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GROUP ACTIONS RELATED TO NON-VANISHING ELEMENTS

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Abstract. We characterize those groups G and vector spaces V such that V is a faithful irreducible G-module and such that each v in V is centralized by a G-conjugate of a fixed non-identity element of the Fitting subgroup $F(G)$ of G. We also determine those V and G for which V is a faithful quasi-primitive G-module and $F(G)$ has no regular orbit. We do use these to show in some cases that a non-vanishing element lies in $F(G)$.

1. Introduction

Suppose a group G acts irreducibly and faithfully on a (finite) vector space V and there is an $x \neq 1$ in $F(G)$ such that every element of V is centralized by a G-conjugate of x (or equivalently, x centralizes an element of each G-orbit on V). If the action of G is primitive, then $|V|$ must be $5^2, 3^4$, or q^2 for a Mersenne prime q and the structure of G is very limited.

Using wreath products and the above results, it is easy to construct imprimitive G-modules with this property. But that is all; i.e. all such actions are subgroups of wreath products involving a quasi-primitive action of this type.

This question arose in [4] (for solvable G), where it was conjectured that a non-vanishing element of a solvable group must lie in the Fitting subgroup of G. Recall x in G is non-vanishing if $\chi(x) \neq 0$ for all $\chi \in \mathrm{Irr}(G)$. A key result of [4] shows that the square of a non-vanishing non-identity element of a solvable group is in $F(G)$. We offer a few modest results towards this conjecture. E.G. This conjecture is true for solvable G except possibly when $|G|$ is divisible by 2 and a Mersenne prime.

The question at hand is related to a question about regular orbits of $F(G)$. In the above situation, $F(G)$ does not have a regular orbit on V. We determine those G that act primitively on a vector

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space in which $F(G)$ has no regular orbit in Theorem 2.2. Here $|V| = 5^2, 3^4, 3^8$, or q^2 for a Mersenne prime q. We do not require solvability for Theorems 2.1 and 2.2. Recall that V is a quasi-primitive G-module if V_N is homogeneous for all normal subgroups N of G. Via Clifford's Theorem, primitive implies quasi-primitive.

If V is a finite vector space of order q^m over a field F of order q, then the elements of V may be labeled by $\mathrm{GF}(q^m)$ consistently with the action of F. The set $\Gamma(q^m) = \{x \to ax^\sigma | a \in \mathrm{GF}(q^m) - \{0\}, \sigma \in \mathrm{Gal}(\mathrm{GF}(q^m)/F)\}$ is a group of linear transformations of V with a normal cyclic group Γ_0 "of multiplications" of order $q^m - 1$. Also $\Gamma/\Gamma_0 \cong \mathrm{Gal}(\mathrm{GF}(q^m)/F)$ is cyclic of order m and a generator of this group induces the automorphism $a \to a^q$ on Γ_0. In particular, $\Gamma(q^2)$ has a cyclic normal subgroup Γ_0 of index two and center of order $q - 1$.

2. **Results**

Theorem 2.1. *Suppose that V is a faithful quasi-primitive G-module and there exists $1 \neq x \in F(G)$ such that each element of V is centralized by a G-conjugate of x. Then*

- *(a) $|V| = q^2$ for a Mersenne prime q and $G = F(G) = P \times S \subseteq \Gamma(V)$ where S is cyclic of odd order and P is the Sylow-2-subgroup of $\Gamma(V)$ that is semi-dihedral of order $4(q+1)$;*
- *(b) $|V| = 5^2$ and $F(G) = QZ$ for normal subgroups $Q \cong Q_8$ and $Z \cong Z_4$ of G with $Q \cap Z = Z(Q)$ while $G/F(G) \cong Z_3$ or S_3;*
- *(c) $|V| = 3^4$ and $F(G)$ is isomorphic to a central product $Q_8 \mathbin{\mathrm{Y}} D_8$ and $G/F(G)$ is isomorphic to Z_5, D_{10}, the Frobenius group F_{20}, A_5, or S_5.*

In all cases, $C_{F(G)}(v) =< x >$ for an involution x whenever $v \in V^\# = V - \{0\}$.

Each of these conclusions does exist. Indeed G acts transitively on the non-zero vectors of V, save possibly in (a) if G is proper in Γ. Hence the semi-direct product VG is a 2-transitive permutation group. Huppert [1] classified and constructed the solvable 2-transitive groups. The three solvable groups in (c) are known as Bucht groups. Note that $Q_8 \mathbin{\mathrm{Y}} D_8$ has a unique faithful irreducible module of order 3^4 and the transitivity of the Bucht group of order $5 \cdot 32$ shows the centralizer in $Q_8 \mathbin{\mathrm{Y}} D_8$ of a non-zero vector has order 2. Note that in (b), that the 10 non-central involutions of $F(G)$ are conjugate in G. In all cases, $|C_{F(G)}(v)| = 2$ for each non-zero vector v.

The question at hand is related to a question about regular orbits of $F(G)$. In Theorem 2.1, observe that $F(G)$ does not have a regular orbit on V. We determine those G that act primitively on a vector space in which $F(G)$ has no regular orbit in Theorem 2.2. Here $|V| = 5^2, 3^4, 3^8$, or q^2 for a Mersenne prime q.

Theorem 2.2. *If V is a quasi-primitive G-module and $F(G)$ has no regular orbit on V; then*

- *(a) $|V| = q^2$ for a Mersenne prime q and $G = F(G) = P \times S \subseteq \Gamma(V)$ where S is cyclic of odd order and $P \in \mathrm{Syl}_2(F(G))$ is either the Sylow-2-subgroup of $\Gamma(V)$ that is semi-dihedral of order $4(q+1)$ or P is dihedral of order $2(q+1)$; or*

(b) *$|V| = 5^2$ and $F(G) = QZ$ for normal subgroups $Q \cong Q_8$ and $Z \cong Z_4$ of G with $Q \cap Z = Z(Q)$ while $G/F(G) \cong Z_3$ or S_3; or*

(c) *$|V| = 3^4$ and $F(G)$ is isomorphic to a central product $Q_8 \curlyvee D_8$ and $G/F(G)$ is isomorphic to Z_5, D_{10}, F_{20}, A_5, or S_5; or*

(d) *$|V| = 3^4$ and $F(G)$ is a central product $Q \curlyvee S$ of normal subgroups $Q \cong Q_8$ and S of G with S semi-dihedral of order 16 or dihedral of order 8. Also $G/F(G) \cong Z_3$ or S_3; or*

(e) *$|V| = 3^8$ and $F(G)$ is a central product $E \curlyvee S$ of an extra-special group E of order 32 and a semi-dihedral group S of order 16. The only element of F that centralizes an element of each G-orbit in V is the identity. Also, if G is solvable, then $F_2(G)/F(G) \cong Z_5, Z_3$, or $Z_3 \times Z_3$.*

We next derive Theorem 2.1 from Theorem 2.2.

Proof of Theorem 2.1. The final statement of Theorem 2.1 follows from comments in the paragraph following the Theorem's statement. We assume that V is a quasi-primitive G-module and there exists $1 \neq x \in F(G)$ such that each $v \in V$ is centralized by a G-conjugate of x. No $v \in V$ is in a regular $F(G)$ orbit and so Theorem 2.2 applies. So V and G satisfy one of the conclusions (a)-(d), but not (e), of Theorem 2.2.

If $|V| = q^2$ for a Mersenne prime q, then the centralizer in $\Gamma(V)$ of a non-zero vector of V has order two. Now $\Gamma(V)$ is nilpotent and its Sylow-2-subgroup is semi-dihedral of order $4(q+1)$. Hence G and its Sylow-2-subgroup P must contain all $q+1$ non-central involutions and they must be conjugate in P. Hence P is a Sylow-2-subgroup of $\Gamma(V)$. Here V and G are as in conclusion (a) of Theorem 2.1. Conclusions (b) and (c) of Theorem 2.2 are precisely those of Theorem 2.1.

Finally, we may assume that $|V| = 3^4$ and that $F(G)$ is a central product $Q \curlyvee S$ of normal subgroups $Q \cong Q_8$ and S of G with S semi-dihedral of order 16. Also $G/F(G) \cong Z_3$ or S_3. There exists $Q_1 \subseteq S$ with Q_1 normal in G and $Q_1 \cong Q_8$. By Theorem 2.10 below, QQ_1 has a regular orbit on V. Since QQ_1 is normal in G; the G-conjugacy class of x lies in $F(G) - QQ_1$. By Proposition 2.8 below; QQ_1 contains 20 of the 24 involutions of $QS = F(G)$. Thus $\mathrm{cl}_G(x)$ has at most 4 elements. But every $v \in V$ is centralized by a conjugate of x and hence $4|C_V(x)| \geq |V|$. But $|C_V(x)| \leq |V|^{1/2}$ by Lemma 2.7 below. This implies $|V| \leq 16$, a contradiction; completing the proof of Theorem 2.1. $\square$

Corollary 2.3. *Suppose V is a faithful irreducible G-module and there exists $1 \neq x \in F(G)$ such that each $v \in V$ is centralized by a G-conjugate of x. Then there exists a faithful irreducible H-module W and a transitive permutation group S of degree n such that G is isomorphic to a subgroup of H wreath S in its action on $V = W^n$. Here H is a factor group of a subgroup of G. Furthermore, we have that*

(i) *$|W| = q^2$; $H = F(H) \subseteq \Gamma(W)$; and the Sylow-2-subgroup of H is semi-dihedral of order $4(q+1)$;*

(ii) *$|W| = 5^2$; $F(H) \cong Q_8 \curlyvee Z_4$; and $H/F(H) \cong Z_3$ or S_3; or*

(iii) *$|W| = 3^4$; $F(H) \cong Q_8 \curlyvee D_8$; with $H/F(H) \cong Z_5, D_{10}, F_{20}, A_5$ or S_5.*

Note $\mathrm{char}(V)$ *is a Mersenne prime or* $\mathrm{char}(V) = 5$ *and 3 divides* $|G|$.

Proof. By Theorem 2.1, we may assume V is not a quasi-primitive G-module and choose M maximal such that M is normal in G and V_M is not homogeneous. Write $V_M = V_1 \oplus \ldots \oplus V_k$ for subspaces V_i of V that are permuted faithfully and transitively by $S = G/M$. Set $H_i = N_G(V_i)$ and $C_i = C_G(V_i)$. We claim there is some $1 \neq z \in F(H_1/C_1)$ that centralizes an element of each H_1-orbit in V_1. If the claim is valid, the conclusion of this corollary holds since G is isomorphic to a subgroup of H_1/C_1 wreath S and since H_1/C_1 satisfies the conclusion of this Corollary (via induction).

We first show $x \in H_i$ for each i. Let L be the smallest normal subgroup of G containing x and note $L \subseteq F(G)$. We may assume that $x \in L - (L \cap H_i)$. Since $L \cap H_i$ is normal in H_i and $L \cap H_i$ is properly contained in the nilpotent group L; it follows that $N_G(L \cap H_i) > H_i$. By choice of M, H_i is a maximal subgroup of G and so $L \cap H_i$ is normal in G. Let $0 \neq w \in V_i$ and choose a conjugate x^g of x that centralizes w. Note x^g must lie in $L \cap H_i$ and so does x by normality of $L \cap H_i$. Hence $x \in H_i$ for all i.

Set $H = H_1/C_1$. Now G is isomorphic to a subgroup of $G^* = H$ wr S in its action on V and certainly x centralizes an element in each G^*-orbit of V. Since $x \in H_i$ for each i and $x \in F(G)$, we have that $xC_i \in F(H_i/C_i)$ for all i and thus $x \in F(G^*)$. To complete the claim, we may assume that $G = G^* = H$ wr S. Write $x = (x_1, \ldots, x_k)$. Since $x \neq 1$, we may assume without loss of generality that $x_1 \neq 1$. Set $z = x_1 \in F(H_1/C_1)$. Let $w \in V_1$. Then $(w, 0, \ldots, 0) \in V$ is centralized by x^g for some $g \in G$. In particular $(w, 0, \ldots, 0)$ is centralized by $(z^g, 0, \ldots, 0)$ and w is centralized by z^g. But $\{(z^g, 0, \ldots, 0) | g \in G\} = \{(z^h, 0, \ldots, 0) | h \in H\}$. Hence, w is centralized by some H-conjugate of z, proving the claim and the Corollary. □

In [4], it was proven that a non-vanishing element x of a solvable group satisfies $x^2 \in F(G)$ and conjectured that $x \in F(G)$. Recall x is a non-vanishing of G if $\chi(x) \neq 0$ for all $\chi \in \mathrm{Irr}(G)$.

Corollary 2.4. *If G is solvable and no Mersenne prime divides $|G|$, then every non-vanishing element of G is in $F(G)$.*

Proof. Assume $x \in G$ is non-vanishing. Note that Nx is a non-vanishing element of G/N whenever $N > 1$ is normal in G. By induction, $Nx \in F(G/N)$ and so x centralizes every chief factor of G/N. If $1 \neq M$ is normal in G and $N \cap M = 1$, then x also centralizes all chief factors of G/M and those of G; whence $x \in F(G)$. So we may assume G has a unique minimal normal subgroup.

Gaschutz's Theorem (1.12 of [5]) states that $F(G)/\Phi(G) = F(G/\Phi(G))$ is a completely reducible and faithful $G/F(G)$-module whenever G is solvable. By the inductive hypothesis and the last paragraph, $\Phi(G) = 1$ while $F(G)$ is the unique minimal normal subgroup of G. Then $F(G)$ and $\mathrm{Irr}(F(G))$ are faithful irreducible $G/F(G)$-modules.

Let $\lambda \in \mathrm{Irr}(F(G))$. Since $\chi(x) \neq 0$ for all $\chi \in \mathrm{Irr}(G|\lambda)$; we have that $\psi^G(x) \neq 0$ for some $\psi \in \mathrm{Irr}(I_G(\lambda))$. This implies that $x^g \in I_G(\lambda)$ for some $g \in G$. By induction $x \in F_2(G)$. So $x \in F(G)$ or the hypotheses of Corollary 2.3 are satisfied with respect to the action of $G/F(G)$ on $\mathrm{Irr}(F(G))$. Since $|G|$ is not divisible by a Mersenne prime, the conclusion of Corollary 2.3 is not satisfied. So $x \in F(G)$. □

Corollary 2.5. *Suppose G is solvable and $F = F(G)$ is a minimal normal subgroup of G. If F is a primitive G/F-module, then every non-vanishing element of G lies in F.*

Proof. Note that by Gaschutz's theorem, F is indeed a faithful irreducible G/F-module (and G even splits over F). If $G^* = G/F$ has a regular orbit on $V = \mathrm{Irr}(F)$; then there exists $\lambda \in \mathrm{Irr}(F)$ with $I_G(\lambda) = F$ and $\lambda^G \in \mathrm{Irr}(G)$ vanishes off F and the conclusion holds. We may assume that G^* and V satisfy the conclusion of Theorem 2.2.

Let $F_i = F_i(G)$. Observe that in all cases of the conclusion of Theorem 2.2 with G solvable and for all $i > 0$, there exists $F_i \subseteq A_i \subseteq F_{i+1}$ such that A_i is normal in G with A_i/F abelian and there exists $\theta \in \mathrm{Irr}(F_{i+1}/F_i)$ that vanishes on $F_{i+1} - A_i$. By normality, every G-conjugate of θ vanishes on $F_{i+1} - A_i$. By Clifford's Theorem, every $\chi \in \mathrm{Irr}(G|\theta)$ vanishes on $F_{i+1} - A_i$. Since A_i/F_i is abelian, there exists $\lambda \in \mathrm{Irr}(F_i)$ in a regular orbit of A_i/F_i. Since A_i is normal in G, it follows that every $\beta \in \mathrm{Irr}(G|\lambda)$ vanishes on $A_i - F_i$. Thus for all $i > 0$, every non-vanishing element of G lying in F_{i+1} indeed lies in F_i. As G is solvable, every non-vanishing element of G lies in F. □

What remains is to prove Theorem 2.2.

If V is a completely reducible and faithful N-module for a nilpotent group N, then a key result of [6, Theorem 3.1] says that N has a regular orbit on $V \oplus V$. Applying this and Clifford's theorem, we have the following:

Lemma 2.6. *Suppose V is an irreducible faithful G-module and N is a normal nilpotent subgroup of G. If N has no regular orbit on V, then $V_N = V_1 \oplus \ldots \oplus V_k$ for non-isomorphic irreducible N-modules $V_i, i = 1, \ldots, k$. In particular, if V is quasi-primitive and N has no regular orbit, then V_N is irreducible.*

Lemma 2.7. *If V is a faithful quasi-primitive G-module and P is a normal p-subgroup of G; then*

- *(i) $P = EZ$ where $Z = Z(P)$ and E is an extra-special p-group (allowing $|E| = p$). Note $P \cap Z = Z(E)$. Also $\exp(E) = p$ if p is odd. or*
- *(ii) $p = 2, P = ET$ where T is quaternion, dihedral or semi-dihedral with $|T| > 8$. Also E is extra-special (allowing $|E| = 2$) and $E \cap T = Z(E)$. Also T has a unique cyclic subgroup U of index two and both U and EU are characteristic in P.*

For a non-central element x of P of order p; we have $|C_V(x)| \leq |V|^{1/p}$. Also, if V_0 is a irreducible EU-submodule of V, then $(V_0)_U \cong eW$ for an irreducible U-submodule W where $e^2 = |E/Z(E)|$.

Proof. That (i) or (ii) holds follows from Theorem 1.2 of [5]. Note, for $p = 2$, this is slightly different from 1.9 or 1.10 of [5], which require solvability of G. In particular, we do not assume E or T is normal in G (in (ii)) and E maybe different from that of Theorem 1.9 of [5]. Let x be a non-central element of P of order p. If $x \in EU$ which is normal in G, then the Brauer character afforded $< x >$ is a multiple of the regular $\rho_{<x>}$ and so $|C_V(x)| = |V|^{1/p}$. So we may assume that $p = 2$ and $x \notin U$. But $< Ux >$ is normal in P and x is conjugate to some ux for $1 \notin u \in U$. Since $C_V(u) = 0$, it follows that $C_V(x) \cap C_V(ux) = 0$. Since $|C_V(x)| = |C_V(ux)|$; it follows that $|C_V(x)| \leq |V|^{1/2}$. □

If p is odd in the above Lemma, then all elements of order p lie in E. If $p = 2$ and we are in part (i) above, then all elements of order 2 lie in EZ_0 where Z_0 has order 4. It is a bit more complicated in (ii). We do a count, which above lets us know how many involutions are in certain subgroups of P. For a subset X of a group, we let $\sigma_2(X) = |\{x \in X | x^2 = 1\}|$ and $\tau_4(X)$ denote the number of elements of X of order 4.

Proposition 2.8. *(i) If E is extra-special of order 2^{2n+1}; then $\sigma_2(E) = 2^{2n} \pm 2^n$ and $\tau_4(E) = 2^{2n} \mp 2^n$ (respectively);*

(ii) If EU is a non-trivial central product of an extra-special group E of order 2^{2n+1} and a cyclic 2-group U with $|U|$ at least 4; then $\sigma_2(EU) = 2^{2n+1}$; and

(iii) If ET is a non-trivial central product of an extra-special 2-group E and a 2-group T that is dihedral, quaternion, or semi-dihedral; then $\sigma_2(ET) = 2^{2n+1} + |U|2^{2n-1}$ if T semi-dihedral and $\sigma_2(ET) = 2^{2n+1} + |U|(2^{2n-1} \pm 2^{n-1})$ otherwise where U is a cyclic subgroup of T on index 2.

Proof. First suppose that A and B are 2-groups with unique central involutions, say a and b respectively. Also assume that $x^2 = a$ (respectively $y^2 = b$) whenever $x \in A$ has order 4 (respectively $y \in B$ has order 4). If G is the non-trivial central product of A and B; i.e. $G = (AB)/Z$ where $Z =< ab >$; then

$$\sigma_2(G) = \frac{1}{2}(\tau_4(A)\tau_4(B) + \sigma_2(A)\sigma_2(B)). \tag{2.1}$$

Note that equation (2.1) follows from the following two observations: If $w = (x, y)$ is in $A \times B$, then w^2 is in Z if and only if both $x^2 = a$ and $y^2 = b$ or both $x^2 = 1$ and $y^2 = 1$. Also, w^2 is in Z if and only if $(zw)^2$ is in Z.

Since an extra-special two-group is a central product of extra-special two-groups of order 8 (i.e. D_8 and Q_8) and since $\{\sigma_2(E), \tau_4(E)\} = \{2, 6\}$ when $|E| = 8$, part (i) can be proven by induction using equation (2.1). Note the two equalities in (i) are equivalent as $|E| = 2^{2n+1}$ and $\exp(E) = 4$. Parts (ii) and (iii) follow from part (i) and equation (2.1). For (iii), we use that every one of the $|U|$ elements of $T - U$ has order 2 or 4 and the number of elements of order 2 in $T - U$ is $|U|$ (when T is dihedral), 0 (when T is quaternion) or $|U|/2$ (if T semi-dihedral). □

Corollary 2.9. *If EU is a non-trivial central product of an extra-special group E of order 2^{2n+1} and a cyclic 2-group U with $|U|$ at least 4; then the number of (elementary abelian) subgroups of order 4 that intersects U trivially is $(2^{2n+1} - 2) \cdot (2^{2n-1} - 2)/3$.*

Proof. Let x be a non-central involution. Then $C_{EU}(x)$ has index two in EU and is a union of U-cosets. But each U-coset has two elements of order 4 and two of order 2 or 1. Thus $C_{EU}(x)$ has $|EU|/4 = 2^{2n}$ elements of order two or one including 1 and x. For each involution $y \neq x$ that commutes with x, there is a unique subgroup of order 4 containing x and y (and xy). Exactly one of these elementary abelian subgroups intersects U non-trivially. Thus there are exactly $((2^{2n} - 2)/2) - 1$ elementary abelian subgroups of order 4 of EU containing x that "avoid" U. As EU has $2^{2n+1} - 2$ non-central involutions, the Corollary follows. □

Theorem 2.10. *Suppose that V is a faithful irreducible P-module and $P = EZ$ for $Z = Z(P)$ and an extra-special 2-group E. If P has no regular orbit on V, then*

(i) $|V| = 9$ and $P = E \cong D_8$;

(ii) $|V| = 5^2; |E| = 8, |Z| = 4$, and $|P| = 16$; or

(iii) $|V| = 3^4$ and $P = E$ is a central product of Q_8 and D_8.

In all cases, $C_E(v) =< t >$ for an involution t whenever $v \in V^{\#}$.

Note. Let $q = \mathrm{char}(V)$. There are two extra-special groups E of order 2^{2n+1} for each n. For each such E; there is a unique faithful irreducible E-module over the field of q elements.

Proof. First consider the case $P = E$ has order 8, so that $P \cong Q_8$ or D_8. The only involution in Q_8 is central and so every Q_8-orbit on $V^{\#}$ is regular. Assume $E \cong D_8$ and note that $|V| = q^2$ and each of the four non-central involutions centralizes $q-1$ non-zero vectors of V with no overlap. Then E has a regular orbit unless $q^2 - 1 = 4(q-1)$; i.e. $q = 3$. In the case $q = 3, C_E(v) =< t >$ for an involution t for all $v \in V^{\#}$. The result holds when $P = E$ has order 8.

Still assume that $|E| = 8$. Now $P = Q \times S$ for a cyclic $2'$-group S and a 2-group Q with $Q = EZ_0$ where $Z_0 = Z(Q)$. Every S-orbit on $V^{\#}$ is regular and $(|S|, |Q|) = 1$. So P has no regular orbit if and only if Q has no regular orbit. If $|Q| = 8$, then $|V| = 9$ by the last paragraph, whence $S = 1$ and $P = EZ = E$ and conclusion (i) holds. So we may assume that 4 divides $|Z_0|$. By Proposition 2.8, Q has exactly 6 non-central involutions, each of which is contained in the subgroup Q_0 of Q of order 16 that contains E. So Q_0 has no regular orbit and each $v \in V^{\#}$ is centralized by one and only one involution. Now $|C_V(t)| = |V|^{1/2}$ for each non-central involution (Lemma 2.7); and so $|V| - 1 = 6(|V|^{1/2} - 1)$. Then $|V| = 5^2; S = 1$ and $|Z| = 4$; as in conclusion (ii).

When $|V| = 3^4$ and E is a central product of Q_8 and D_8 (here E has 10 non-central involutions); then $E = F(B)$ for a Bucht group $B \subseteq GL(V)$ of order $2^5 \cdot 5$. Since B acts transitively on $V^{\#}$; whence $C_B(v) = C_E(v)$ has order two for all $v \in V^{\#}$. Note that $Z = C_P(E)$ must have order 2 and conclusion (iii) holds.

As above, $P = Q \times S$ for a cyclic $2'$-group S and a 2-group $Q \supseteq E$ with S acting Frobeniusly on V. Since P has no regular orbit on V, each $v \in V$ is centralized by an involution in Q. Hence $V = \cup C_V(x)$ where the union is over all non-central involutions of Q. By Lemma 2.7 , $|C_V(x)| \leq |V|^{1/2}$ for all involutions in Q, which is a central product of E and $Z(Q)$. If t is the number of non-central involutions of Q, then $t \geq |V|^{1/2}$.

Now $|P : Z| = |Q : Z(Q)| = |E : Z(E)| = e^2$ where $e = 2^n$ for an integer $n > 0$. By the second paragraph, we may assume that $n > 1$ and $e \geq 4$. Now V is an irreducible P-module by Lemma 2.6 and $V_Z \cong W^e$ for a faithful irreducible Z-module W. Note $|Z|$ divides $q-1$ where $q = |W|$. Now $t \geq q^{e/2}$ by the last paragraph.

By Proposition 2.8, we see $2e^2 = 2^{2n+1} \geq t \geq q^{e/2}$. Since $2e^2 \geq q^{e/2}$ and $e \geq 4$; we have that $e = 8$ and $q = 3$ or that $e = 4$ and q is 3 or 5. Note that $|Z|$ divides $q - 1$.When $e = 8$ and $q = 3$, then $P = Q = E$ and Proposition 2.8 shows that $72 = e^2 + e \geq t \geq 3^4$, a contradiction. Similarly, we have a contradiction when $P = Q = E, e = 4$, and $q = 5$. Hence we may assume that $e = 4$ and with $q = 3$

or 5. When $q = 5$, then $|Z| = 4$. When $q = 3, |Z| = 2$ and $P = E \cong Q_8 \,\mathrm{Y}\, Q_8 \cong D_8 \,\mathrm{Y}\, D_8$ by the third paragraph. Applying Proposition 2.8, note that $t = 30$ when $q = 5$ and $t = 18$ when $q = 3$.

For $v \in V^{\#}, C_P(v)$ is isomorphic to a subgroup of P/Z as $C_Z(v) = 1$. So $C_P(v)$ is elementary abelian of order at most 4. Hence $C_P(v)$ has 0,1, or 3 involutions. Since P has no regular orbit; $C_P(v)$ has exactly 1 or 3 involutions for $v \in V^{\#}$. Let a_i denote the number of v in V centralized by exactly i involutions in P. Since each non-central involution centralizes $q^2 - 1$ elements of $V^{\#}$; it follows that $a_1 + a_3 = q^4 - 1$ and that $a_1 + 3a_3 = t(q^2 - 1)$. For $q = 3$, we then have $a_1 = 48$ and $a_3 = 32$. For $q = 5$, then $a_1 = 576$ and $a_3 = 48$.

If B is any elementary abelian subgroup of P of order 4 that avoids Z (i.e. $B \cap Z = 1$), then the Brauer character afforded B by V is the regular character ρ_B. In this case, $C_V(B)$ is a one dimensional and there are $q - 1$ v in $V^{\#}$ centralized by B. In fact, $B = C_P(v)$ for these $q - 1$ vectors. No non-zero vector is centralized by two such B. Thus there exist at most $a_3/(q - 1)$ (i.e. 16 or 12) elementary abelian subgroups of P that avoid Z. If $|Z| = 4$, then Corollary 2.9 shows that there are 60 elementary subgroups of order 4 that avoid Z, a contradiction. Thus $|Z| = 2, q = 3$, and $P = E \cong Q_8 \,\mathrm{Y}\, Q_8 \cong D_8 \,\mathrm{Y}\, D_8$. Now P is a normal subgroup of a group $G \subseteq GL(V)$ with $G/P \cong Z_3 \times Z_3$ (here G is a central product of two copies of $SL(2, 3)$). Now G/P has a regular orbit on $P/Z(P)$. So there is an $x \in P - Z(P)$ whose centralizer in G is in P and hence the G-conjugacy class of x has 18 elements. Since P has 18 non-central involutions (and 12 elements of order 4), the involutions in P are G-conjugate. If O is the set of vectors in V fixed by exactly one involution, it follows that t divides $|O|$, a contradiction as $t = 18$ and $|O| = 48$. This completes the proof. $\square$

Alex Turull has an inclusion-exclusion counting arguments that determines the exact number of regular orbits above, although the argument is a bit different for when $|U| = 2$ or not and they are a bit longer than current arguments.

Proof of Theorem 2.2. We assume that V is a faithful irreducible and quasi-primitive G-module and that $F = F(G)$ has no regular orbit on V. We must show V and G satisfy one of the conclusions (a)-(e) of Theorem 2.2. In particular, we must show $|V| = 5^2, 3^4, 3^8$ or q^2 for a Mersenne prime.

Applying Lemma 2.7, $F = ET$ where the Sylow-subgroups of E are extra-special (or of prime order). Also $T = C_F(E)$ is cyclic or there exists U normal in G with $|T : U| = 2$, U cyclic, and 8 dividing $|U|$. In all cases $T \cap E = Z(E)$. Since F has no regular orbit, V is an irreducible F-module by Lemma 2.6. Also $V_T \cong eW$ for a faithful irreducible T-module W and integer e where $e^2 = |E : Z(E)| = |F : T|$. Note that $|U|$ divides $|W| - 1$.

First suppose that $F = T$; i.e. that $e = 1$. Since U acts fixed-point-freely on V and F has no regular orbit on V, each $v \in V^{\#}$ is fixed by a unique involution in $T - U$. Indeed each $v \in V^{\#}$ is fixed by a unique involution $T_0 - U_0$ where T_0 and U_0 are the Sylow-2-subgroups of T and U respectively. It follows from Lemma 2.7 that T_0 is dihedral or semi-dihedral and so the number of involutions in $T_0 - U_0$ is either $|U_0|$ or $|U_0|/2$. Now $|C_V(x)| = |V|^{1/2}$ for all involutions x in $T_0 - U_0$. By uniqueness, $|V| - 1 = t(|V|^{1/2} - 1)$ where t is the number of involutions in $T_0 - U_0$. Thus $|V|^{1/2} + 1$ is a power of 2. Since $|V|^{1/2}$ is a power of a prime, it follows that $|V| = q^2$ for a Mersenne prime q. Also $t = q + 1$

and $|T_0| = 2|U_0|$ is $2t$ or $4t$. If W_0 is an irreducible U-submodule of V and $0 \neq w \in W_0$, then w is centralized by an involution in $T - U$; whence W_0 is invariant under $< U, x >= T = F$. So V is an irreducible U-module and since U is normal and abelian, it follows that $G \subseteq \Gamma(V)$ (see [5, Theorem 2.1]). Here V and G are as in conclusion (a) of Theorem 2.2. So we assume that $F > T$ and $e > 1$.

Since F has no regular orbit on V, we have that $V = \cup C_V(x)$ where the union is taken over all non-central elements x of prime order in F. Now $|C_V(x)| \leq |V|^{1/p}$ if $x \in F$ has order p, a prime by Lemma 2.7. Let s denote the number of non-central subgroups of odd prime order of F and let t be the number of non-central involutions in F. Then

$$s \cdot |V|^{1/3} + t \cdot |V|^{1/2} \geq |V| \text{ and } s + t \geq |V|^{1/2}. \tag{2.2}$$

In particular, $|F(G)| > s + t \geq |V|^{1/2}$. Also, $|V| = |W|^e$. On the other hand, $|F| = e^2|U||T : U|$ with $|T : U| \leq 2$. Also 8 divides $|U|$ if $|T : U| = 2$. Also $|U|$ divides $|W| - 1$. In particular, $2e^2|W| \geq |F| > s + t \geq |V|^{1/2} \geq |W|^{e/2}$ and $|W| < (2e^2)^{2/(e-2)}$. As $|W| \geq 3$, it follows that $e < 13$. Since each prime divisor of e divides $q - 1$; we further have that $e \leq 4; e = 5$ with $|W| = 11; e = 6$ with $|W| = 7; e = 8$ with $|W| = 3$ or 5; or $e = 9$ with $|W| = 4$.

In all the cases where $e > 4$, note that 8 does not divide $|U|$ and so T is cyclic. In the case where $e = 8$, now Theorem 2.10 shows that F does have a regular orbit, a contradiction. When e is 5, 6, or 9; note that

$$|F| = e^2|U| \leq e^2(|W| - 1) \leq |W|^{e/2} \leq |V|^{1/2} \leq s + t,$$

a contradiction. Hence we may assume e is 2, 3, or 4.

When $e = 3$, the Hall-$2'$-subgroup S of F is a central product of an extra-special group of order 27 and exponent 3 with $Z(S)$. Hence $s = 12$. If $t = 0$, then equation (2.2) yields that $12 \geq |V|^{2/3} \geq |W|^2$, a contradiction as 3 divides $|W| - 1$. So $t > 0$ and the Sylow-2-subgroup P of F is dihedral or semi-dihedral. Note that P has a cyclic subgroup of order $|P|/2$ and $3|P|/2$ divides $|W| - 1$. Also $t \leq |P|/2 \leq (|W| - 1)/3 \leq |V|^{1/3}/3$. By equation (2.2),

$$12|V|^{1/3} + |V|^{5/6}/3 > |V|.$$

This implies $|V| < 64$ and so $|W| = |V|^{1/3} < 7$, a contradiction as 6 divides $|W| - 1$. Hence $e = 2$ or 4.

Since $e = 2$ or 4, the Hall-$2'$-subgroup S of F is cyclic, $s = 0$, and S acts Frobeniusly on V. Thus the Sylow-2-subgroup P of F has no regular orbit on V and so V_P is irreducible by Lemma 2.6. In particular, $C_G(P)$ is cyclic (see [5, Lemma 2.9]) and $C_G(P) \subseteq F$. If T is cyclic, then Theorem 2.10 applies to F and we have that one of the following:

(i) $|V| = 9$ and $F = E \cong D_8$;
(ii) $|V| = 5^2; |E| = 8, |Z| = 4$, and $|F| = 16$; or
(iii) $|V| = 3^4$ and F is a central product of Q_8 and D_8.

In (i), $\text{Aut}(F)$ is a 2-group; so G is a 2-group and $G \cong D_8$. Here V is not quasi-primitive, although $G \subseteq \Gamma(V) = \Gamma(3^2)$ and conclusion (a) is still satisfied. In (ii), it is easy to see that $G/C_G(P)$ is solvable and applying Theorem 1.9 and Corollary 1.10 of [5] shows that F is a central product of Z

and a normal subgroup $Q \cong Q_8$ and also that $G/F \cong A_3$ or S_3. Conclusion (b) holds in (ii). Assume that $|V| = 3^4$ and F is a central product of Q_8 and D_8. Note that G/F acts faithfully on F/Z where $Z = Z(F)$ and G/F permutes the 10 non-central involutions of F. For x a non-central element of F, the F-conjugacy class of x is $\{x, xz\}$ where $Z = < z >$. It follows that G/F faithfully permutes the 5 conjugacy classes of non-central involutions of F. Hence G/F is isomorphic to a subgroup of S_5. If G is non-solvable, then G/F is isomorphic to A_5 or S_5 and conclusion (c) of this Theorem is satisfied. If G is solvable and $|G|$ is divisible by 5, a Zsigmondy prime divisor of $3^4 - 1$, then G is a Bucht group (see [5, Lemma 6.7]) and conclusion (c) of this Theorem is satisfied. If G/F is a $\{2, 3\}$-group then Theorem 1.9 of [5] shows that $F/Z = Q/Z \times D/Z$ for normal subgroups $Q \cong Q_8$ and $D \cong D_8$ with Q/Z a faithful G/F-module. So $G/F \cong A_3$ or S_3 and conclusion (d) of the Theorem is satisfied. The conclusion of this Theorem is satisfied whenever T is cyclic.

We can now assume that T and $T_0 \in \mathrm{Syl}_2(T)$ are not cyclic. Hence T_0 is dihedral, quaternion, or semi-dihedral and $|T_0| \geq 16$. Now $\dim(W)$ is even as W is an irreducible T_0-module.

Suppose that $e = 2$ and $|W| = 9$. Now $S = 1$ and $F = P$. Then $T = T_0 \in \mathrm{Syl}_2(GL(2, 3))$ is semi-dihedral of order 16. Now V and F are as in conclusion (e) of Theorem 2.2. We still need to show that G/F is isomorphic to A_3 or S_3. Since U and EU are characteristic in F, it follows that $\mathrm{Aut}(P)$ is a $\{2, 3\}$-group. Since V_P is irreducible, $C_G(P) \subseteq F(G) = P$ and G is a $\{2, 3\}$-group, hence G is solvable. We may apply Theorem 1.9 of [5] to assume (with no loss of generality) that E and T are normal in G, that $E \cong Q_8$ and there exists $F \subseteq A \subseteq G$ such that A/F acts faithfully and completely reducible on the chief factor F/T and $|G/A| \leq 2$. As F is a 2-group, it follows that $G/F \cong A_3$ or S_3. Conclusion (d) is satisfied here.

Assume that $e = 2$. Since $\dim(W)$ is even, we may assume from the last paragraph that $|W|$ is at least 25. By Proposition 2.8; $t \leq 6 + 3|U_0|$. If $(|W| - 1)/|U_0| \geq 5$; then

$$t \leq 6 + 3(|W| - 1)/5 < |W| \leq |V|^{1/2};$$

contradicting equation (2.2). So $(|W| - 1)/|U_0| \leq 4$ and so $|W| - 1$ is a power of two or three times a power of 2. Since $|W|$ is a prime power but not a prime nor 9, elementary arguments [5, Propositions 3.1 and 3.2] show that $|W| = 5^2$ or 7^2. Also $|U_0| = (|W| - 1)/3 = 8$ or 16 (respectively) and $|T_0| = 16$ or 32 (respectively). Now W is an irreducible U_0-module with U_0 normal and abelian in T, we have $T \subseteq \Gamma(W)$ and $T_0 \in \mathrm{Syl}(\Gamma(W))$. Now T_0 induces the automorphism $a \to a^q$ (for $q = 3$ or 5 (resp.) of U_0; we see that T_0 is semi-dihedral. Now Proposition 2.8 shows that

$$t = 6 + 2|U_0| < |W| \leq |V|^{1/2}$$

contradicting equation (2.2). Hence we may assume that $e \neq 2$.

We now have that $e = 4$. We have from Proposition 2.8 (note $n = 2$) that

$$30 + 10|U| \leq t \leq |W|^2 \leq |V|^{1/2}.$$

As $|U|$ divides $|W| - 1$ and $\dim(W)$ is even, we see that $|W| = 9$ and $|U| = 8$. Then $S = 1, F = P$, and $T = T_0 \in \mathrm{Syl}_2(GL(2, 3))$ is semi-dihedral of order 16. So F is a central of the semi-dihedral group T

and E, an extra-special of order 32. The first statement of conclusion (e) of Theorem 2.2 is satisfied. We verify the last two statements.

Now EU is characteristic in F and so EU is normal in G. By Theorem 2.10, EU has a regular orbit on V. Suppose $1 \neq x \in F$ centralizes an element of each G-orbit on V. So does every power of x. Since EU has a regular orbit on V and $|F : EU| = 2$, we see that $< x > \cap EU = 1$ and x is an involution. Every element of V is centralized by a conjugate of x and so $V = \cup_{g \in G} C_V(x^g)$. Thus $|\operatorname{cl}_G(x)||C_V(x)| \geq |V|$ and so $|\operatorname{cl}_G(x)| \geq |V|^{1/2} = 81$. By Proposition 2.8 ((ii),(iii)); we see that $|\operatorname{cl}_G(x)| \leq$ the number of involution in $F - EU = 64$. By this contradiction, no such x exists.

Finally assume that G is solvable. Applying Corollary 1.10 of [5], we may assume that E and T are normal in G and that $E/Z(E)$ is a faithful G/F-module that is irreducible (of order 16) or the direct sum of two irreducible modules of order 4. Since G/F preserves a symplectic form on $E/Z(E)$, then G/F has no element of order 15 (see [3, Lemma 9.2.4]. Hence $F_2(G)/F$ is isomorphic to Z_5, Z_3, or $Z_3 \times Z_3$ and conclusion (e) of the Theorem is satisfied. □

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