
\]
\[
[\beta_{j+\frac{n}{2}}, \tau \beta]^{[\beta_{j+\frac{n}{2}}]} \beta_{\beta_{\beta}} \quad \text{(7.24)} \quad [\beta_{j}, \tau \beta][\beta_{\beta}] \quad \text{(7.24)} \quad [\beta_{j+\frac{n}{2}}, \tau \beta];
\]
\[
[\beta_{0}, \tau \beta]^{[\beta_{\beta}]} \quad \text{(7.14)} \quad [\beta_{\frac{n}{2}}, \tau \beta][\beta_{\beta}] \quad \text{(7.14)} \quad [\beta_{0}, \tau \beta];
\]
\[
[\beta_{\beta}, \tau \beta]^{[\beta_{\beta}]} = [\beta_{\beta}, \tau \beta]^{[\beta_{\beta}]} \tau^{-1} \tau \beta \quad \text{(7.11)} \quad [\beta_{\beta}, \tau \beta];
\]

(II) \(\tau \beta_{0} \in C_{H}(U)\):

\[
\beta_{j}^{[\beta_{j}]} = \beta_{j}^{[\beta_{j}]} \quad \text{(7.21)} \quad \beta_{j}^{[\beta_{j}] \beta_{\beta}} \quad \text{(7.21)} \quad \beta_{j}^{[\beta_{j}]};
\]
\[
(\beta_{j+\frac{n}{2}})^{[\beta_{0}, \tau \beta]} \beta_{j+\frac{n}{2}} \quad \beta_{j+\frac{n}{2}}^{[\beta_{j+\frac{n}{2}}]};
\]
\[
[\beta_{0}, \tau \beta]^{[\beta_{\beta}]} \quad \text{(7.11)} \quad [\beta_{\frac{n}{2}}, \tau \beta][\beta_{\beta}] \quad \text{(7.14)} \quad [\beta_{0}, \tau \beta];
\]
\[
[\beta_{\beta}, \tau \beta]^{[\beta_{\beta}]} \quad \text{(7.14)} \quad [\beta_{\beta}, \tau \beta]^{[\beta_{\beta}]} \quad \text{(7.14)} \quad [\beta_{\beta}, \tau \beta];
\]
\[
[\beta_{j}, \tau \beta]^{[\beta_{\beta}]} \quad \text{(7.24)} \quad [\beta_{j}, \tau \beta]^{[\beta_{\beta}]} \quad \text{(7.24)} \quad [\beta_{j}, \tau \beta];
\]

(III) \(\tau \beta_{0} \in C_{H}(\beta_{\beta})\):

\[
(\beta_{\beta}^{[\beta_{0}]} \beta_{0})^{[\beta_{0}]} = \beta_{0}^{-2} \tau^{-1} \beta_{0} \beta_{0}^{[\beta_{0}]} \quad \text{(7.15)} \quad \beta_{0}^{-2} \tau^{-1} \beta_{0} \beta_{0}^{-1} \beta_{0} \beta_{0}^{[\beta_{0}]} \beta_{0}
\]
\[
= \beta_{0}^{-2} \tau^{-1} \beta_{0} \beta_{0}^{-1} \beta_{0} \beta_{0}^{-1} \beta_{0} \beta_{0} = (\tau \beta_{0}^{[\beta_{0}]} \beta_{0}^{-1} \beta_{0} \beta_{0}^{-1} \beta_{0} \beta_{0}^{[\beta_{0}]} \beta_{0} \beta_{0}) \quad \text{(7.15)} \quad \beta_{0}^{-2} \tau^{-1} \beta_{0} \beta_{0}^{-1} \beta_{0} \beta_{0} \beta_{0}^{[\beta_{0}]} \beta_{0}
\]
\[
= \beta_{0}^{-2} \tau^{-1} \beta_{0} \beta_{0}^{-1} \beta_{0} \beta_{0} \beta_{0} = 1.
\]
(IV) $\langle \tau \beta |^2_o \rangle^J \leq V$:

$$(\tau \beta |^2_0)^{\tau^k} = \tau (\beta |^2_0)^{\tau^k} = \tau \beta |^2_o [\beta |^2_0, \tau^k] = \tau \beta |^2_o [\beta |^2_0, \tau^k]$$. 

$$(\tau \beta |^2_0)^{[\beta |^2_0]} = \beta |^2_0^{-1} \tau \beta |^2_0 \beta |^2_0 = \tau \beta |^2_0^{-1} \tau \beta |^2_0 \beta |^2_0 = \tau [\beta, \beta |^2_0]$$

$$(7.41) \quad (\tau \beta |^2_0)^{[\beta |^2_0]} = \beta |^2_0 [\beta, \beta |^2_0] = \tau \beta |^2_0 [\beta |^2_0, \tau^{\beta |^2_0}]$$

$$(\tau \beta |^2_0)^{[\beta |^2_0]} = \beta |^2_0 \tau, \beta |^2_0 = \tau \beta |^2_0 [\beta |^2_0, \tau^{\beta |^2_0}]$$

$$(\tau \beta |^2_0)^{\beta |^2_0} = (\tau \beta |^2_0)^{[\beta |^2_0]} [\beta |^2_0, \tau^{\beta |^2_0}]$$

(V) $\langle \beta |^2_0 \beta |^2_0 \rangle^J \leq V$:

$$(\beta |^2_0 \beta |^2_0)^{\tau^k} = \beta |^2_0 \beta |^2_0 [\beta |^2_0, \tau^k] = \beta |^2_0 \beta |^2_0 [\beta |^2_0, \tau^k]$$

$$(7.42) \quad (\beta |^2_0 \beta |^2_0)^{\beta |^2_0} = \beta |^2_0^{-1} \beta |^2_0 \beta |^2_0 = \beta |^2_0^{-1} \beta |^2_0^{-1} \tau \beta |^2_0$$

$$(7.43) \quad \beta |^2_0 \beta |^2_0 (7.42) (\beta |^2_0 \beta |^2_0)^{-1}$$

$$(7.44) \quad (\beta |^2_0 \beta |^2_0)^{\beta |^2_0} = (\beta |^2_0 \beta |^2_0)^{-1} (\beta |^2_0 \beta |^2_0)^{-1}$$
8. Solvable groups for $n = 4$

Let $B$ be an abelian subgroup of $A_4 = Aut(T_4)$ normalized by $\tau$ and let $\beta \in B$. Then, by Proposition 4.1, $\sigma_\beta \in D = \langle (0,1,2,3), (0,2) \rangle$, the unique Sylow 2-subgroup of $\Sigma_4$ which contains $\sigma = \sigma_\tau = (0,1,2,3)$.

The normalizer of $\langle \tau \rangle$ here is $\Gamma_0 = N_{A_4} \langle \langle \tau \rangle \rangle = \langle \Psi, \iota \rangle$ where $\Psi$ is the monic normalizer and where $\iota = \iota^{(1)}(0,3)(1,2)$ inverts $\tau$.

Given a group $W$, the subgroup generated by squares of its elements is denoted by $W^2$.

**Lemma 8.1.** Let $L = L(D)$ be the layer closure of $D$ above. If $\gamma \in L^2$ then $\gamma \tau$ is conjugate to $\tau$.

*Proof.* If $\alpha \in L$ then $\sigma_{\alpha^2} \in \langle \sigma^2 \rangle$ and the product in some order of the states $(\alpha^2)_{i\in\{0,1,2,3\}}$ belongs to $S = L^2$.

Let $\gamma \in S$. Then $\gamma \tau$ is transitive on the 1st level of the tree and $(\gamma \tau)^3$ is inactive with conjugate 1st level states, where the first state is

$$(\gamma|_0)(\gamma|_1)(\gamma|_2)(\gamma|_3)\tau \text{ if } \sigma_\gamma = e,$$

and

$$(\gamma|_0)(\gamma|_3)(\gamma|_2)(\gamma|_1)\tau \text{ if } \sigma_\gamma = \sigma^2,$$

in both cases the element is contained in $S\tau$. Therefore, $\gamma \tau$ is transitive on the 2nd level of the tree. Now use induction to prove that $\gamma \tau$ is transitive on all levels of the tree. As $\gamma \tau$ is transitive on all levels of the tree, then $\gamma \tau$ is conjugate to $\tau$. \hfill \Box

8.1. **Cases** $\sigma_\beta \in \{(0,3)(1,2), (0,1)(2,3)\}$. We will show that these cases cannot occur. We note that $\sigma_\tau$ conjugates $(0,1)(2,3)$ to $(0,3)(1,2)$. Since the argument for $\beta$ applies to $\beta^\tau$, it is sufficient to consider the first case.

Suppose $\sigma_\beta = (0,1)(2,3)$. Then,

$$\beta^\tau = (\tau^{-1}(\beta|_3), \beta|_0, \beta|_1, \beta|_2\tau)(\sigma_\beta)^{\sigma^\tau}.$$

On substituting $\alpha = \beta^\tau$ in $\theta = [\beta, \alpha]$ and in (2.5)

(8.1) \hspace{1cm} $\theta|_{(i)\sigma_\alpha} = (\beta|_{(i)\sigma_\alpha})^{-1} (\alpha|_{(i)})^{-1} (\alpha|_{(i)\sigma_\beta})$, $\forall i \in Y.$

we get $\theta = e$ and

(8.2) \hspace{1cm} $e = (\beta|_{(i)\sigma_\beta})^{-1} (\beta^\tau|_{(i)})^{-1} (\beta|_{(i)})^{-1} (\beta^\tau|_{(i)\sigma_\beta})$, $\forall i \in Y$

and so for the index $i = 0$, we obtain

$$e = (\beta|_3)^{-1} (\tau^{-1}(\beta|_3))^{-1} (\beta|_0)(\beta|_0),$$

$$e = (\beta|_3)^{-2} \tau (\beta|_0)^2$$

which is impossible.
8.2. Cases $\sigma_{\beta} \in \{(0, 2), (1, 3)\}$.

**Lemma 8.2.** Let $\alpha, \gamma \in \text{Aut}(T_4)$ be such that
\[
\sigma_{\alpha}, \sigma_{\gamma} \in \langle (0, 1, 2, 3), (0, 2) \rangle,
\]
\[
\tau^{-1}\alpha^2 = \gamma^2 \tau,
\]
\[
[\alpha, \tau^k] \gamma = [\gamma, \tau^k]
\]
for all $k \in \mathbb{Z}$. Then,
\[
\sigma_{\alpha}, \sigma_{\gamma} \in \langle \sigma \rangle, \quad \sigma_{\alpha} \sigma_{\gamma} = \sigma^{\pm 1}.
\]

**Proof.** From the second and third equations above, we have $\sigma_{\alpha}^2 \sigma_{\gamma} = \sigma_{\alpha}^2 \sigma$ and $[\sigma_{\alpha}, \sigma_{\gamma}^k]_{\gamma} = [\sigma_{\gamma}, \sigma_{\gamma}^k]$.

(i) Suppose $\sigma_{\gamma}^2 = e$. Then $\sigma_{\alpha}^2 = \sigma^2$ and therefore, $\sigma_{\alpha} = \sigma^{\pm 1}$, $[\sigma_{\alpha}, \sigma_{\gamma}^k]_{\gamma} = [\sigma_{\gamma}, \sigma_{\gamma}^k]$ for all $k$; thus, $\sigma_{\gamma} \in \langle \sigma \rangle$ and $\sigma_{\gamma} \in \langle \sigma^2 \rangle$, $\sigma_{\alpha} \sigma_{\gamma} = \sigma^{\pm 1}$ follows.

(ii) Suppose $o(\sigma_{\gamma}) = 4$. Then, $\sigma_{\gamma} = \sigma^{\pm 1}$ and $\sigma_{\alpha}^2 = e$. Since $[\sigma_{\alpha}, \sigma_{\gamma}^k]_{\gamma} = e$ for all $k$, we obtain $\sigma_{\alpha} \in \langle \sigma \rangle$, $\sigma_{\alpha}^2 = e$ and $\sigma_{\alpha} \in \langle \sigma^2 \rangle$. Therefore, $\sigma_{\alpha} \sigma_{\gamma} = \sigma^{\pm 1}$. □

(1) Suppose $\sigma_{\beta} = (0, 2)$. Then by the analysis in Section 7.2, we conclude
\[
V = \langle [\beta_i, \tau^k], \beta_1, \beta_3, \beta_2 \beta_0, \tau \beta_0^2 \mid i \in Y, k \in \mathbb{Z} \rangle
\]
is an abelian normal subgroup of $H$.

By Lemma 8.1, $\tau \beta_0^2 = \mu$ is a conjugate of $\tau$. As $V$ is abelian, there exist $\xi, t_1, t_2 \in \mathbb{Z}_4$ such that
\[
\mu = \tau \beta_0^2, \beta_2 = \mu \xi, \beta_1 = \mu t_1, \beta_3 = \mu t_2.
\]

Therefore,
\[
\beta_2 = \mu \xi^{-1} \beta_0^{-1}, \tau = \mu \beta_0^{-2}.
\]

On substituting $\gamma = \beta_0$ and $\alpha = \beta_2$ in (7.14) and (7.15), by Lemma 8.2 we obtain $\sigma_{\alpha} \gamma = \sigma_{\beta_2} \beta_0 = \sigma^{\pm 1}$. Thus, from $\beta_2 \beta_0 = \mu \xi$, we reach $\xi \in U(\mathbb{Z}_4)$.

By (7.15), we have
\[
\beta_2 \tau^{-1} = \tau \beta_0^2.
\]

It follows then that
\[
\mu \xi \beta_0^{-1} \mu \xi \beta_0^{-1} \beta_0^2 \mu^{-1} = \mu,
\]
\[
\left(\mu \xi \right)^{\beta_0} = \mu^{2-\xi}.
\]

Therefore,
\[
(8.3) \quad \mu \beta_0 = \mu \frac{2-\xi}{\xi}
\]
where $\frac{2-\xi}{\xi} \in \mathbb{Z}_4^1$.

By Equation (7.23) we have
\[
\beta_1 \beta_0 = \beta_3.
\]
It follows that

\[
(\mu t_1)^{\beta_0} = \mu t_2, \quad \mu t_1^{2-\xi \tau} = \mu t_2, \quad t_2 = t_1 \frac{2-\xi}{\xi}.
\]

We have reached the form of \(\beta\),

\[
\beta = (\beta_0, \mu t_1, \mu \xi \beta_0^{-1}, \mu t_1^{2-\xi \tau})(0, 2)
\]

where \(\mu = \tau^\alpha\) for some \(\alpha \in \text{Aut}(T_4)\).

Since \(\mu \beta_0 = \mu^{2-\xi \tau}\), we have \(\beta_0 = \left(\lambda_{2-\xi} \tau^m\right)^\alpha\) for some \(m \in \mathbb{Z}_4\).

Hence,

\[
\mu t_1 = (\tau t_1)^\alpha,
\]

\[
\mu \xi \beta_0^{-1} = \left(\tau \left(\lambda_{2-\xi} \tau^m\right)^{-1}\right)^\alpha
\]

\[
= \left(\lambda_{\frac{2-\xi}{\xi}} \tau^{(\xi-m)\frac{2-\xi}{\xi}}\right)^\alpha.
\]

Thus

\[
\beta = \left(\lambda_{2-\xi} \tau^m, \tau t_1, \lambda_{\frac{2-\xi}{\xi}} \tau^{(\xi-m)\frac{2-\xi}{\xi}}, \tau t_1^{2-\xi \tau}\right)^\alpha(0, 2)
\]

and

\[
\tau = \mu \beta_0^{-2}
\]

\[
= \left(\tau \left(\lambda_{2-\xi} \tau^m\right)^{-2}\right)^\alpha
\]

\[
= \left(\lambda_{\frac{2-\xi}{\xi}} \tau^{(1-2m)\frac{2-\xi}{\xi}}\right)^\alpha.
\]

We note that in case \(\xi = 1\), \(\beta\) has the form

\[
\beta = (\tau^m, \tau t_1, \tau^{1-m}, \tau t_1)^\alpha(0, 2)
\]

where \(\tau = (\tau^{1-2m})^\alpha\); therefore,

\[
\beta = (\tau^{\frac{m}{1-2m}}, \tau^{\frac{1-m}{1-2m}}, \tau^{\frac{1-m}{1-2m}}, \tau^{\frac{1-m}{1-2m}})(0, 2).
\]

(2) Suppose \(\sigma \beta = (1, 3)\). Then, \(\gamma = \beta^\tau\) satisfies \([\gamma, \gamma^k] = e\). Therefore, the previous case applies and

\[
\gamma = \left(\lambda_{\frac{2-\xi}{\xi}} \tau^m, \tau t_1, \lambda_{\frac{2-\xi}{\xi}} \tau^{(\xi-m)\frac{2-\xi}{\xi}}, \tau t_1^{2-\xi \tau}\right)^\alpha(0, 2),
\]

where

\[
\tau = \left(\lambda_{\frac{2-\xi}{\xi}} \tau^{(1-2m)(\frac{2-\xi}{\xi})^2}\right)^\alpha = (e, e, e, \left(\lambda_{\frac{2-\xi}{\xi}} \tau^{(1-2m)(\frac{2-\xi}{\xi})^2}\right)^\alpha_\tau)
\]

Hence, \(\beta\) has the form

\[
\beta = \gamma^{\tau^{-1}} = (\tau t_1, \lambda_{2-\xi} \tau^{1+m-\xi}, \tau t_1^{2-\xi \tau}, \lambda_{\frac{2-\xi}{\xi}} \tau^{(1-m)\frac{2-\xi}{\xi}})^\alpha(1, 3).
\]
8.3. **The case** \( \sigma_\beta = (\sigma_\tau)^2 = (0, 2) (1, 3) \). We know that

\[
V = \left\langle N, \beta_i | \beta_{i+2}, \beta_j^2 \tau^{-\Delta(j, j+2)} | i, j \in Y \text{ and } k \in \mathbb{Z} \right\rangle
\]

is an abelian normal subgroup of \( H \) and

\[
(8.4) \quad \tau^{\Delta(i, j)} \beta_{i+2} \beta_j^{\tau^{\Delta(i, j)}} = \beta_{j+2}^i,
\]

by analysis of the case [7.1]

From Lemmas 8.1 and 8.2, we have

\[
\tau \beta_0 = \mu, \quad \beta_2 = \mu \xi_0, \quad \beta_1 = \mu \xi_1, \quad \tau \beta_1^2 = \mu \xi_2
\]

where \( \mu = \tau^\alpha \) and \( \xi_0, \xi_1, \xi_2 \in U(\mathbb{Z}_4) \). Therefore,

\[
(8.5) \quad \tau = \mu \beta_0^{\alpha-2}
\]

\[
(8.6) \quad \beta_2 = \mu \xi_0 \beta_0^{\alpha-1}
\]

\[
(8.7) \quad \beta_1 = \mu \xi_1 \beta_1^{\alpha-1}
\]

\[
(8.8) \quad \tau = \mu \xi_2 \beta_1^{\alpha-2}.
\]

Now, we let \( i, j \) take their values from \( Y \) in (8.4). Note that \((i, j) \) and \((j, i)\) produce equivalent equations and the case where \( i = j \) is a tautology. Thus we have to treat the cases \((i, j) = (0, 1), (0, 2), (1, 3), (2, 3), (0, 3), (1, 2)\). Indeed, the last two cases turn out to be superfluous.

(i) Substitute \( i = 0, j = 2 \) in (8.4), to obtain

\[
(8.9) \quad \beta_2^2 \tau^{-1} = \tau \beta_0^2
\]

Use (8.5) and (8.6) in (8.9) to get

\[
\mu \xi_0 \beta_0^{\alpha-1} \mu \xi_0 \beta_0^{\alpha-1} \beta_0^{\alpha-2} \mu^{-1} = \mu
\]

and so,

\[
(\mu \xi_0)^{\beta_0} = \mu^{2-\xi_0}.
\]

Therefore,

\[
(8.10) \quad \mu^{\beta_0} = \mu^{2-\xi_0}
\]

Since \( \frac{2-\xi_0}{\xi_0} \in \mathbb{Z}_4 \), we find

\[
(8.11) \quad \beta_0 = \left( \lambda \frac{2-\xi_0}{\xi_0} \tau m_0 \right)^\alpha.
\]

From (8.6),
\[ (8.12) \quad \beta|_2 = \mu^{\xi_0} \beta|_0^{-1} = \left( \tau^{\xi_0 - m_0} \lambda \frac{\xi_0}{\xi_1} \right)^{\alpha} = \left( \lambda \frac{\xi_0}{\xi_1} \tau^{(\xi_0 - m_0) \frac{\xi_0}{\xi_1}} \right)^{\alpha}. \]

(ii) Substitute \( i = 1, j = 3 \) in \( (8.4) \) to get
\[ (8.13) \quad \beta|_3^2 \tau^{-1} = \tau \beta|_1^2. \]

On using \( (8.7) \) and \( (8.8) \) in \( (8.13) \), we obtain
\[ \mu^{\xi_1} \beta|_1^{-1} \mu^{\xi_1} \beta|_1^{-1} \beta|_2^{-1} \mu^{-\xi_2} = \mu^{\xi_2} \]
and so,
\[ (\mu^{\xi_1})\beta|_1 = \mu^{2\xi_2 - \xi_1}. \]

Therefore,
\[ (8.14) \quad \mu^\beta|_1 = \mu^{\frac{2\xi_2 - \xi_1}{\xi_1}}. \]

Since \( \frac{2\xi_2 - \xi_1}{\xi_1} \in \mathbb{Z}_4 \), we have
\[ (8.15) \quad \beta|_1 = \left( \lambda \frac{\xi_1}{2\xi_2 - \xi_1} \tau^{m_1} \right)^{\alpha}. \]

By \( (8.7) \), we find
\[ (8.16) \quad \beta|_3 = \mu^{\xi_1} \beta|_1^{-1} = \left( \tau^{\xi_1 - m_1} \lambda \frac{\xi_1}{2\xi_2 - \xi_1} \right)^{\alpha} = \left( \lambda \frac{\xi_1}{2\xi_2 - \xi_1} \tau^{(\xi_1 - m_1) \frac{\xi_1}{2\xi_2 - \xi_1}} \right)^{\alpha}. \]

(iii) Substitute \( i = 0, j = 1 \) in \( (8.4) \) to get
\[ (8.17) \quad \beta|_2 \beta|_1 = \beta|_3 \beta|_0. \]

Use \( (8.11), (8.12), (8.15) \) and \( (8.16) \) in \( (8.17) \), to obtain
\[ \lambda \frac{\xi_0}{2 \xi_0} \tau^{(\xi_0 - m_0) \frac{\xi_0}{2 \xi_0}} \lambda \frac{\xi_0}{2 \xi_0} \tau^{m_1} = \lambda \frac{\xi_0}{2 \xi_0} \tau^{(\xi_1 - m_1) \frac{\xi_0}{2 \xi_0}} \lambda \frac{\xi_0}{2 \xi_0} \tau^{m_0} \]
and so,
\[ \lambda \frac{\xi_0}{2 \xi_0} \tau^{(\xi_0 - m_0) \frac{\xi_0}{2 \xi_0}} = \lambda \frac{\xi_0}{2 \xi_0} \tau^{(\xi_1 - m_1) \frac{\xi_0}{2 \xi_0}} \tau^{\xi_0} \lambda \frac{\xi_0}{2 \xi_0} \tau^{m_0}. \]

Therefore,
\[ (8.18) \quad \left( \frac{\xi_1}{2 \xi_2 - \xi_1} \right)^2 = \left( \frac{\xi_0}{2 - \xi_0} \right)^2 \]
and
\[ (8.19) \quad (\xi_0 - m_0) \frac{\xi_0}{2 - \xi_0} \frac{2 \xi_2 - \xi_1}{\xi_1} + m_1 = (\xi_1 - m_1) \frac{\xi_1}{2 \xi_2 - \xi_1} \frac{2 - \xi_0}{\xi_0} + m_0. \]
(iv) Substitute $i = 2, j = 3$ in (8.4) to get

\[(8.20)\] \[\beta|_{0} = \beta|_{2}.\]

Use (8.11), (8.12), (8.15) and (8.16) in (8.20), to obtain

\[\lambda_{2-i_0} \tau^{m_0} \lambda_{\xi_{2-i_1}} \tau^{(\xi_{1-m_1})} \frac{\xi_{2-i_1}}{\xi_{2-i_1}} = \lambda_{2-i_2} \tau^{m_1} \lambda_{\xi_{2-i_0}} \tau^{(\xi_{0-m_0})} \frac{\xi_{2-i_0}}{\xi_{2-i_0}}\]

and so,

\[\lambda_{\xi_{2-i_0}} \tau^{m_0} \frac{\xi_{1}}{\xi_{1}} + (\xi_{1-m_1}) \frac{\xi_{1}}{\xi_{2-i_1}} = \lambda_{\xi_{2-i_0}} \tau^{m_1} \tau^{(\xi_{0-m_0})} \frac{\xi_{0}}{\xi_{2-i_0}}\]

Therefore,

\[
\left(\frac{\xi_{1}}{2\xi_{2} - \xi_{1}}\right)^2 = \left(\frac{\xi_{0}}{2 - \xi_{0}}\right)^2
\]

and

\[(8.21)\] \[m_0 \xi_{2-i_0} + (\xi_{1} - m_1) \frac{\xi_{1}}{2\xi_{2} - \xi_{1}} = m_1 \frac{\xi_{0}}{2 - \xi_{0}} + (\xi_{0} - m_0) \frac{\xi_{0}}{2 - \xi_{0}}.\]

We have from (8.18)

\[(8.22)\] \[\frac{\xi_{0}}{2 - \xi_{0}} = \pm \frac{\xi_{1}}{2\xi_{2} - \xi_{1}}.\]

(a) If

\[\frac{\xi_{0}}{2 - \xi_{0}} = \frac{\xi_{1}}{2\xi_{2} - \xi_{1}},\]

then

\[2\xi_{2} \xi_{0} - \xi_{1} \xi_{0} = 2\xi_{1} - \xi_{1} \xi_{0},\]

and so,

\[(8.23)\] \[\xi_{2} = \frac{\xi_{1}}{\xi_{0}}.\]

From (8.19), we get

\[(8.24)\] \[m_{1} = \frac{\xi_{1} - \xi_{0}}{2} + m_{0}.\]

(b) If

\[\frac{\xi_{0}}{2 - \xi_{0}} = - \frac{\xi_{1}}{2\xi_{2} - \xi_{1}},\]

then by (8.19) and (8.21),

\[m_{0} - \xi_{0} + m_{1} = m_{1} - \xi_{1} + m_{0}\]

\[m_{0} + \xi_{1} - m_{1} = -m_{1} - \xi_{0} + m_{0},\]

which implies $\xi_{1} = \xi_{0} = 0$, which is impossible.

Now by (8.23) and (8.24), we have

\[(8.25)\] \[\beta|_{1} = \left(\lambda_{2-i_{0}} \tau^{\xi_{1} - \xi_{0}} \frac{\xi_{1}}{2} + m_{0}\right)^{\alpha}\]

and

\[(8.26)\] \[\beta|_{3} = \left(\lambda_{\xi_{2-i_{0}}} \tau^{\xi_{1} - \xi_{0}} \frac{\xi_{1} + \xi_{0}}{2} - m_{0}\right)^{\alpha}.\]
Therefore,

$$\beta = (\beta |_0, \beta |_1, \beta |_2, \beta |_3)(0, 2)(1, 3)$$

where $\beta |_0, \beta |_1, \beta |_2$ and $\beta |_3$ are described in (8.11), (8.25), (8.12) and (8.26), respectively, and

$$\tau = \mu \beta |_0^{-2} = \left( \tau \left( \frac{1 - \xi_0}{\xi_0} \right) \right)^{\alpha} \left( \frac{\xi_0}{1 - \xi_0} \right)^{-\tau} \left( \frac{\xi_0}{1 - \xi_0} \right)^{\alpha}.$$  

(v) The cases $(i, j) = (1, 2), (0, 3)$ in (8.4) do not add any more information about $\beta$. Summarizing, we have found

$$\beta |_0 = \left( \frac{1 - \xi_0}{\xi_0} \right)^{\alpha}, \beta |_1 = \left( \frac{1 - \xi_0}{\xi_0} \right)^{\alpha}$$

$$\beta |_2 = \left( \frac{1 - \xi_0}{\xi_0} \right)^{\alpha}, \beta |_3 = \left( \frac{1 - \xi_0}{\xi_0} \right)^{\alpha}$$

(8.27)

$$\tau = \left( \frac{1 - \xi_0}{\xi_0} \right)^{\alpha} \left( \frac{\xi_0}{1 - \xi_0} \right)^{\alpha}.$$  

(8.28)

In the particular case where $\xi_0 = 1$, $\beta$ has the form

$$\beta = \left( \tau, \tau, \tau, \tau \right)(0, 2)(1, 3)$$

where $\tau = \left( \frac{1 - \xi_0}{\xi_0} \right)^{\alpha}$.

8.4. Cases $\sigma_\beta \in \{ e, \sigma_\tau, \sigma_\tau^{-1} \}$. (1) Suppose $\sigma_\beta = e$ and let $\beta$ stabilize the $k$th level of the tree. Then by Proposition 4.3, we have

$$[\beta |_u, \beta |_v^{\tau^k}] = e, \text{ for all } u, v \in M \text{ with } |u| = |v| = k.$$  

Therefore, $\hat{N} = \langle \beta |_w | |w| = k, w \in M \rangle$ is abelian and so is its normal closure $\hat{M}$ under $\langle \hat{N}, \tau \rangle$. Also, active elements in $\hat{M}$ are characterized in 8.1, 8.2, 8.3 and 8.4. In particular, there exists $\kappa \in \hat{M}$ such that $\sigma_\kappa = (0, 2)(1, 3)$ and $\beta \in \times_{\kappa} C(\kappa)$.

(2) Suppose $\sigma_\beta = \sigma_\tau = (0, 1, 2, 3)$. Then, clearly the element

$$\beta^2 = (\beta |_0 \beta |_1, \beta |_1 \beta |_2, \beta |_2 \beta |_3, \beta |_3 \beta |_0)(0, 2)(1, 3)$$

satisfies $[\beta^2, (\beta^2)^{\tau^k}] = e$ for all $k \in \mathbb{Z}_4$. Therefore, by the previous analysis, we have

$$\beta |_0 \beta |_1 = \left( \frac{1 - \xi_0}{\xi_0} \right)^{\alpha}.$$  

(8.30)
\[(8.31) \quad \beta|_1 \beta|_2 = \left( \frac{\xi_2 - \xi_0}{\xi_0} \right)^{\alpha}, \]
\[(8.32) \quad \beta|_2 \beta|_3 = \left( \frac{\xi_2 - \xi_0}{\xi_0} \right)^{\alpha}, \]
\[(8.33) \quad \beta|_3 \beta|_0 = \left( \frac{\xi_0 - \xi_0}{\xi_0} \right)^{\alpha}, \]
\[(8.34) \quad \tau = \left( \frac{\xi_0 - \xi_0}{\xi_0} \right)^{\alpha}. \]

Hence, multiplying (8.30) by (8.32), we obtain
\[(8.35) \quad \beta|_0 \beta|_1 \beta|_2 \beta|_3 = \left( \frac{\xi_0 - \xi_0}{\xi_0} \right)^{\alpha}. \]

We define
\[(8.36) \quad \psi_\eta = \begin{cases} 
\lambda_\eta, & \text{if } \eta \in \mathbb{Z}_4 \\
\theta \lambda_\eta, & \text{if } -\eta \in \mathbb{Z}_4
\end{cases}, \]
\[\theta = \theta^{(1)}(e, \tau^{-1}, \tau^{-1}, \tau^{-1})(1, 3) \]
(an inverter of \(\tau\)) and \(\gamma = (e, (\beta|_0)^{-1}, (\beta|_2)^{-1}, (\beta|_0 \beta|_1 \beta|_2)^{-1}) \left( \alpha^{-1} \psi_{2-\xi_0} \right)^{(1)} \).

We verify, by (8.35), that \(\gamma\) conjugates \(\beta\) to
\[(e, e, e, (\beta|_0 \beta|_1 \beta|_2 \beta|_3)^{(1)}) \left( \alpha^{-1} \psi_{2-\xi_0} \right)^{(1)}\]
which is equal to \(\tau\).

(3) Suppose \(\sigma_\beta = \sigma_\tau^{-1} = (0, 3, 2, 1)\). Then, \(\beta^{-1}\) satisfies the previous case and \(\beta^{-1} = \tau^\gamma\) for some \(\gamma \in \mathcal{A}_4\). Therefore, as \(\theta\) inverts \(\tau\), we have
\[(8.37) \quad \beta = (\beta^{-1})^{-1} = (\tau)^{-1} = (\tau)^{\theta_1}\]

8.5. **Final Step.** We finish the proof of the second part of Theorem A. For the case where the activity of \(\beta\) is a 4-cycle, we use the fact that \(\beta^2 \in B\), which we have already described. Next, from the description of the centralizer of \(\beta^2\), we are able to pin down the form of \(\beta\).

**Proposition 8.3.** Let \(\beta = (\beta|_0, \beta|_1, \beta|_2, \beta|_3)(0, 2)(1, 3)\) be such that \((\beta|_0)(\beta|_2) = \tau^{\theta_1}\) and \((\beta|_1)(\beta|_3) = \tau^{\theta_2}\), for some \(\theta_1, \theta_2 \in \text{Aut}(T_4)\). Then, \(\beta\) is conjugate to \(\tau^2\).
Proof. Let $\alpha = (e, e, \beta|_{0}^{-1}, \beta|_{1}^{-1})$. Then,

\begin{equation}
\beta^\alpha = (e, e, \beta|_{0}\beta|_{2}, \beta|_{1}\beta|_{3})(0, 2)(1, 3).
\end{equation}

Therefore, substituting $\beta|_{0}\beta|_{2} = \tau^{\theta_1}$ and $\beta|_{1}\beta|_{3} = \tau^{\theta_2}$ in the above equation, we have

\[ \beta^\alpha = (e, e, \tau^{\theta_1}, \tau^{\theta_2})(0, 2)(1, 3). \]

Conjugating $\beta^\alpha$ by $\gamma = (\theta_1^{-1}, \theta_2^{-1}, \theta_1^{-1}, \theta_2^{-1})$ we produce

\[ \beta^\alpha\gamma = \tau^2. \]

We show below that active elements of $B$ produce within $B$ elements conjugate to $\tau^2$.

**Proposition 8.4.** Let $\beta \in B$ with nontrivial $\sigma_\beta$. Then

(i) If $\sigma_\beta = \sigma_\tau^2$, then $\beta$ is a conjugate of $\tau^2$.

(ii) If $\sigma_\beta \in \{0, 2\}$, then $\beta\tau$ is a conjugate $\tau^2$.

(iii) If $\sigma_\beta \in \{\sigma_\tau, \sigma_\tau^{-1}\}$, then $\beta^2$ is a conjugate of $\tau^2$.

**Proof.** It is enough to prove (i), since (ii), (iii) are just special cases.

If $\sigma_\beta = \sigma_\tau^2$, then

\begin{equation}
\beta|_{0} = \left(\lambda_{2-\xi_0} \tau^{m_0}\right)^\alpha, \beta|_{1} = \left(\lambda_{2-\xi_0} \tau \frac{\xi_{1-\xi_0} + m_0}{2-\xi_0}\right)^\alpha,
\end{equation}

\begin{equation}
\beta|_{2} = \left(\lambda_{2-\xi_0} \tau \frac{(\xi_{0-m_0}) \xi_{0}}{2-\xi_0}\right)^\alpha, \beta|_{3} = \left(\lambda_{2-\xi_0} \tau \frac{\xi_{1+m_0} - m_0}{2-\xi_0}\right)^\alpha,
\end{equation}

\begin{equation}
\tau = \left(\lambda_{2-\xi_0} \tau \frac{(1-2m_0)}{2-\xi_0} \frac{\xi_0}{2-\xi_0}\right)^2,
\end{equation}

where $\xi_0, \xi_1 \in U(\mathbb{Z}_4)$, $m_0 \in \mathbb{Z}_4$.

Therefore,

\[ \beta|_{0}\beta|_{2} = \left(\lambda_{2-\xi_0} \tau^{m_0} \lambda_{2-\xi_0} \tau \frac{(\xi_{0-m_0}) \xi_{0}}{2-\xi_0}\right)^\alpha = \left(\lambda_{2-\xi_0} \tau\right)^\alpha = \tau \left(\lambda_{2-\xi_0} \tau\right)^\alpha = \tau \left(\psi_{2-\xi_0}\right)^\alpha \]

\[ \beta|_{1}\beta|_{3} = \left(\lambda_{2-\xi_0} \tau \frac{\xi_{1-\xi_0} + m_0}{2-\xi_0} \lambda_{2-\xi_0} \tau \frac{\xi_{1+m_0} - m_0}{2-\xi_0}\right)^\alpha = \left(\lambda_{2-\xi_0} \tau\right)^\alpha = \tau \left(\psi_{2-\xi_0}\right)^\alpha \]

It follows from Proposition \[8.3\] that $\beta$ is a conjugate of $\tau^2$.  

**Corollary 8.5.** Suppose $\beta \in B$ is an active element. Then, $B$ is conjugate to a subgroup of the centralizer $C(\tau^2)$. 

Proposition 8.6. Let $\gamma \in C(\tau^2)$. Then,
\begin{equation}
\gamma = (\tau^{m_0}, \tau^{m_1}, \tau^{m_0+\delta((0)\sigma_\gamma, 2)}, \tau^{m_1+\delta((1)\sigma_\gamma, 2)})\sigma_\gamma,
\end{equation}
where $m_0, m_1 \in \mathbb{Z}_4, \sigma_\gamma \in C_{\Sigma_4}(\sigma^2)$.

Proof. Write $\gamma = (\gamma_0, \gamma_1, \gamma_2, \gamma_3)\sigma_\gamma$. Then $\tau^2 \gamma = \gamma \tau^2$ translates to
\begin{equation}
(e, e, \tau, \tau)(0, 2)(1, 3)(\gamma_0, \gamma_1, \gamma_2, \gamma_3)\sigma_\gamma
= (\gamma_0, \gamma_1, \gamma_2, \gamma_3)\sigma_\gamma(e, e, \tau, \tau)(0, 2)(1, 3),
\end{equation}
and this in turn translates to
\begin{align*}
(\gamma_2, \gamma_3, \tau \gamma_0, \tau \gamma_1)(0, 2)(1, 3)\sigma_\gamma
&= (\gamma_0, \gamma_1, \gamma_2, \gamma_3)(\tau^{\delta((0)\sigma_\gamma, 2)}, \tau^{\delta((1)\sigma_\gamma, 2)}, \tau^{\delta((2)\sigma_\gamma, 2)}, \tau^{\delta((3)\sigma_\gamma, 2)})\sigma_\gamma(0, 2)(1, 3) \\
&= (\gamma_0 \tau^{\delta((0)\sigma_\gamma, 2)}, \gamma_1 \tau^{\delta((1)\sigma_\gamma, 2)}, \gamma_2 \tau^{\delta((2)\sigma_\gamma, 2)}, \gamma_3 \tau^{\delta((3)\sigma_\gamma, 2)})\sigma_\gamma(0, 2)(1, 3)
\end{align*}
Thus, we have
\begin{align*}
\gamma_2 &= \gamma_0 \tau^{\delta((0)\sigma_\gamma, 2)} \\
\gamma_3 &= \gamma_1 \tau^{\delta((1)\sigma_\gamma, 2)} \\
\tau \gamma_0 &= \gamma_2 \tau^{\delta((2)\sigma_\gamma, 2)} \\
\tau \gamma_1 &= \gamma_3 \tau^{\delta((3)\sigma_\gamma, 2)}.
\end{align*}
Hence,
\begin{align*}
\gamma_2 &= \gamma_0 \tau^{\delta((0)\sigma_\gamma, 2)} \\
\gamma_3 &= \gamma_1 \tau^{\delta((1)\sigma_\gamma, 2)} \\
\tau \gamma_0 &= \tau^{\delta((0)\sigma_\gamma, 2)+\delta((2)\sigma_\gamma, 2)} = \tau, \tau \gamma_1 &= \tau^{\delta((1)\sigma_\gamma, 2)+\delta((3)\sigma_\gamma, 2)} = \tau.
\end{align*}
Therefore, there exist $m_0, m_1 \in \mathbb{Z}_4$ such that
\begin{align*}
\gamma_0 &= \tau^{m_0}, \gamma_1 = \tau^{m_1}, \\
\gamma_2 &= \tau^{m_0+\delta((0)\sigma_\gamma, 2)}, \gamma_3 = \tau^{m_1+\delta((1)\sigma_\gamma, 2)}.
\end{align*}
Hence, $\gamma$ has the form
\begin{equation}
\gamma = (\tau^{m_0}, \tau^{m_1}, \tau^{m_0+\delta((0)\sigma_\gamma, 2)}, \tau^{m_1+\delta((1)\sigma_\gamma, 2)})\sigma_\gamma,
\end{equation}
where $\sigma_\gamma \in C_{\Sigma_4}(\sigma^2)$. 

Corollary 8.7. The centralizer of $\tau^2$ in $A_4$ is
\begin{equation*}
C(\tau^2) = \langle (e, e, \tau, \tau)(0, 2), \tau, (\tau^{m_0}, \tau^{m_1}, \tau^{m_0}, \tau^{m_1}) | m_0, m_1 \in \mathbb{Z}_4 \rangle.
\end{equation*}

Corollary 8.8. Let $\gamma \in C(\tau^2)$ be such that $\sigma_\gamma \in \langle (0, 2)(1, 3) \rangle$. Then
\begin{equation*}
\gamma \in \langle (\tau^{m_0}, \tau^{m_1}, \tau^{m_0}, \tau^{m_1}), \tau^2 | m_0, m_1 \in \mathbb{Z}_4 \rangle.
\end{equation*}
Proposition 8.9. Let $\hat{H} = \langle (\tau^{m_0}, \tau^{m_1}, \tau^{m_0}, \tau^{m_1}), \tau^2 \mid m_0, m_1 \in \mathbb{Z}_4 \rangle$. Then the normalizer $N_{A_4}(\hat{H})$ is the group
\[
\langle C(\tau^2), (\psi_{2m_0+1}, \psi_{2m_1+1}, \psi_{2m_0+1}\tau^{m_0}, \psi_{2m_1+1}\tau^{m_1}) \mid m_0, m_1 \in \mathbb{Z}_4 \rangle,
\]
where, for each $\eta \in U(\mathbb{Z}_4)$, $\psi_\eta$ is defined by \[8.36\] and
\[
\tau^{\psi_\eta} = \tau^\eta.
\]

Proof. As
\[
(8.44) \quad \alpha = (\psi_{2m_0+1}\psi_{2m_1+1}, \psi_{2m_0+1}\tau^{m_0}, \psi_{2m_1+1}\tau^{m_1}),
\]
conjugates $\tau^2$ to
\[
(\tau^{m_0}, \tau^{m_1}, \tau^{m_0+1}, \tau^{m_1+1})(0, 2)(1, 3),
\]
where $m_0, m_1 \in \mathbb{Z}_4$, and any other element in $N_{A_4}(\hat{H})$ is equal to an element in $C(\tau^2)$ times an element of the form \[8.44\], then $N_{A_4}(\hat{H})$ is the desired subgroup. \hfill $\Box$

Theorem 8.10. Let $G$ be a solvable subgroup of $\text{Aut}(T_4)$ which contains $\tau$. Then, $G$ is a subgroup of
\[
\times_4 \cdots (\times_4 (\times_4 T^\alpha \times S_4) \times S_4) \cdots) \times S_4
\]
for some $\alpha \in A_4$, where $T$ is the normalizer in $A_4$ of $C(\tau^2)$.

Proof. As in the case $n = p$, we assume $G$ has derived length $d \geq 2$ and let $B$ be the $(d - 1)$th term of the derived series of $G$. Then, $B$ is an abelian group normalized by $\tau$. On analyzing the case \[8.4\] and the final step, there exists a level $t$ such that $B$ is a subgroup of $\hat{V} = \times_4 C(\mu^2)$, where $\mu = \tau^\alpha$ for some $\alpha \in A_4$ and where $\sigma_{\mu^2} = (0, 2)(1, 3)$. There also exists $\beta \in B$ such that $\beta|_u = \mu^2$ for some index $u \in \mathcal{M}$.

Moreover, if $T$ is the normalizer of $C(\tau^2)$, then clearly, $T^\alpha$ is the normalizer of $C(\mu^2)$.

We will show now that $G$ is a subgroup of
\[
\hat{J} = \times_4 \cdots (\times_4 (\times_4 T^\alpha \times S_4) \times S_4) \cdots) \times S_4
\]
where the cartesian product $\times_4$ appears $t$ times.

Let $\gamma \not\in \hat{J}$. Since $\gamma \not\in \hat{J}$, there exists $w \in \mathcal{M}$ having $|w| = t$ and $\gamma|_w \not\in T^\alpha$. Since $\tau$ is transitive on all levels of the tree, by Corollary \[8.8\] we can conjugate $\beta$ by an appropriate power of $\tau$ to get $\theta \in B$ such that
\[
\theta|_w = \mu^2 \text{ or } \theta|_w = (\mu^2)^\tau = ((\tau^{m_0}, \tau^{m_1}, \tau^{m_0+1}, \tau^{m_1+1})(0, 2)(1, 3))^\alpha,
\]
where $m_0, m_1 \in \mathbb{Z}_4$. Thus, for $v = w^\gamma$ we have
\[
(\theta^\gamma)|_v \not\in \hat{V} \text{ and } \gamma|_w \not\in C(\mu^2)
\]
which implies $\theta^\gamma \not\in B \leq \hat{V}$ and $\gamma \not\in G$. Hence, $G$ is a subgroup of $\hat{J}$. \hfill $\Box$
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