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GROUPS WITH MINIMAX COMMUTATOR SUBGROUP

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Abstract. A result of Dixon, Evans and Smith shows that if G is a locally (soluble-by-finite) group whose proper subgroups are (finite rank)-by-abelian, then G itself has this property, i.e. the commutator subgroup of G has finite rank. It is proved here that if G is a locally (soluble-by-finite) group whose proper subgroups have minimax commutator subgroup, then also the commutator subgroup G' of G is minimax. A corresponding result is proved for groups in which the commutator subgroup of every proper subgroup has finite torsion-free rank.

1. Introduction

A group G is said to have *finite rank* r if every finitely generated subgroup of G can be generated by at most r elements, and r is the least positive integer with such property. It follows from results of V.V. Belyaev and N.F. Sesekin [1] that if G is any (generalized) soluble group of infinite rank whose proper subgroups are finite-by-abelian, then also the commutator subgroup G' of G is finite. More recently, M.R. Dixon, M.J. Evans and H. Smith [4] proved that if G is a locally (soluble-by-finite) group in which the commutator subgroup of every proper subgroup has finite rank, then G' has likewise finite rank.

The aim of this paper is to give a further contribution to this topic, investigating the structure of groups whose proper subgroups have minimax commutator subgroup. Recall that a group G is called *minimax* if it has a series of finite length in which every factor satisfies either the minimal or the maximal condition on subgroups. Obviously, locally (soluble-by-finite) groups satisfying the

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maximal condition on subgroups are polycyclic-by-finite, and it is also well-known that all locally finite groups satisfying the minimal condition on subgroups are Černikov groups (see [10] Part 1, p.98). It follows that every minimax locally (soluble-by-finite) group G is soluble-by-finite and has finite rank. Moreover, the finite residual J of G is a direct product of finitely many Prüfer subgroups, the Fitting subgroup F/J of G/J is nilpotent with finite torsion subgroup and G/F is finitely generated and abelian-by-finite (see [10] Part 2, Theorem 10.33). Our main result is the following theorem.

Theorem A *Let G be a locally (soluble-by-finite) group whose proper subgroups have minimax commutator subgroup. Then the commutator subgroup G' of G is a soluble-by-finite minimax group.*

If G is a locally (soluble-by-finite) group whose proper subgroups have periodic commutator subgroup, it has recently been proved that also G' is periodic (see [3], Corollary 3.5). Thus Theorem A has the following direct consequence, that was independently proved by O.A. Yarovaya [11] in a paper which is not easily available.

Corollary A1 *Let G be a locally (soluble-by-finite) group whose proper subgroups have Černikov commutator subgroup. Then the commutator subgroup G' of G is a Černikov group.*

A series of recent papers shows that in many cases the behaviour of proper subgroups of infinite rank has a strong influence on the structure of the group (see for instance [3],[4],[5],[7]). We prove here the following corollary of our main theorem.

Corollary A2 *Let G be a locally (soluble-by-finite) group whose proper subgroups of infinite rank have minimax commutator subgroup. Then either G has finite rank or its commutator subgroup G' is minimax.*

Recall that a group G has *finite torsion-free rank* if it has a series of finite length whose factors are either periodic or infinite cyclic. Although the structure of a locally soluble group with finite torsion-free rank can be complicated, it turns out that torsion-free locally nilpotent groups with finite torsion-free rank are nilpotent and have finite rank (see for instance [10] Part 2, Theorem 6.36). For such reason we will prove our next theorem within the universe of *radical groups*, i.e. groups admitting an ascending series with locally nilpotent factors.

Theorem B *Let G be a radical group in which the commutator subgroup of every proper subgroup has finite torsion-free rank. Then the commutator subgroup G' of G has finite torsion-free rank.*

Most of our notation is standard and can be found in [10].

2. Proofs

Our first lemma is an easy consequence of a result on groups whose abelian subgroups are minimax, which was proved independently by R. Baer and D.I. Zaicev. Recall here that the *finite residual* of a group G is the intersection of all subgroups of finite index of G, and that G is *residually finite* if its finite residual is trivial.

Lemma 2.1. *Let G be a radical group, and let A be a torsion-free abelian minimax normal subgroup of G. Then $G/C_G(A)$ is a soluble minimax residually finite group.*

Proof. As $G/C_G(A)$ is isomorphic to a group of automorphisms of a torsion-free abelian minimax group, all its abelian subgroups are minimax (see [10] Part 2, Corollary to Lemma 10.37), and hence the radical group $G/C_G(A)$ is soluble and minimax (see [10] Part 2, Theorem 10.35). Moreover, it is well-known that every periodic linear group over the field of rational numbers is finite, and so it follows from the structure of soluble minimax groups that $G/C_G(A)$ is residually finite. □

Next lemma deals with an embedding property of minimax normal subgroups in radical groups.

Lemma 2.2. *Let G be a radical group, and let J be the finite residual of G. If N is a minimax normal subgroup of G, then $[N, \underbrace{J, \dots, J}_{k}] = \{1\}$ for some positive integer k.*

Proof. It follows from the structure of soluble minimax groups that N has a finite series

$$\{1\} \le D \le T = A_0 \le A_1 \le \dots \le A_m \le N,$$

consisting of characteristic subgroups, such that D is a direct product of finitely many Prüfer subgroups, T/D is finite, A_{i+1}/A_i is torsion-free abelian for all $i = 0, \dots, m-1$, and the last factor N/A_m is polycyclic. Clearly, the group $G/C_G(D)$ is residually finite, and it is known that $G/C_G(N/A_m)$ is polycyclic (see [10] Part 1, Theorem 3.27) and so also residually finite. Moreover, it follows from Lemma 2.1 that $G/C_G(A_{i+1}/A_i)$ is residually finite for every $i = 0, 1, \dots, m-1$. Therefore J stabilizes the above series of N, and hence

$$[N, \underbrace{J, \dots, J}_{k}] = \{1\}$$

for some positive integer k. □

Corollary 2.3. *Let G be a radical group with no proper subgroups of finite index, and let N be a minimax normal subgroup of G. Then N is contained in some term with finite ordinal type of the upper central series of G.*

Our next elementary result shows that if G is any group in which all proper normal subgroups are abelian, then G is soluble, provided that it is either locally soluble or radical.

Lemma 2.4. *Let G be a group whose proper normal subgroups are abelian. Then either G is soluble or it has a simple non-abelian homomorphic image.*

Proof. Assume that the group G is not soluble, so that it must be perfect. Then for each pair (H, K) of proper normal subgroups of G, the product HK is properly contained in G, so that it is abelian and hence $[H, K] = \{1\}$. It follows that the subgroup M, generated by all proper normal subgroups of G, is a maximal normal subgroup, and so G/M is a simple non-abelian group. □

Lemma 2.5. *Let G be a locally soluble group whose proper normal subgroups have minimax commutator subgroup. Then G is soluble.*

Proof. As minimax locally soluble groups are soluble, we have that every proper normal subgroup of G is soluble. It follows that every non-trivial homomorphic image of G contains an abelian non-trivial normal subgroup, and hence G is hyperabelian. Assume now for a contradiction that G is a (non-trivial) perfect group, so that in particular G has no proper subgroups of finite index. Application of Grün's lemma yields that $Z(G)$ is the last term of the upper central series of G. Let N be any proper normal subgroup of G. Then the commutator subgroup N' of N is a minimax normal subgroup of G, and so it is contained in $Z(G)$ by Corollary 2.3. Therefore all proper normal subgroups of $G/Z(G)$ are abelian, and hence G is soluble by Lemma 2.4. □

Lemma 2.6. *Let A be an abelian non-periodic group. Then A contains a subgroup B such that A/B is a periodic group of finite rank with three non-trivial primary components.*

Proof. Choose a positive integer n which is the product of three distinct prime numbers. Let E be a maximal free abelian subgroup of A, and put $C = E^n$. Then A/C is a periodic group with at least three non-trivial primary components. The statement is now a consequence of the fact that every primary abelian non-trivial group has a locally cyclic non-trivial direct factor. □

Recall that a group class $\mathfrak{X}$ is said to be $\boldsymbol{N}_0$*-closed* if in any group G the product of two normal $\mathfrak{X}$-subgroups is likewise an $\mathfrak{X}$-group. Clearly, any class of groups which is closed with respect to extensions and homomorphic images is likewise $\boldsymbol{N}_0$-closed. In particular, the class of minimax groups and that of groups with finite torsion-free rank have such property.

Lemma 2.7. *Let $\mathfrak{X}$ be an $\boldsymbol{N}_0$-closed class of groups, and let G be a group such that G/G' is not periodic. If the commutator subgroup of every proper normal subgroup of G belongs to $\mathfrak{X}$, then also the commutator subgroup G' of G is an $\mathfrak{X}$-group.*

Proof. As G/G' is a non-periodic abelian group, it follows from Lemma 2.6 that G contains normal subgroups U_1, U_2, U_3 such that $G = U_1U_2U_3$ and $U_iU_j \neq G$ for all i, j. In particular, the commutator subgroup of U_iU_j is a normal $\mathfrak{X}$-subgroup of G for any choice of i and j, and hence also the commutator subgroup

$$G' = (U_1U_2)'(U_1U_3)'(U_2U_3)'$$

belongs to the class $\mathfrak{X}$. □

Recall that a group G is said to be a *CC-group* (or to have *Černikov conjugacy classes*) if $G/C_G(\langle x\rangle^G)$ is a Černikov group for every element x of G. Such groups were introduced by Y.D. Polovickiĭ [9], and their behaviour has been investigated by several authors. In particular, it is known that a periodic group G has Černikov conjugacy classes if and only if $[G, x]$ is a Černikov subgroup for all x in G (see [10] Part 1, Theorem 4.36). Groups whose proper subgroups have Černikov conjugacy classes have been studied by J. Otal and J.M. Peña [8], and in fact they proved that there are no soluble minimal non CC-groups.

Lemma 2.8. *Let G be a soluble group whose proper subgroups have Černikov conjugacy classes. Then G is a CC-group.*

In our proofs we will need also the following lemma, which was proved in [6].

Lemma 2.9. *Let G be a periodic CC-group containing a normal subgroup N such that N' and G/N are Černikov groups. Then also the commutator subgroup G' of G is a Černikov group.*

We are now in a position to prove our main result.

Proof of Theorem A $-$ The commutator subgroup G' of G has finite rank (see [4]). Suppose first that the group G is locally soluble, and so even soluble by Lemma 2.5, and assume for a contradiction that the statement is false. As G' is properly contained in G, the normal subgroup G'' is minimax, so that G'/G'' is not minimax and G/G'' is also a counterexample. Replacing G by the factor group G/G'', it can be assumed without loss of generality that G' is abelian. Moreover, the factor group G/G' must be periodic by Lemma 2.7. If G is periodic, then every proper subgroup of G has Černikov commutator subgroup, and in particular it has Černikov conjugacy classes, so that G itself is a CC-group by Lemma 2.8. Consider a non-central element x of G. Thus $G/C_G(\langle x\rangle^G)$ is a Černikov group and $C_G(\langle x\rangle^G)$ is properly contained in G, so that also $C_G(\langle x\rangle^G)'$ is a Černikov group. Application of Lemma 2.9 yields that G' is a Černikov group, and this contradiction shows that G cannot be periodic, so that G' is not periodic.

Suppose that G contains a proper normal subgroup H such that the index $|G : H|$ is finite. Then H' is a minimax normal subgroup of G, so that G'/H' is not minimax, and replacing G by G/H', it can be assumed that H is abelian. Let A be a maximal free abelian subgroup of $H \cap G'$. Then G'/A is periodic, since the index $|G' : H \cap G'|$ is finite. Moreover, as G' has finite rank and A has finitely many conjugates in G, the abelian normal subgroup A^G is finitely generated. The previous argument shows that the periodic group G/A^G cannot be a counterexample, so that G'/A^G is a Černikov group, and hence G' is minimax. This contradiction proves that G has no proper subgroups of finite index, so that in particular the periodic abelian group G/G' is divisible. Let T be the subgroup consisting of all elements of finite order of G'. Then G'/T is a torsion-free abelian group of finite rank, and so the periodic group $G/C_G(G'/T)$ is linear over the field of rational numbers. Thus $G/C_G(G'/T)$ is finite, so that $C_G(G'/T) = G$ and hence the factor group G/T is nilpotent. It follows that G/T is even abelian, and so $G' = T$ is periodic. This last contradiction proves the statement when G is locally soluble.

Suppose now that G is an arbitrary locally (soluble-by-finite) group whose proper subgroups have minimax commutator subgroup. Since G' has finite rank, it contains a locally soluble subgroup of finite index (see [2]), and so it is soluble-by-finite by Lemma 2.5. Therefore the group G is soluble-by-finite. If S is the largest soluble normal subgroup of G, its commutator subgroup S' is minimax by the first part of the proof, and so it can be assumed that S is a proper subgroup of G. Clearly, it is enough to show that G'/S' is minimax, and so replacing G by G/S' we may also suppose that S is abelian.

Let $x_1, \dots, x_k$ be elements of G such that

$$G = \langle S, x_1, \dots, x_k \rangle.$$

For each $i = 1, \dots, k$, the subgroup $\langle S, x_i \rangle$ is properly contained in G, so that $[S, x_i]$ is a minimax subgroup of S, and hence also its normal closure $L_i = [S, x_i]^G$ is minimax. Thus

$$L = \langle L_1, \dots, L_k \rangle$$

is a minimax normal subgroup of G. Moreover, S/L is contained in the centre of G/L, so that G/L is finite over its centre, and hence its commutator subgroup G'/L is finite by Schur's theorem (see [10] Part 1, Theorem 4.12). Therefore G' is a minimax group. □

Let $\mathfrak{X}$ be a class of groups which is closed with respect to subgroups. We shall say that $\mathfrak{X}$ is *persistent* within a certain universe $\mathfrak{U}$ if every $\mathfrak{U}$-group, whose proper subgroups belong to $\mathfrak{X}$, likewise lies in $\mathfrak{X}$. Of course, this is equivalent to the requirement that there are no minimal non-$\mathfrak{X}$ groups in the class $\mathfrak{U}$.

Lemma 2.10. *Let $\mathfrak{X}$ be a group class which is persistent within a universe $\mathfrak{U}$, and let G be an $\mathfrak{U}$-group whose proper subgroups of infinite rank belong to $\mathfrak{X}$. If the factor group G/G' has infinite rank, then G belongs to $\mathfrak{X}$.*

Proof. As G/G' has infinite rank, G' is contained in a subgroup H of G such that both H/G' and G/H have infinite rank. Let X be any subgroup of finite rank of G. Then XH is a proper subgroup of infinite rank of G, and hence it belongs to $\mathfrak{X}$. As $\mathfrak{X}$ is closed with respect to subgroups, also X is an $\mathfrak{X}$-group. It follows that all proper subgroups of G lie in $\mathfrak{X}$, and so G belongs to the persistent class $\mathfrak{X}$. □

Proof of Corollary A2 – Clearly, the commutator subgroup of any proper subgroup of G has finite rank, and hence also G' has finite rank. Assume that G has infinite rank, so that also G/G' has infinite rank. It follows from Theorem A that the class of groups with minimax commutator subgroup is persistent within the universe of locally (soluble-by-finite) groups, and hence the statement is a direct consequence of Lemma 2.10. □

Our last lemma provides informations on normal subgroups with finite torsion-free rank in perfect groups.

Lemma 2.11. *Let G be a radical group with no periodic non-trivial normal subgroups, and let N be a normal subgroup of G with finite torsion-free rank. If G is perfect, then N is contained in the centre of G.*

Proof. Let H be the Hirsch-Plotkin radical of N. Then H is torsion-free, and so it is nilpotent of finite rank. It follows that the i-th factor $Z_{i+1}(H)/Z_i(H)$ of the upper central series of H is a torsion-free abelian group of finite rank for each non-negative integer i (see [10] Part 1, Theorem 2.25). Thus the radical group $G/C_G(Z_{i+1}(H)/Z_i(H))$ is linear over the field of rational numbers, so that it is soluble (see for instance [10] Part 1, p.78), and hence even trivial, because G is perfect. Therefore H is contained in $Z_k(G)$ for some non-negative integer k. On the other hand, it follows from Grün's lemma that $Z(G)$ is the last term of the upper central series of G, and hence H lies in $Z(G)$. In particular, $H = Z(N)$, and so N/H cannot have locally nilpotent non-trivial normal subgroups. Therefore $N = H$, and the statement is proved. □

Proof of Theorem B – Let T be the largest periodic normal subgroup of G. Clearly, it is enough to prove that the statement holds for the factor group G/T, and hence replacing G by G/T we may suppose without loss of generality that G has no periodic non-trivial normal subgroups. Assume for a contradiction that G is perfect (and non-trivial), and let N be any proper normal subgroup of G. Then the commutator subgroup N' of N has finite torsion-free rank, and so it is contained in $Z(G)$ by Lemma 2.11. It follows that every proper normal subgroup of $G/Z(G)$ is abelian, contradicting Lemma 2.4, since G has no simple non-abelian homomorphic images.

Therefore the commutator subgroup G' is properly contained in G, and so G'' has finite torsion-free rank. Let K/G'' be the subgroup consisting of all elements of finite order of the abelian group G'/G''. Then also K has finite torsion-free rank, and replacing G by G/K it can be assumed that G' is torsion-free abelian. It follows that the commutator subgroup of any proper subgroup of G has finite rank, and hence G' itself has finite rank. The statement is proved. □

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