CO-PROLONGATIONS OF A GROUP EXTENSION

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ABSTRACT. The aim of this paper is to study co-prolongations of central extensions. We construct the obstruction theory for co-prolongations and classify the equivalence classes of these by kernels of homomorphisms between 2-dimensional cohomology groups of groups.

1. Introduction

A description of group extensions by means of factor sets leads to a close relationship between the group extension problem and the group cohomology theory [1, 5, 9]. Analogously, the extension problems of a type of algebras (such as rings, k-split algebras) were dealt with by appropriate cohomology theories (such as Mac Lane cohomology, Hochschild cohomology) (see [4, 5]). Quang et al used the group cohomology to study the prolongation of central extensions in [8]. In this paper we consider the dual of that problem.

Let $G$ be a group and $A$ be a $G$-module. Consider a short exact sequences of groups

$$E : 0 \longrightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} G \longrightarrow 1,$$

in which the left action by $G = B/A$ on $A$ induced by $B$-conjugation on the commutative normal subgroup $A$ is the given $G$-module structure on $A$. We say that $B$ is an extension of the group $A$ by the group $G$. A morphism between two extensions is a triple of homomorphisms $(\alpha, \beta, \gamma)$ such that the following diagram commutes

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For a given extension $E$ and a homomorphism $\gamma$, there exists an extension $E_0$ and a morphism $(id_A, \beta, \gamma)$ (then, $B$ is the fibred product, or the pull-back $A \times_{G_0} G$) (see [3] Ch. IV, [7]). This shows that $\text{Ext}(G, A)$ is a contravariant functor in terms of the first variable, $G$, ([5] Ch. III). Given two extensions $E_0, E$ and two morphisms $\alpha, \gamma$, Proposition 5.1.1 [9] indicates a necessary and sufficient condition for there to exist a homomorphism $\beta$ such that $(\alpha, \beta, \gamma)$ is a morphism.

Given an extension $E_0$ and a homomorphism $\gamma : G_0 \to G$, the problem here is that of finding whether there is any corresponding extension $E$ such that $E_0 = E\gamma$. A particular case when $E_0$ is a central extension and $\gamma : G_0 \to G$ is a normal monomorphism is studied in [8]. In this case $E$ is said to be a prolongation of the extension $E_0$. The obstruction theory and the classification of prolongations of $E_0$ were dealt with in the case of $\alpha$ is surjective.

The objective of this paper is to solve the problem of prolongations in the case $\gamma$ is surjective and $\alpha$ is an identity. Then, $E$ is termed a co-prolongation of the extension $E_0$. In Section 2 we show some necessary conditions for the existence of co-prolongations. We also prove that each such co-prolongation is just a central extension and it induces a crossed module. After stating the problem of $\gamma$-co-prolongations of a group extension, we construct the obstruction theory for this concept (Theorem 3.1) in Section 3, and then classify co-prolongations (Theorem 3.3) of the extension $E_0$ by the kernel of the inducing homomorphism

$$\bar{\gamma} : H^2(G, A) \to H^2(G_0, A).$$

2. Co-prolongations of a group extension

In this paper we fix the system $(E_0, \gamma)$, where $E_0$ is the extension

$$0 \to A \xrightarrow{i_0} B_0 \xrightarrow{p_0} G_0 \to 0,$$

and $\gamma : G_0 \to G$ is injective. Further, for short, we identify the abelian group $A$ with the normal subgroup $i_0(A)$ of $B_0$; the operations in $B_0, G_0$ are denoted by the addition, even though they are are non-necessarily abelian.

Now, suppose that $\alpha = id_A$ in the diagram (1.1). Then, the extension $E$ is call a co-prolongation by $\gamma$ (or $\gamma$-co-prolongation) of the extension $E_0$.

Remark. In the commutative diagram (1.1), since $\alpha = id_A$ and $\gamma$ is surjective, $\beta$ also is a surjection.
Proposition 2.1. If the extension $E$ is a $\gamma$-co-prolongation of the extension $E_0 \in \text{Opext}(G_0, A, \varphi_0)$, then there uniquely exists a homomorphism $\varphi : G \to \text{Aut} A$ such that the following diagram commutes:

\begin{align}
G_0 & \xrightarrow{\gamma} G, \quad \varphi \gamma = \varphi_0. \\
\varphi_0 & \xrightarrow{} \text{Aut} A \xleftarrow{} \varphi
\end{align}

Proof. The homomorphism $\varphi : G \to \text{Aut} A$ induced by the extension $E$ is determined by

\begin{align}
i[(\varphi pb)a] = b + ia - b, \quad b \in B, a \in A.
\end{align}

It is easy to check that $\varphi$ satisfies (2.1). If $\varphi' : G \to \text{Aut}(A)$ is a homomorphism satisfying (2.1), then $\varphi' = \varphi$ since $\gamma$ is surjective. \hfill \blacksquare

Proposition 2.2. If the extension $E$ is a $\gamma$-co-prolongation of the extension $E_0$, there exists an isomorphism $j : \text{Ker}\gamma \to \text{Ker}\beta$ such that $p_0 j = \text{id}_{\text{Ker}\gamma}$.

Proof. Since the right hand side square of the diagram (1.1) commutes, $p_0(\text{Ker}\beta) \subset \text{Ker}\gamma$.

Thus, the homomorphism $p_0$ induces a homomorphism $p^* : \text{Ker}\beta \to \text{Ker}\gamma$. We show that $p^*$ is surjective. Take $c \in \text{Ker}\gamma$, then $c = p_0(x_0)$, where $x_0 \in B_0$. Then,

\begin{align}
0 = \gamma(c) = \gamma p_0(x_0) = p\beta(x_0).
\end{align}

It follows that $\beta(x_0) \in \text{Ker} p = A$. Set $a = \beta(x_0)$. Since the left hand side square commutes, $\beta(a) = a = \beta(x_0)$, hence $x_0 - a \in \text{Ker}\beta$. One obtains

\begin{align}
c = p_0(x_0) = p_0(x_0 - a) \in p_0(\text{Ker}\beta),
\end{align}

hence $p^*$ is surjective. Also, $A \cap \text{Ker}\beta = 0$, or $\text{Ker}(p_0) \cap \text{Ker}\beta = 0$, which implies that $p^*$ is injective. Then, $j = (p^*)^{-1} : \text{Ker}\gamma \to \text{Ker}\beta$ is the required isomorphism. \hfill \blacksquare

A monomorphism $j : C \to D$ is said to be normal if $jC$ is a normal subgroup in $D$.

Lemma 2.3. If there exists a normal monomorphism $j : \text{Ker}\gamma \to B_0$ such that $p_0 j = \text{id}_{\text{Ker}\gamma}$, then $p_0^{-1}(\text{Ker}\gamma) = A \times j(\text{Ker}\gamma)$ and the following diagram commutes

\begin{align}
(2.3) \quad E' : 0 \longrightarrow A \xrightarrow{i'} A \times \text{Ker}\gamma \xrightarrow{p'} \text{Ker}\gamma \longrightarrow 0 \quad & \xrightarrow{\nu} \\
E_0 : 0 \longrightarrow A \xrightarrow{i} B_0 \xrightarrow{p} G_0 \longrightarrow 0,
\end{align}

where $A \times \text{Ker}\gamma$ is the direct product, $\nu$ is an inclusion, $i' : a \mapsto (a, 0)$, $p' : (a, c) \mapsto c$, and $\varepsilon : (a, c) \mapsto a + j(c)$. 
Proof. Let \( b \in p_0^{-1}(\text{Ker} \, \gamma) \). Then there exists \( c \in \text{Ker} \, \gamma \) such that \( p_0(b) = c = p_0(j(c)) \). It follows that \( b - j(c) = a \in \text{Ker} \, p_0 = A \). Thus, \( b = a + j(c) \in A + j(\text{Ker} \, \gamma) \). It is easy to see that \( A \cap j(\text{Ker} \, \gamma) = 0 \), hence \( p_0^{-1}(\text{Ker} \, \gamma) = A \times j(\text{Ker} \, \gamma) \). The map \( \alpha : (a, c) \mapsto a + j(c) \) is a homomorphism and diagram (2.3) commutes.

**Definition 2.4.** A crossed module is a quadruple \( \mathcal{M} = (B, D, d, \theta) \) in which \( d : B \to D, \theta : D \to \text{Aut}B \) are group homomorphisms such that

\begin{align*}
C_1 & : \theta d = \mu, \\
C_2 & : d(\theta_x(b)) = \mu_x(d(b)), \ x \in D, b \in B, \\
\end{align*}

where \( \mu_x \) is an inner automorphism given by conjugation with \( x \).

A crossed module \( (B, D, d, \theta) \) is sometimes denoted by \( B \overset{d}{\to} D \), or \( B \to D \).

Crossed modules over groups are introduced by Whitehead [10] (see also [1] Ch. IV, [6]). The problem of group extensions of the type of a crossed module presented in [2]. This closely relates to the problem of prolongations (see [8]). Now, we show that each co-prolongation of a group extension determines a crossed module.

**Proposition 2.5.** If there exists a \( \gamma \)-co-prolongation of \( E_0 \), then:

1. \( E_0 \) is a central extension,

2. \( E_0 \) induces a homomorphism \( \theta : G_0 \to \text{Aut}(A \times \text{Ker} \, \gamma) \) such that \( (A \times \text{Ker} \, \gamma, G_0, \nu \rho', \theta) \) is a crossed module.

**Proof.** It follows from Proposition 2.2 and Lemma 2.3 that the diagram (2.3) commutes. In this diagram, since \( A \subset Z(A \times \text{Ker} \, \gamma) \), \( E_0 \) is a \((id_A, \nu)\)-prolongation of the extension \( E' \) in the sense of [8]. According to Theorem 10 [8], \( E_0 \) is a central extension.

2) This follows from Proposition 2 [8]. The homomorphism \( \theta : G_0 \to \text{Aut}(A \times \text{Ker} \, \gamma) \) is given by

\[ \theta_g = \phi_{b_0}, \quad p_0(b_0) = g_0, \]

\[ \phi_{b_0}(x) = \varepsilon^{-1} \mu_{b_0}(\varepsilon x), \ x \in A \times \text{Ker} \, \gamma. \]

3. **The obstruction of a co-prolongation**

In this section, suppose that the isomorphism \( \varphi : G \to \text{Aut} \, A \) of the system \((E_0, \gamma)\) satisfies (2.1). The “co-prolongation problem” is that of finding whether there is any extension \( E \) of \( A \) by \( G \) which is a co-prolongation of the extension \( E_0 \) and, if so, how many of these exist.

Let \( \{u(x_0), x_0 \in G_0\}, \ u(0) = 0, \) be a set of representatives of \( G_0 \) in \( B_0 \). This set induces a homomorphism \( \varphi_0 : G_0 \to \text{Aut} \, A \) by (2.2) and a factor set \( f_0 : (G_0)^2 \to A \) by

\[ f_0(x_0, y_0) = u(x_0) + u(y_0) - u(x_0 + y_0). \]

According to [5] Ch. IV, the function \( f_0 : G_0 \times G_0 \to A \) is uniquely defined in terms of \( B^2_{\varphi_0}(G_0, A) \), that means the element

\[ \overline{f_0} = f_0 + B^2_{\varphi_0}(G_0, A) \]
is completely determined. Thanks to the relation (2.1), the homomorphism $\gamma : G_0 \to G$ induces one

$$\gamma : H^2_\varphi(G, A) \to H^2_\varphi(G_0, A)$$

$$h \mapsto \gamma^* h,$$

where $(\gamma^* h)(x_0, y_0) = h(\gamma x_0, \gamma y_0), \forall x_0, y_0 \in G_0$. Then, the element

$$\tilde{f}_0 = \overline{f_0} + \text{Im}(\overline{\gamma}) \in \text{Coker}(\overline{\gamma}),$$

is not dependent on the choice of the representative $u(x_0)$. We call $\tilde{f}_0$ the obstruction of $\gamma$-co-prolongation of the extension $E_0$.

To prove Theorem 3.1 and Corollary 3.2, one represents the factor set $f_0$ with respect to the factor set $s : G^2 \to \text{Ker} \gamma$ of the extension $G_0 \xrightarrow{\gamma} G$. Under the hypothesis of Lemma 2.3, choose a set of representatives $\{u(x_0), x_0 \in G_0\}$ in $B_0$ as follows. Firstly, choose a set of representatives $\{v(x), x \in G\}, v(1) = 0$, of $G$ in $G_0$ whose the corresponding factor set is $s : G^2 \to \text{Ker} \gamma$. For each $x \in G$, choose an element $u_x$ in $B_0$ such that

$$p_0(u_x) = v(x), \ u_1 = 0.$$

Since the element $x_0 \in G_0$ is uniquely written as

$$x_0 = c + v(x), \ c \in \text{Ker} \gamma, \ x \in G,$$

we set

$$(3.1) \quad u(x_0) = jc + u_x.$$

Let $f_0$ be the factor set of $B_0$ corresponding to this factor set. For $x_0, y_0 \in G_0$, one has

$$x_0 = c + v(x), \ y_0 = d + v(y), \ c, d \in \text{Ker} \gamma, \ x, y \in G.$$

It follows that

$$x_0 + y_0 = (c + v(x)) + (d + v(y))$$

$$= c + \mu v(x)(d) + v(x) + v(y)$$

$$= c + \mu v(x)(d) + s(x, y) + v(xy).$$

Then,

$$(3.2) \quad c_0 = c + \mu v(x)(d) + s(x, y) \in \text{Ker} \gamma,$$

hence the relation (3.1) implies

$$u(x_0 + y_0) = jc_0 + u_{xy}.$$

Simple calculations lead to

$$(3.3) \quad f_0(x_0, y_0) = jc + \mu u_x(jd) + (u_x + u_y - u_{xy}) - jc_0.$$

**Theorem 3.1.** Co-prolongations of $E_0$ exist if and only if $\tilde{f}_0$ vanishes on Coker(\overline{\gamma}).
Proof. Necessary condition. Let $E$ be a $\gamma$-co-prolongation of $E_0$. Choose in $B_0$ a set of representatives $u(x_0), x_0 \in G_0$, such that the induced obstruction $\tilde{f}_0$ vanishes in $\text{Coker}(\gamma)$.

If $j : \text{Ker} \gamma \to \text{Ker} \beta$ is the isomorphism mentioned in Proposition 2.2, then $p_0 j = id_{\text{Ker} \gamma}$. The set of representatives $u(x_0), x_0 \in G_0$, chosen by (3.1) in $B_0$, gives a factor set $f_0$ satisfying (3.3).

Since

\[ p\beta(u_x) = \gamma p_0(u_x) = \gamma(v(x)) = x, \]

hence $\{r(x) = \beta(u_x), x \in G\}$ is a set of representatives of $G$ in $B$ (clearly, $r(1) = 0$). Let $f$ be a factor set of $B$ corresponding to this set of representatives, we prove that

\[ f_0 = \gamma^* f. \]

Since $jc, jc_0, \mu_{ux}(jd)$ are in Ker $\beta$, act $\beta$ on two sides of the equality (3.3) (note that $\beta|_A = id_A$), one has

\[ f_0(x_0, y_0) = \beta u_x + \beta u_y - \beta u_{xy} = r(x) + r(y) - r(xy) = f(x, y). \]

It follows that

\[ (\gamma^* f)(x_0, y_0) = f(\gamma x_0, \gamma y_0) = f(x, y) = f_0(x_0, y_0), \]

that is $\gamma^* f = f_0$, and hence $\tilde{f}_0$ vanishes in $\text{Coker}(\gamma)$.

Sufficient condition. Let $f_0 = 0 \in \text{Coker}(\gamma)$, where $f_0$ is a factor set of $E_0$. There exists $f \in Z^2(G, A)$ such that

\[ f_0 = \gamma^* (f) + \delta, \delta \in B^2(G_0, A). \]

If $\{u(x_0), x_0 \in G_0\}$ is a set of representatives corresponding to the factor set $f_0$, then one can choose a set of representatives $u'(x_0) = u(x_0) - t(x)$ so that one obtains a new factor set

\[ f_0'(x_0, y_0) = (\gamma^* f)(x_0, y_0). \]

According to [5] Ch. IV, there exists an extension $E$ of the crossed product $\mathcal{B} = [A, \varphi, f, G]$. This is a $\gamma$-co-prolongation of the extension $E_0$. Indeed, consider the diagram

\[
\begin{array}{cccccc}
E_0: & 0 & \longrightarrow & A & \overset{i_0}{\longrightarrow} & B_0 & \overset{p_0}{\longrightarrow} & G_0 & \longrightarrow & 0 \\
E: & 0 & \longrightarrow & A & \overset{i}{\longrightarrow} & \mathcal{B} & \overset{p}{\longrightarrow} & G & \longrightarrow & 1 \\
\end{array}
\]

where $i : a \mapsto (a, 1); p : (a, x) \mapsto x; \beta : a + u'(x_0) \mapsto (a, \gamma x_0)$. Clearly, $\beta$ is a group homomorphism making the above diagram commute, that means $E$ is a co-prolongation of $E_0$. \qed

Corollary 3.2. Let $(E_0, \gamma)$ and $\varphi : G \to \text{Aut} A$ satisfy (2.1). If the onto-homomorphism $\gamma : G_0 \to G$ is split and there is a normal monomorphism $j : \text{Ker} \gamma \to B_0$ such that $p_0 j = id_{\text{Ker} \gamma}$, then $\gamma$-co-prolongations of the extension $E_0$ exist.
Thus, \( \tilde{f} \) is a homomorphism induced by \( \gamma \).

Act \( p_0 \) on two sides of the above equality, we have \( v(x) + d - v(x) = d' \). Since \( G_0 = \text{Ker} \gamma \times \text{Inv} \), \( d' = d \), that means \( \mu_{a_x}(jd) = jd \). Since \( s = 0 \), the relation \( (3.2) \) becomes \( c_0 = c + d \). Besides,

\[
\begin{align*}
u_x + u_y - u_{xy} &= u(v(x)) + u(v(y)) - u(v(x)v(y)) = f_0(v(x), v(y)) \in A.
\end{align*}
\]

Then, it follows from \( A \cap j(\text{Ker} \gamma) = 0 \) that each element of \( A \) commutes with each element of \( j(\text{Ker} \gamma) \). Thus, equality \( (3.3) \) turns into

\[
(3.4) \quad f_0(x_0, y_0) = f_0(v(x), v(y)).
\]

Now, define a function \( f : G^2 \to A \) by

\[
(3.5) \quad f(x, y) = f_0(v(x), v(y)), \quad x, y \in G.
\]

The relation \( (2.1) \) and the fact that \( f_0 \in Z^3_{\varphi_0}(G_0, A) \) imply \( f \in Z^3_{\varphi}(G, A) \). Clearly,

\[
(\gamma^*f)(x_0, y_0) = f(\gamma x_0, \gamma y_0) = f((x, y) \quad 4.3 \quad f_0(v(x), v(y)) \quad 4.3 \quad f_0(x_0, y_0).
\]

Thus, \( \tilde{f}_0 = 0 \) in \( \text{Coker}(\overline{\varphi}) \), and hence by Theorem \( 3.1 \) there exist co-prolongations of \( E_0 \) by \( \gamma \).

\[ \square \]

**Theorem 3.3** (Classification theorem). *If the system \((E_0, \gamma)\) together with the homomorphism \( \varphi : G \to \text{Aut} A \) satisfying the relation \( (2.1) \) have \( \gamma \)-co-prolongations, then the set of equivalence classes of \( \gamma \)-co-prolongations is a torseur under the group \( K = \text{Ker}(\overline{\varphi}) \), where*

\[
\overline{\varphi} : H^2_{\varphi}(G, A) \to H^2_{\varphi_0}(G_0, A)
\]

*is a homomorphism induced by \( \gamma \).*

**Proof.** Firstly, observe that each extension of \( G \) by \( A \) inducing \( \varphi \) is isomorphic to the extension of the crossed product \([A, \varphi, f, G]\), where \( f \) is uniquely determined up to a coboundary \( \delta t \in B^2_{\varphi}(G, A) \).

Let \( U \) be the set of equivalence classes of co-prolongations of the extension \( E \). To prove that \( U \) is a torseur under \( K = \text{Ker}(\overline{\varphi}) \), one constructs a map

\[
\Lambda : \text{Ker}(\overline{\varphi}) = K \to \text{Aut}(U)
\]

by the formula

\[
\Lambda(\overline{t})(\text{cls}[A, \varphi, f, G]) = \text{cls}[A, \varphi f, h, G].
\]
Thanks to the above observation, this formula is not dependent on the representative element of the class $\overline{h}$, as well as the representative $f$. Thus, $\Lambda$ is well defined. Further, $\Lambda(\overline{h})$ is actually an element of the group of transformations of $U$. Clearly, $\Lambda$ is a group homomorphism.

It remains to prove that for any two co-prolongations $E_1, E_2$ of the extension $E$, there uniquely exists an element $\overline{h} \in \text{Ker}(\gamma)$ such that

$$\text{cls}E_2 = \Lambda(\overline{h})(\text{cls}E_1).$$

Indeed, one has $\text{cls}E_1 = \text{cls}[G, \varphi, f_i, A], i = 1, 2$, where $\overline{\gamma f_i} = \overline{f}$. By setting $h = -f_1 + f_2$, the proof is completed. $\square$

REFERENCES


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