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MAXIMAL SUBSETS OF PAIRWISE NON-COMMUTING ELEMENTS OF p-GROUPS OF ORDER LESS THAN p^6

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Abstract. Let G be a non-abelian group of order p^n, where $n \leq 5$ in which G is not extra special of order p^5. In this paper we determine the maximal size of subsets X of G with the property that $xy \neq yx$ for any x, y in X with $x \neq y$.

1. **Introduction**

Let G be a non-abelian group. A subset X of G is said to be a subset of pairwise non-commuting elements of G if $xy \neq yx$ for any x, y in X with $x \neq y$. The maximal cardinality of these subsets is denoted by $\omega(G)$. Also $\omega(G)$ is the maximal clique size in the non-commuting graph of a finite group G. Let $Z(G)$ be the center of G. The non-commuting graph of a group G is a graph with $G \backslash Z(G)$ as the set of vertices and two vertices are joined if and only if they do not commute. By a famous result of B. H. Neumann [11], answering a question of P. Erdös, the finiteness of $\omega(G)$ in G is equivalent to the finiteness of the factor group $G/Z(G)$. Furthermore many attempts have been made to find $\omega(G)$ for some groups G, see for example [1], [2], [3], [5], [6], [7], [11] [12]. In this paper we find $\omega(G)$ for all non-abelian p-groups G with $|G| \leq p^5$ in which G is not extra special of order p^5. To prove this we use the classification of p-groups by James [10], which based on isoclinism. Following [4, §29], two groups are isoclinic if their commutator subgroups and central quotients are isomorphic and their commutator operations are essentially the same.

Throughout this paper the following notation is used. All groups are assumed to be finite. The letter p denotes a prime number. $\mathcal{C}_G(x)$ is the centralizer of an element x in a group G. The terms

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of the lower central series of G are denoted by $\gamma_i = \gamma_i(G)$. The minimal number of generators of G is denoted by $d(G)$. We write $[a, b]$ for $a^{-1}b^{-1}ab$. Also $\mathbb{Z}_n$ is the cyclic group of order n and $(\mathbb{Z}_n)^k$ is the direct product of k copies of $\mathbb{Z}_n$. We write SmallGroup(n,m) for the mth group of order n as quoted in the "Small Groups" library in GAP[8]. All unexplained notation is standard and follows that of [4].

2. **Some Basic Results**

In this section we give some basic results that are needed for the main results of the paper. A group G is called an AC-group if the centralizer of every non-central element of G is abelian. In this section first we give some results for AC-group G and find $\omega(G)$ in some cases.

Lemma 2.1. [13, Lemma 3.2] *The following are equivalent on a group G.*

(i) *G is an AC-group.*

(ii) *If $[x, y] = 1$, then $\mathcal{C}_G(x) = \mathcal{C}_G(y)$, where $x, y \in G \backslash Z(G)$.*

Lemma 2.2. *Let G be an AC-group.*

(i) *If $a, b \in G\backslash Z(G)$ with distinct centralizers, then $\mathcal{C}_G(a) \cap \mathcal{C}_G(b) = Z(G)$.*

(ii) *If $G = \cup_{i=1}^{k} \mathcal{C}_G(a_i)$, where $\mathcal{C}_G(a_i)$ and $\mathcal{C}_G(a_j)$ are distinct for $1 \leq i < j \leq k$, then $\{a_1, \ldots, a_k\}$ is a set of pairwise non-commuting elements in G of maximal cardinality.*

Proof. This is straightforward. See also [6, Lemma 2.2]. □

Lemma 2.3. [6, Lemma 2.3] *Let G be a group of order p^n with the central quotient of order p^2, where p is a prime number. Then G is an AC-group and $\omega(G) = p + 1$.*

Lemma 2.4. *Let G be a group of order p^n with the central quotient of order p^3, where p is a prime number. Then*

(i) *G is an AC-group.*

(ii) *If G possesses no abelian maximal subgroup, then $\omega(G) = p^2 + p + 1$.*

(iii) *If G possesses an abelian maximal subgroup, then G has exactly one abelian maximal subgroup and $\omega(G) = p^2 + 1$.*

Proof. See [3, Lemma 3.2(i) and Theorem 3.3].

□

Lemma 2.5. *Let $G = A \times B$, where A is an abelian subgroup of G. Then*
$\omega(G) = \omega(B)$.

Proof. This is clear. □

Now following [4, §29], we define isoclinism between two groups and we state that isoclinic groups have the same maximal cardinality of subsets of pairwise non-commuting elements. Two groups G

and H are said to be isoclinic provided there exist two isomorphisms $f : G/Z(G) \to H/Z(H)$ and $f' : G' \to H'$ such that if $f(g_1Z(G)) = h_1Z(H)$ and $f(g_2Z(G)) = h_2Z(H)$, then $f'([g_1, g_2]) = [h_1, h_2]$. In this case we write $G \sim H$.

Lemma 2.6. *Let $G \sim H$. Then*

(i) *G is an AC-group if and only if H is an AC-group,*
(ii) *$\omega(G) = \omega(H)$.*

Proof. (i) Let G be an AC-group. We claim that $\mathcal{C}_H(h)$ is abelian for any $h \in H\backslash Z(H)$. Assume that $h_1, h_2 \in \mathcal{C}_H(h)$. By the above definition we see that there exist g, g_1 and g_2 in G, such that $f(gZ(G)) = hZ(H)$, $f(g_1Z(G)) = h_1Z(H)$ and $f(g_2Z(G)) = h_2Z(H)$. Moreover $f'([g, g_i]) = [h, h_i] = 1$ for $1 \leq i \leq 2$. This implies that $g_1, g_2 \in \mathcal{C}_G(g)$, which completes the proof.
(ii) Let $X = \{a_1, \ldots, a_n\}$ be a subset of pairwise non-commuting elements of G of maximal cardinality. Therefore $a_iZ(G) \neq a_jZ(G)$ for $1 \leq i < j \leq n$. This implies that $f(a_iZ(G)) \neq f(a_jZ(G))$. Let $f(a_iZ(G)) = b_iZ(H)$ for $1 \leq i \leq n$. Therefore $\{b_1, \ldots, b_n\}$ is a subset of pairwise non-commuting elements of H since $1 \neq f'([a_i, a_j]) = [b_i, b_j]$. This yields that $\omega(G) \leq \omega(H)$. Now by the same argument we see that $\omega(H) \leq \omega(G)$, as desired. □

For the rest of this section we give some results for p-groups of maximal class. Let G be a p-group of maximal class and order p^n $(n \geq 4)$, where p is a prime. Following [9], we define the 2-step centralizer K_i in G to be the centralizer in G of $\gamma_i(G)/\gamma_{i+2}(G)$ for $2 \leq i \leq n-2$ and define $P_i = P_i(G)$ by $P_0 = G$, $P_1 = K_2$, $P_i = \gamma_i(G)$ for $2 \leq i \leq n$. The degree of commutativity $l = l(G)$ of G is defined to be the maximum integer such that $[P_i, P_j] \leq P_{i+j+l}$ for all $i, j \geq 1$ if P_1 is not abelian and $l = n-3$ if P_1 is abelian.

Lemma 2.7. *Let G be a p-group of maximal class which possesses an abelian maximal subgroup. Then P_1 is abelian.*

Proof. See [6, Lemma 3.1]. □

Theorem 2.8. [9, Theorem 3.2.11] *Let G be a p-group of maximal class and order p^n where n is odd and $5 \leq n \leq 2p+1$, then G has positive degree of commutativity.*

Corollary 2.9. *Let G be a p-group of maximal class and order p^5.*

(i) *If G possesses an abelian maximal subgroup, then $\omega(G) = p^3 + 1$.*
(ii) *If G possesses no abelian maximal subgroup, then $\omega(G) = p^3 + p + 1$.*

Proof. By Theorem 2.8, we see that $[P_1, P_3] = 1$. Therefore the result follows from [6, Theorems 3.4 and 3.7]. □

3. Main Results

In this section we determine $\omega(G)$ for all non-abelian groups G of order p^n, where $n \leq 5$ in which G is not an extra special group of order p^5. For $n = 3$, we see that $\omega(G) = p+1$ by Lemma 2.3. First for $p = 2$ we use the following program in GAP, which computes $\omega(G)$:

```
LoadPackage("grape");
N:=function(a,b)
return(IsAbelian(Group(a,b)));
end;

NonCommutingGraph:=function(g)
local k, x, y;
k:=Graph(g,Difference(g,Center(g)),OnPoints,function(x,y) return N(x,y)=false;end);
return k;
end;

clique:=function(x)
local G1,G2;
G1:=NonCommutingGraph(x);
G2:=ComplementGraph(G1);
return Size(IndependentSet(G2));
end;

CliqueNumber:=function(x)
local c, t, M;
c:=clique(x);
while c>0  do
t:=c;
M:=CompleteSubgraphsOfGivenSize(NonCommutingGraph(x),c+1,0);
c:=Size(M);
if c=0 then  return(t);  fi;
od;
end;
```

Lemma 3.1. *Let G be a non-abelian group of order 2^4*

(i) *If G is of maximal class, then $\omega(G) = 5$.*

(ii) *If G is of class two, then $\omega(G) = 3$.*

Lemma 3.2. *Let G be a non-abelian group of order 2^5. Then $G = \mathrm{SmallGroup}(2^5, i)$, where $1 \leq i \leq 51$ and $i \notin \{1, 3, 16, 21, 36, 45, 51\}$.*

(i) $\omega(G) = 3$, *where* $i \in \{2,4,5,12,17,22,23,24,25,26,37,38,46,47,48\}$.

(ii) $\omega(G) = 5$, *where* $i \in \{9,10,11,13,14,15,27,28,29,30,31,32,33,34,$
$35,39,40,41,42,49,50\}$.

(iii) $\omega(G) = 6$, *where* $i \in \{6,7,8,43,44\}$.

(iv) $\omega(G) = 9$, *where* $i \in \{18,19,20\}$.

For the rest of the paper we assume that $p > 2$. Now by [10], all groups of order p^n, where $n \in \{4,5\}$ are classified in isoclinic families. In fact we see that there are two non-isoclinic families of non-abelian groups of order p^4 and by Lemma 2.6, it is enough to select one group G from each family and find $\omega(G)$. By the notation of [10], we select $H_1 = \Phi_2(1^4)$ and $H_2 = \Phi_3(1^4)$, which are non-isoclinic groups of order p^4. Similarly by [10], we have eight non-isoclinic families of non-abelian groups of order p^5, which are not extra special. So we select one group G_i from each non-isoclinic families for $1 \le i \le 8$. By using the notation of [10], we select G_i as below:

$G_1 = \Phi_2(311)a$, $G_2 = \Phi_3(1^5)$, $G_3 = \Phi_4(221)a$, $G_4 = \Phi_6(1^5)$,
$G_5 = \Phi_7(1^5)$, $G_6 = \Phi_8(32)$, $G_7 = \Phi_9(2111)a$ $G_8 = \Phi_{10}(1^5)$.

Moreover following [10], $\alpha_{i+1}^{(p)}$ will denote the word $\alpha_{i+1}^{p}\alpha_{i+2}^{\binom{p}{2}}\ldots\alpha_{i+k}^{\binom{p}{k}}\ldots\alpha_{i+p}$ where i is a positive integer and $\alpha_{i+2}\ldots\alpha_{i+p}$ are suitably defined. For economy of space, all relations of the form $[\alpha,\beta] = 1$ (with α,β generators) have been omitted from the any given presentation and should be assumed when reading it. By using Table 4.1 of [10], we list the following properties of groups H_1, H_2 and G_i for $1 \le i \le 8$ in the table below. Moreover $\omega(G)$ is listed bellow for these groups and the proofs are in the following lemmas.

Table 1

Group	$\lvert G/Z(G)\rvert$	G'	$\mathrm{cl}(G)$	$\omega(G)$
H_1	p^2	$\mathbb{Z}_p$	2	$1+p$
H_2	p^3	$(\mathbb{Z}_p)^2$	3	$1+p^2$
G_1	p^2	$\mathbb{Z}_p$	2	$1+p$
G_2	p^3	$(\mathbb{Z}_p)^2$	3	$1+p^2$
G_3	p^3	$(\mathbb{Z}_p)^2$	2	$1+p^2$
G_4	p^3	$(\mathbb{Z}_p)^3$	3	p^2+p+1
G_5	p^4	$(\mathbb{Z}_p)^2$	3	$p(p+1)$
G_6	p^4	$\mathbb{Z}_{p^2}$	3	$p(p+1)$
G_7	p^4	$(\mathbb{Z}_p)^3$	4	p^3+1
G_8	p^4	$(\mathbb{Z}_p)^3$	4	p^3+p+1

Lemma 3.3. *We have* $\omega(H_1) = 1+p$ *and* $\omega(H_2) = 1+p^2$.

Proof. By Lemma 2.3, we have $\omega(H_1) = 1+p$ since $|H_1/Z(H_1)| = p^2$. Moreover by using Table 1, we have $H_2' \cong (\mathbb{Z}_p)^2$ and $|Z(H_2)| = p$ and so we see that $H_2/\mathcal{C}_{H_2}(H_2') \hookrightarrow \mathrm{GL}(2,p)$. Hence we deduce that $\mathcal{C}_{H_2}(H_2')$ is an abelian subgroup of order p^3 in H_2. Therefore Lemma 2.4(iii) completes the proof. □

Corollary 3.4. *Let* G *be a non-abelian group of order* p^4.

(i) *If G is of maximal class, then $\omega(G) = 1 + p^2$.*

(ii) *If G is of class two, then $\omega(G) = 1 + p$.*

Proof. We see that $G \sim H_2$ when G is of maximal class and $G \sim H_1$ when G is of class two by [10, 4.4]. Now the result follows from Lemma 2.6(ii). □

Lemma 3.5. *We have*

(i) $\omega(G_1) = 1 + p$,

(ii) $\omega(G_2) = 1 + p^2$,

(iii) $\omega(G_3) = 1 + p^2$.

Proof. (i) This is evident from Lemma 2.3 and Table 1.
(ii) By [10, 4.4 and 4.5] we may write $G_2 = B \times \mathbb{Z}_p$, where B is of maximal class and order p^4. Therefore $\omega(B) = 1 + p^2$ by Corollary 3.4(i). Hence the result follows from Lemma 2.5.
(iii) By [10, 4.5], we have the following presentation for G_3 :
$G_3 = \langle \alpha, \alpha_1, \alpha_2, \beta_1, \beta_2 | [\alpha_i, \alpha] = \beta_i, \alpha^p = \beta_2, {\alpha_1}^p = \beta_1, {\alpha_2}^p = {\beta_i}^p = 1 \ (i = 1, 2)\rangle$. On Setting $H = \langle \alpha_1, \alpha_2, \beta_1, \beta_2\rangle$, we see that H is an abelian subgroup of order p^4. Hence we may conclude the result by Lemma 2.4(iii) and Table 1. □

Lemma 3.6. *We have $\omega(G_4) = p^2 + p + 1$.*

Proof. By Table 1, we deduce that the Frattini subgroup of G_4 is equal to G_4' and so $\mathrm{d}(G_4) = 2$. Therefore G_4 has $1 + p$ distinct maximal subgroups. By [10, 4.5], G_4 has the following presentation:

$$G_4 = \langle \alpha_1, \alpha_2, \beta, \beta_1, \beta_2 | [\alpha_1, \alpha_2] = \beta, [\beta, \alpha_i] = \beta_i, \alpha_i^p = \beta^p = \beta_i^p = 1 \ \ (i = 1, 2)\rangle.$$

On setting $M_0 = \langle \alpha_2, \beta, \beta_1, \beta_2\rangle$ and $M_i = \langle \alpha_1\alpha_2^i, \beta, \beta_1, \beta_2\rangle$ for $1 \le i \le p$. It is easy to check that these non-abelian maximal subgroups are distinct. Therefore the result follows from Lemma 2.4(ii). □

Lemma 3.7. *We have $\omega(G_5) = p(p + 1)$.*

Proof. By [10, 4.5], G_5 has the following presentation :
$G_5 = \langle \alpha, \alpha_1, \alpha_2, \alpha_3, \beta | [\alpha_i, \alpha] = \alpha_{i+1}, [\alpha_1, \beta] = \alpha_3, \alpha^p = \alpha_1^{(p)} = \alpha_{i+1}^p = \beta^p = 1 \ \ (i = 1, 2)\rangle$. For $p = 3$, by using GAP we see that $G_5 = \mathrm{SmallGroup}(243, 58)$ and $\omega(G_5) = 12$. For $p > 3$, we have $\alpha_1^{(p)} = \alpha_1^p = 1$ and by using Table 1, we see that $G_5' = \langle \alpha_2, \alpha_3\rangle$, $C_{G_5}(G_5') = \langle \alpha_1, \alpha_2, \alpha_3, \beta\rangle$ and $Z(G_5) = \langle \alpha_3\rangle$. Now on setting $M_0 = \langle \alpha_1, \alpha_2, \alpha_3, \beta\rangle$ and $M_i = \langle \alpha\alpha_1^i, \alpha_2, \alpha_3, \beta\rangle$ for $1 \le i \le p$, it is easy to check that these non-abelian maximal subgroups are distinct and $M_i \cap M_j = K$ for $0 \le i \ne j \le p$, where $K = \langle \alpha_2, \alpha_3, \beta\rangle$ and so $G = \bigcup_{i=0}^{p} M_i$. Obviously $Z(M_0) = \langle \alpha_2, \alpha_3\rangle$, $Z(M_i) = \langle \alpha_2^i\beta, \alpha_3\rangle$ for $1 \le i \le p$ and $Z(M_i) \cap Z(M_j) = Z(G)$ for $0 \le i \ne j \le p$. Hence $K = \bigcup_{i=0}^{p} Z(M_i)$. By Lemma 2.3, we have $\omega(M_i) = 1 + p$. Let $T = \{x_1, \ldots, x_m\}$ be a subset of pairwise non-commuting elements in G_5. We claim that $|T \cap (M_i \setminus K)| \le p$, for otherwise suppose that $x_1, \ldots, x_{1+p} \in M_i \setminus K$. Hence $M_i = \bigcup_{j=1}^{p+1} \mathcal{C}_{M_i}(x_j)$ since $\omega(M_i) = 1 + p$. By the fact that $x_j \notin K$ we have $Z(M_i) \le \mathcal{C}_{M_i}(x_j) \cap K \lneq K$ and so $Z(M_i) = \mathcal{C}_{M_i}(x_j) \cap K$ for $1 \le j \le 1 + p$. Therefore $K = K \cap M_i = \bigcup_{j=1}^{p+1}(\mathcal{C}_{M_i}(x_j) \cap K) = Z(M_i)$, which is impossible. By the fact that K is abelian we have $|T \cap K| \le 1$. If $|T \cap K| = 1$ then $m \le p^2 + 1$

since $K = \bigcup_{i=0}^{p} Z(M_i)$ and if $|T \cap K| = 0$ we deduce that $m \leq p(p+1)$. Therefore $\omega(G_5) \leq p(p+1)$. On setting $A = \{\alpha\alpha_1^j\beta^i, \alpha_1\beta^i, \alpha\alpha_2^i\beta\}$ for $0 \leq i \leq p-1$ and $1 \leq j \leq p-1$ it is easy to check that A is a subset of pairwise non-commuting elements of G_5 of order $p(p+1)$, which completes the proof. □

Lemma 3.8. *We have*

(i) $\omega(G_6) = p(p+1)$,

(ii) $\omega(G_7) = p^3 + 1$,

(iii) $\omega(G_8) = p^3 + p + 1$.

Proof. (i) By [10, 4.5], we have the following presentation for G_6:

$$G_6 = \langle \alpha_1, \alpha_2, \beta | [\alpha_1, \alpha_2] = \beta = {\alpha_1}^p, \beta^{p^2} = {\alpha_2}^{p^2} = 1 \rangle.$$

Therefore $G_6 = \langle \alpha_1, \alpha_2 \rangle$, $\langle \alpha_1 \rangle \trianglelefteq G_6$ and $G_6/\langle \alpha_1 \rangle$ is cyclic. The rest follows from Table 1 and [7, Theorem 1.1], .

(ii) By Table 1, G_7 is of maximal class. Also by using [10, 4.5], we see that $G_7 = \langle \alpha, \alpha_1, \alpha_2, \alpha_3, \alpha_4 | [\alpha_i, \alpha] = \alpha_{i+1}, \alpha^p = \alpha_4, {\alpha_1}^{(p)} = {\alpha_{i+1}}^{(p)} = 1 \ (i = 1, 2, 3) \rangle$. On Setting $H = \langle \alpha_1, \alpha_2, \alpha_3, \alpha_4 \rangle$, we deduce that H is abelian and $|G_7/H| = p$, which completes the proof by using Corollary 2.9(i).

(iii) By Table 1, G_8 is of maximal class. Also by [10, 4.5], we have the following presentation for G_8: $G_8 = \langle \alpha, \alpha_1, \alpha_2, \alpha_3, \alpha_4 | [\alpha_i, \alpha] = \alpha_{i+1}, [\alpha_1, \alpha_2] = \alpha_4, \alpha^p = {\alpha_1}^{(p)} = {\alpha_{i+1}}^{(p)} = 1 \ (i = 1, 2, 3) \rangle$. Moreover $\gamma_i(G) = \langle \alpha_i, \ldots, \alpha_4 \rangle$ for $2 \leq i \leq 4$ and so $P_1 = \langle \alpha_1, \alpha_2, \alpha_3, \alpha_4 \rangle$ since $[\alpha_1, \gamma_2(G)] \leq \gamma_4(G)$. This shows that P_1 is not abelian and the result follows from corollaries 2.7 and 2.9(ii).

□

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