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NON-NILPOTENT GROUPS WITH THREE CONJUGACY CLASSES OF NON-NORMAL SUBGROUPS

HAMID MOUSAVI

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ABSTRACT. For a finite group G let $\nu(G)$ denote the number of conjugacy classes of non-normal subgroups of G. The aim of this paper is to classify all the non-nilpotent groups with $\nu(G) = 3$.

1. **Introduction**

Let G be a finite group. We denote by $\nu(G)$ the number of conjugacy classes of non-normal subgroups of G. Obviously, $\nu(G) = 0$ if and only if G is Hamiltonian. In 1995, Brandl [1] classified finite groups with $\nu(G) = 1$. In 1977, R. La Haye [2] showed that for a group G, $|G'| \leq \rho(G)^{\nu(G)+\varepsilon}$ and $|G/Z(G)| \leq \rho(G)^{\nu(G)+\varepsilon+1}$ where $\rho(G)$ denotes the largest prime p such that G has an element of order p, and $\varepsilon = 1$ if G has an element of order 2, and $\varepsilon = 0$ otherwise. This goes to say that $\nu(G)$ can play an important role in the structure of finite groups, so many authors work on this concept but mostly in p-groups.

The present author in [3] gave a complete classification of finite groups with $\nu(G) = 2$. The aim of this paper is classifying finite non-nilpotent groups with $\nu(G) = 3$. The following theorem is the main result of this paper.

Theorem 1.1. *Let G be a non-nilpotent finite group with $\nu(G) = 3$. Then G is isomorphic to one of the following groups:*

(i) $SL(2, 3)$.
(ii) $\mathbb{Z}_3 \times S_3$.

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(iii) $\langle x, y, z \mid x^{p^n}, y^{q^2}, z^r, [x, y], [x^p, z], [y, z], z^x = z^i \rangle$, *where* p, q *and* r *are different primes,* $p \mid r-1$ *and* $i^p \equiv 1 (\mod r)$.

(iv) $\langle x, y, z \mid x^p, y^q, z^r, [x, y], z^x = z^i, z^y = z^j \rangle$, *where* p, q *and* r *are different primes,* $pq \mid r-1$ *and* $i^p \equiv 1 (\mod r)$, $j^q \equiv 1 (\mod r)$.

(v) $\langle x, y, z \mid x^{p^n}, y^q, z^r, [x^p, y], [x^p, z], [y, z], y^x = y^i, z^x = z^j \rangle$ *where* p, q *and* r *are different primes,* $i^p \equiv 1 (\mod q)$, $j^p \equiv 1 (\mod r)$ *and* $p \mid (q-1, r-1)$.

(vi) $\langle x, y \mid x^{p^n}, y^{q^3}, [x^p, y], y^x = y^i \rangle$, *where* p, q *are primes,* $p \mid (q-1)$ *and* $i^p \equiv 1 \mod q^3$.

(vii) $D_{4q} = \langle x, y, z \mid x^2, y^2, z^q, (xz)^2, [x, y], [y, z] \rangle$, *where* $q \neq 2$ *is prime.*

(viii) $\langle x, y, z \mid x^4, y^2, z^q, z^x z, [y, z], [x, y] \rangle$, *where* $q \neq 2$ *is prime.*

(ix) $\langle x, y, z \mid x^4, y^4, z^q, x^2y^2, z^x z, x^y x, [z, y] \rangle$, *where* $q \neq 2$ *is prime.*

(x) $\langle x, y \mid x^{p^n}, y^q, [x^{p^3}, y], y^x = y^i \rangle$, *where* p, q *are primes,* $p^3 \mid q-1$ *and* $i^{p^3} \equiv 1 \mod q$, $i^{p^2} \not\equiv 1 \mod q$.

(xi) $\langle x, y, z \mid x^9, y^2, z^2, y^x z, z^x yz, [z, y] \rangle$.

(xii) $(\mathbb{Z}_q \times \mathbb{Z}_q) \rtimes \mathbb{Z}_p$, *where* p, q *are odd primes and* $q = 2p-1$ *and* P *acts irreducibly.*

(xiii) $(\mathbb{Z}_q \times \mathbb{Z}_q \times \mathbb{Z}_q) \rtimes \mathbb{Z}_p$, *where* p, q *are primes and* $p = q^2 + q + 1$ *and* P *acts irreducibly.*

Our notation is standard and can be found in [4]. Throughout this paper, we use $\mathbb{Z}_n$, D_{2n} and Q_8 for cyclic group of order n, the dihedral group of order $2n$ and quaternion group of order 8, respectively.

2. **Preliminaries**

We will repeatedly use the following simple lemmas without even mentioning them.

Lemma 2.1. *Let* G *be a finite group and* $H \ntrianglelefteq G$. *Then* H *has at most one maximal subgroup which is normal in* G.

Lemma 2.2. *Let* G *be a finite group and* $H \leqslant G$ *has normal complement* N *in* G. *Then for any normal subgroup* K *of* H, $KN \trianglelefteq G$.

Lemma 2.3 (Burnside)**.** [4, Theorem, 7.50] *Let* G *be a finite group and* P *be a Sylow* p*-subgroup of* G. *If* P *contained in the center of its normalizer, then* P *has a normal complement in* G.

We first show that every group presented in Theorem 1.1 has three conjugacy classes of non-normal subgroups.

Theorem 2.4. *For every group presented in Theorem 1.1,* $\nu(G) = 3$.

Proof. Let G be one of the groups presented in Theorem 1.1. Then there exists a Sylow p-subgroup P of G such that $P \ntrianglelefteq G$ and P has a normal complement N; as well as P is cyclic except the groups (vii), (viii) and (ix) (in these cases $P \cong \mathbb{Z}_2 \times \mathbb{Z}_2$, $\mathbb{Z}_4 \times \mathbb{Z}_2$ and $P = Q_8$ respectively) and N is cyclic except groups (i), (ii), (iv), (xi), (xii) and (xiii) (in this cases $N \cong Q_8$, $Z_3 \times \mathbb{Z}_3$, $\mathbb{Z}_r \rtimes \mathbb{Z}_q$, $\mathbb{Z}_2 \times \mathbb{Z}_2$, $\mathbb{Z}_q \times \mathbb{Z}_q$ and $\mathbb{Z}_q \times \mathbb{Z}_q \times \mathbb{Z}_q$, respectively). Also G' is cyclic except the groups (i), (xi), (xii) and (xiii), and $G' = N$ except the groups (ii), (iii), (iv), in which cases $G' \lneqq N$ is of prime order and group (ix) that is $N \lneqq G'$.

Let H be a non-normal subgroup which is not conjugate with P. Then $H = (H \cap P)(H \cap N)$. In the groups (xii) and (xiii), P is maximal of order p and acts irreducibly on proper subgroups of N; hence $H \leqslant N$. For the group (xii), $|H| = q$ and N contains two conjugacy class of non-normal subgroups of size p; and for the group (xiii), $|H| = q$ or q^2 and in each case N contains just one class of non-normal subgroups of size p. We can see obviously that the group (xi) has non-normal subgroups of order 2 and 6.

In the groups (vii),..., (x), N is normal of order q (in the group (vii), (viii) and (x), $N = G'$ and in the group (ix), N is characteristic in G'); so $H \leqslant P$. For the group (x), H is a maximal or second maximal subgroup of P, which in either case constitutes just one conjugacy class of size q; and for the groups (vii), (viii) and (ix), P has three maximal subgroups and two of them can not be normal. Assume that M_1 and M_2 are maximal subgroups of P which are non-normal in G. Then M_1 and M_2 can not be conjugate, otherwise $P = M_1M_2 \leqslant NM_1 \trianglelefteq G$ so $NM_1 = G$ which is a contradiction.

Since the maximal subgroup of P in the group (vi) is normal and N is cyclic of order q^3, then any subgroup of P and N is normal in G. Therefore H must contain P and so $|H| = p^nq$ or p^nq^2, hence H can not be normal and constitutes just one conjugacy class of size q, in each case.

In the group (v) every subgroup of P and N is normal in G then H must contain P. Therefore H is one of two maximal subgroups of G of index q or r. Since $P = \mathcal{N}_G(P)$, in each case $H \ntrianglelefteq G$ and constitutes just one conjugacy class of size q or r.

Obviously, in the group (iv), G' is a group of order r in N. Hence, either $H \leqslant N$ or $P \leqslant H$. Therefore, either H is the normalizer of P which is a maximal subgroup of index r or a subgroup of order q in N.

In the group (iii), again every proper subgroup of P and N is normal in G and $N_G(P)$ is a maximal subgroup of index r. Therefore, $G = \mathcal{N}_G(P)G'$ and $P \lneqq H \leqslant \mathcal{N}_G(P)$. Hence, either $H = \mathcal{N}_G(P)$ or H is a subgroup of index rq which is not normal.

Trivially, in the group (ii), $N = Z(G) \times G'$ so either $H = Z(G) \times P = \mathcal{N}_G(P)$ which is maximal of index 3 or H is maximal in N which is not equal with Z(G) or G'. In the latter case $N = \mathcal{N}_G(H)$ is a maximal subgroup of index 2.

Finally, in the group (1), $Z(G) \lneqq G' = N = Q_8$, hence either $H = Z(G) \times P = \mathcal{N}_G(P)$ which is maximal of index 4 or H is a maximal subgroup of N. □

3. **The Classification Theorems**

Let G be a non-nilpotent finite group and P be a non-normal Sylow p-subgroup of G. Then $\mathcal{N}_G(P) \ntrianglelefteq G$. Suppose that M is a maximal subgroup of G containing $\mathcal{N}_G(P)$. By the Frattini argument $M \ntrianglelefteq G$. Now we can distinguish four cases as follows:

(1) $P \lneqq \mathcal{N}_G(P) \lneqq M$;

(2) $P \lneqq \mathcal{N}_G(P) = M$;

(3) $P = \mathcal{N}_G(P) \lneqq M$;

(4) $P = \mathcal{N}_G(P) = M$.

Since $\nu(G) = 3$, in cases (1), (2) and (3), P has at most one maximal subgroup which is non-normal in G, and in case (4), at most two maximal subgroups. Hence in cases (1), (2) and (3), P is cyclic and in case (4), P is cyclic if p is an odd number or $P/\Phi(P) \ncong \mathbb{Z}_2 \times \mathbb{Z}_2$. So in cases (1), (2) and (3), P has normal complement N in G. Indeed, let L be a complement of P in $\mathcal{N}_G(P)$. If $L \trianglelefteq G$ then $L \leqslant \mathcal{C}_G(P)$, otherwise P must be maximal in $\mathcal{N}_G(P)$ and for any $y \in G\backslash M$, $\langle y\rangle \trianglelefteq G$ so the Hall p'-subgroup of $\langle y\rangle$ must be contained in $\mathcal{C}_G(P)$. Therefore $P \lneqq \mathcal{C}_G(P)$, so we have $P \leqslant Z(\mathcal{N}_G(P))$. In case (4) if P is abelian then has a normal complement. Also in cases $P \lneqq \mathcal{N}_G(P)$ we have $L = N \cap \mathcal{N}_G(P)$ is a normal complement of P in $\mathcal{N}_G(P)$.

We fixed the above notation for the following theorems.

Theorem 3.1. *Let G be a finite non-nilpotent group with $\nu(G) = 3$. Then case (1) can not occur.*

Proof. In this case P is a maximal subgroup of $\mathcal{N}_G(P)$ and $\mathcal{N}_G(P)$ is a maximal subgroup of M. Since $\mathcal{N}_G(P) = PL$, then $|L| = q$ for some prime number $q \neq p$. If N has a subgroup L_1 of prime order different from L, we must have $PL_1 \trianglelefteq G$ so $G = L_1\mathcal{N}_G(P)$ and $\mathcal{N}_G(P)$ must be maximal in G, a contradiction. Therefore, $L \leqslant N$ is the only subgroup of prime order. Now from $L \leqslant \mathcal{C}_N(P)$ we have $N = [N,P]\mathcal{C}_N(P)$, again this is a contradiction because N can be neither cyclic nor quaternion. □

Theorem 3.2. *Let G be a finite non-nilpotent group with $\nu(G) = 3$ which satisfies (2). Then G is isomorphic to one of the following groups:*

(i) $SL(2,3)$.

(ii) $\mathbb{Z}_3 \times S_3$.

(iii) $\langle x,y,z \mid x^{p^n}, y^{q^2}, z^r, z^x = z^i, [x^p,z], [x,y], [z,y]\rangle$, *where p, q and r are different primes, $p \mid r-1$ and* $i^p \equiv 1 \mod r$.

(iv) $\langle x,y,z \mid x^p, y^q, z^r, [x,y], z^x = z^i, z^y = z^j\rangle$, *where p, q and r are different primes, $pq \mid r-1$ and* $i^p \equiv 1 \mod r$, $j^q \equiv 1 \mod r$.

Proof. Let H be a non-normal subgroup of G which is not conjugate with P and M. Firstly we assume that M dose not contain any conjugate of H, then $L \trianglelefteq G$ and $[P,L] = 1$. If K is a non-trivial proper subgroup of L, then $PK \trianglelefteq G$ so $P \trianglelefteq G$, which is imposable. Therefore $|L| = q$ for some prime number $q \neq p$. Also we can write $H = H_PH_N$ where H_P is a Sylow p-subgroup of H with normal complement H_N. If $H_P \neq 1$, then any subgroup of N is normal in G so $H_N \trianglelefteq G$ and we must have $H_P = P^g$ for some $g \in G$ (otherwise $H_P \trianglelefteq G$ and $H \trianglelefteq G$). Since $M^g \nleqslant H$, then $L \cap H = 1$, hence $N = L \times H_N$ is abelian. Therefore $L \leqslant Z(G)$ and $H = PH_N \trianglelefteq G$, a contradiction.

Now we assume that $H_P = 1$. Then $H \leqslant N$ and N is non-cyclic. If $\Phi(P) \neq 1$, then $H\Phi(P) \trianglelefteq G$ and so $H = N \cap H\Phi(P) \trianglelefteq G$, again we have contradiction, hence $|P| = p$. Since any subgroup of N which contains L can not be normal in G, then either L is maximal in N and $L \cap H = 1$ or $L \leqslant H$, H is cyclic of order q^2 and $|N| = q^3$. In the later case $L = \Omega_1(N) \trianglelefteq G$ and $N \cong Q_8$, so $p = 3$, because $(P \hookrightarrow S_4)$. Therefore $G \cong SL(2,3)$.

In the first case $|H| = r$ is prime. If $r \neq q$, as $H \ntrianglelefteq N$, then $|G : \mathcal{N}_G(H)| = q$, therefore $\mathcal{N}_G(H)$ must be normal in G, hence $H \trianglelefteq G$ and we reach a contradiction. Therefore $r = q$ and $L \leqslant Z(G)$. As $|G : \mathcal{N}_G(H)| = p$ and N contains $q + 1$ subgroups of order q, we must have $p = q - 1$, because N dose not contains two maximal subgroups L_1 and L_2 such that PL_1 and PL_2 are normal in G (otherwise $P = PL_1 \cap PL_2 \trianglelefteq G$ a contradiction). Therefore $p = 2$, $q = 3$ and $G \cong L \times [N, P]P \cong \mathbb{Z}_3 \times S_3$.

Now we consider M contains a conjugate of H. Without lose of generality we can assume that $H \leqslant M$. Let $|L|$ is not prime and $K \leqslant L$ be of prime order. Since $PK \ntrianglelefteq G$, then we can assume that $H = PK$. As H is maximal and normal in M, then H is the only maximal subgroup of M which contains P, therefore L is a cyclic normal subgroup of order q^2. Since M is maximal in G and any subgroup of N is normal in G, so L is a maximal subgroup of prime index r in N. If $r = q$, then $|\Omega_1(N)| = q^2$ and we can find subgroups L_1 and L_2 different from L, such that PL_1 and PL_2 are normal in G, which implies that $P \trianglelefteq G$. Therefore $r \neq q$ and N is cyclic. So that $G \cong [N, P]P \times L$ and is a group presented in (iii).

Now we assume that $|L| = q$ is a prime. Hence P is maximal in M. If $L \nleqslant H$, then $H \leqslant P$, as P is cyclic and normal in M, then $H \trianglelefteq M$. So LH is normal in G and contains H as a characteristic subgroup, hence $H \trianglelefteq G$, a contradiction. Therefore $L \leqslant H$. If $L \trianglelefteq G$ then $H = (H \cap P)L \trianglelefteq G$ which is impossible, therefore $H = L \ntrianglelefteq G$ is of prime order q. If $\Phi(P) \neq 1$ then $H\Phi(P) \trianglelefteq G$ which implies the contradiction $H \trianglelefteq G$, because $[L, P] = 1$. Therefore, $\Phi(P) = 1$ and $|P| = p$. Since L is maximal in N and N can not be abelian (otherwise $L \trianglelefteq G$), so $|N| = qr$ for some prime number $r \neq q$. Assume that $Z \leqslant N$ is of prime order r, then $Z \trianglelefteq G$. Therefore, $G \cong Z \rtimes (P \times L)$ is a group presented in (iv). □

Theorem 3.3. *Let G be a finite non-nilpotent group with $\nu(G) = 3$ which satisfies (3). Then G is isomorphic to one of the following groups:*

(v) $\langle x, y, z \mid x^{p^n}, y^q, z^r, [x^p, y], [x^p, z], [y, z], y^x = y^i, z^x = z^j \rangle$ *where p, q and r are different primes, $i^p \equiv 1 \mod q$, $j^p \equiv 1 \mod r$ and $p \mid (q - 1, r - 1)$.*

(vi) $\langle x, y \mid x^{p^n}, y^{q^3}, [x^p, y], y^x = y^i \rangle$, *where p, q are primes, $p \mid (q - 1)$ and $i^p \equiv 1 \mod q^3$.*

Proof. Assume first that any subgroup of M which is not conjugate with P, is normal in G. Then P is a maximal subgroup of M and $N \cap M$ is a normal subgroup of G of prime order $q \neq p$. We set $K = M \cap N$, so $M = PK$ and for any $g \in G$, $M^g = P^gK$. Since $P = \mathcal{N}_G(P)$, than $Z(G) \leqslant P$ and any subgroup of G which contains P must be non-normal. Therefore either N just contains two subgroups which are normal in G or K is the only P-invariant subgroups of N. In the latter case N contains a non-normal subgroup of G, otherwise N is cyclic of order q^2 and $\nu(G) = 2$. Hence N is non-cyclic, also similar to proof of pervious theorem we have $\Phi(P) = 1$. If N is a prime power order then either $K = \Omega_1(N)$, so N is generalized quaternion, thus $K \leqslant Z(G)$, a contradiction, or $N = \Omega_1(N)$ then $N \cong \mathbb{Z}_q \times \mathbb{Z}_q$ and contains $q + 1$ subgroups of order q, then N contains at least two P-invariant subgroups, because $K \trianglelefteq G$. We reach a contradiction. If N is not prime power order, then N contains a non-normal subgroup with q conjugates that one of them must be P-invariant, again we have a contradiction.

Therefore N contains exactly two proper non-trivial subgroups which are normal in G. So $|N| = qr$ for some prime $r \neq q$ and $p \mid (q-1, r-1)$. Therefore, G is a group presented in (v).

Now let $H \lneqq M$ be a non-normal subgroup of G which is not conjugate with P. If N is not prime power order then N contains two subgroups K_1 and K_2 of distinct prime order q and r, respectively. Since one of the K_1 or K_2 must be normal in G, say K_1, then $PK_1 \ntrianglelefteq G$ so $K_1 \leqslant M$. Either $M = PK_1$ then $r \nmid |M|$ and we have $K_2 \trianglelefteq G$ or $PK_1 \lneqq M$ then again $K_2 \trianglelefteq G$, because $\nu(G) = 3$, therefore $PK_2 \trianglelefteq G$ and we reach a contradiction by Frattini argument. Hence N is of prime power order. Let K be a maximal subgroup of N. If $K \trianglelefteq G$ then $PK \ntrianglelefteq G$ so $P^gK \leqslant M$ for some $g \in G$, hence $K = M \cap N$. Otherwise $K \ntrianglelefteq G$ so $H = K^x \leqslant M$ for some $x \in P$, again we must have $K = M \cap N$. Therefore $M \cap N$ is the only maximal subgroup of N, which implies that N is cyclic.

If P is maximal in M, then $|M \cap N| = q$ is a prime number. If $M \cap N \leqslant H$ than $H = H_P H_N \trianglelefteq G$, because $H_N = M \cap N$ and $|H_P| < |P|$. So $H_N = 1$ and $H(M \cap N) \trianglelefteq G$. By Frattini argument $G = (M \cap N)\mathcal{N}_G(H)$. Hence $N = (M \cap N)(N \cap \mathcal{N}_G(H))$, as N is cyclic, so $G = \mathcal{N}_G(H)$, which is impossible. Therefore H is a maximal subgroup of M which contains P. Since H must be maximal in M so $|N| = q^3$ and G is a group presented in (vi). □

Theorem 3.4. *Let G be a finite non-nilpotent group with $\nu(G) = 3$ which satisfies (4). Then G is isomorphic to one of the following groups:*

(vii) $D_{4q} = \langle x, y, z \mid x^2, y^2, z^q, (xz)^2, [x, y], [y, z]\rangle$, *where $q \neq 2$ is prime.*
(viii) $\langle x, y, z \mid x^4, y^2, z^q, z^x z, [y, z], [x, y]\rangle$, *where $q \neq 2$ is prime.*
(ix) $\langle x, y, z \mid x^4, y^4, z^q, x^2y^2, z^x z, x^y x, [z, y]\rangle$, *where $q \neq 2$ is prime.*
(x) $\langle x, y \mid x^{p^n}, y^q, [x^{p^3}, y], y^x = y^i\rangle$, *where p, q are primes, $p^3 \mid q-1$ and $i^{p^3} \equiv 1 \mod q$, $i^{p^2} \not\equiv 1 \mod q$.*
(xi) $\langle x, y, z \mid x^9, y^2, z^2, y^x z, z^x yz, [z, y]\rangle$.
(xii) $(\mathbb{Z}_q \times \mathbb{Z}_q) \rtimes \mathbb{Z}_p$, *where p, q are primes and $q = 2p - 1$.*
(xiii) $(\mathbb{Z}_q \times \mathbb{Z}_q \times \mathbb{Z}_q) \rtimes \mathbb{Z}_p$, *where p, q are primes and $p = q^2 + q + 1$.*

Proof. By the assumption $G = PN$. First we consider that $P/\Phi(P) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. Note that P has two maximal subgroups H and K that are not normal in G. Since $\nu(G) = 3$, both H and K are cyclic; and $[\Phi(P), N] = 1$. If P is non-abelian then $Z(P) = \Phi(P)$ and $|P'| = 2$, so P is a minimal non-abelian 2-group. Also $|N| = q$ is a prime, because any subgroup of N must be normal in G. Suppose that x, y and z are generators of H, K and N respectively. Obviously $z^x = z^y = z^{-1}$, hence $[z, xy] = 1$. If $\Phi(P) = 1$ then $x^2 = y^2 = 1$ and $G \cong \mathbb{Z}_q \rtimes (\mathbb{Z}_2 \times \mathbb{Z}_2)$. As $P = \langle x, xy\rangle$ and $[z, xy] = 1$, so G is the group presented in (vii).

Suppose that $\Phi(P) \neq 1$. If $x^2 \neq y^2$ then $\langle y^2x\rangle \ntrianglelefteq G$, because $x^z = xz^2$. Suppose that $x^2 = y^2$ and $x^4 \neq 1$ then $\langle x^{-1}yt\rangle \ntrianglelefteq P$, where $t^2 = [x, y]$, because $(x^{-1}yt)^2 = 1$ and $(x^{-1}yt)^x = x^{-1}yt[x, y]$. Therefore, $x^2 = y^2$ and $x^4 = 1$; also $\Phi(P) = \langle x^2\rangle = Z(G)$. Hence $|P| = 8$ and either $P \cong \mathbb{Z}_4 \times \mathbb{Z}_2$ or $P = Q_8$. In the first case $(xy^{-1})^2 = 1$, $[z, xy^{-1}] = 1$ and $P = \langle x, xy^{-1}\rangle$ and the second case $|xy| = 4$, $[z, xy] = 1$ and $P = \langle x, xy\rangle$. So G is the group presented in (viii) or (ix), respectively.

Now consider the case $P/\Phi(P) \ncong \mathbb{Z}_2 \times \mathbb{Z}_2$, so P is cyclic with normal complement N. Since P is maximal then N does not have any subgroup which is normal in G, hence is characteristically simple. Since N contains at most two conjugacy classes of non-normal subgroups of G, then $|N|$ can have at most two prime divisors, thus N must be solvable and so elementary abelian with $|N| \leq q^3$ for some prime number $q \neq p$. Also, we have $|N| = q$ if and only if the maximal and second maximal subgroup of P are non-normal in G. In the sequel we suppose that H and K are non-conjugate non-normal subgroups of G different from P.

First let any subgroup of N be normal in G, then as P is maximal in G, we have $|N| = q$. We can assume that $H \leqslant K \leqslant P$ and H is maximal in K and K is maximal in P. Suppose that $P = \langle x \rangle$, then $x^{p^3} \in \mathcal{C}_P(N)$. Therefore, $G \cong N \rtimes P$ and is the group presented in (x).

Now suppose that N contains at least a subgroup which is not-normal in G so we can assume that $H \leqslant N$ and $|N| > q$. If $1 \neq L \leqslant P$ such that $L \trianglelefteq G$, from $[L, N] = 1$ we have LH contains H as characteristic subgroup, so $LH \ntrianglelefteq G$. Now we can assume that $K = LH$. Therefore $|P| = p^2$ and also $|N| = q^2$, because H must be maximal in N. Since P acts transitively on $q + 1$ subgroups of N and $[L, H] = 1$ we must have $p = q + 1$ and so $p = 3$, $q = 2$. Therefore $|G| = 36$ and $G \cong (\mathbb{Z}_2 \times \mathbb{Z}_2) \rtimes \mathbb{Z}_9$ is the group presented in (xi).

Finally, suppose that $L = 1$, so $|P| = p$; then both of H and K are subgroups of N and we can assume that $|H| = q$. If all non-normal subgroups of G except P have same order q, then $|N| = q^2$ and $q + 1 = 2p$. Since P acts irreducibly on N, then p must be an odd prime number. Therefore, $G \cong (\mathbb{Z}_q \times \mathbb{Z}_q) \rtimes \mathbb{Z}_p$ and G is the group presented in (xii). Otherwise, $|N| = q^3$ and K is maximal in N. So $q^2 + q + 1 = p$ and $G \cong (\mathbb{Z}_q \times \mathbb{Z}_q \times \mathbb{Z}_q) \rtimes \mathbb{Z}_p$ is the group presented in (xiii). □

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Hamid Mousavi

Department of Mathematics, University of Tabriz, P.O.Box 5166617766, Tabriz, Iran

Email: `hmousavi@tabrizu.ac.ir`