GROUP ACTIONS RELATED TO NON-VANISHING ELEMENTS

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Abstract. We characterize those groups $G$ and vector spaces $V$ such that $V$ is a faithful irreducible $G$-module and such that each $v$ in $V$ is centralized by a $G$-conjugate of a fixed non-identity element of the Fitting subgroup $F(G)$ of $G$. We also determine those $V$ and $G$ for which $V$ is a faithful quasi-primitive $G$-module and $F(G)$ has no regular orbit. We do use these to show in some cases that a non-vanishing element lies in $F(G)$.

1. Introduction

Suppose a group $G$ acts irreducibly and faithfully on a (finite) vector space $V$ and there is an $x \neq 1$ in $F(G)$ such that every element of $V$ is centralized by a $G$-conjugate of $x$ (or equivalently, $x$ centralizes an element of each $G$-orbit on $V$). If the action of $G$ is primitive, then $|V|$ must be $5^2$, $3^4$, or $q^2$ for a Mersenne prime $q$ and the structure of $G$ is very limited.

Using wreath products and the above results, it is easy to construct imprimitive $G$-modules with this property. But that is all; i.e. all such actions are subgroups of wreath products involving a quasi-primitive action of this type.

This question arose in [4] (for solvable $G$), where it was conjectured that a non-vanishing element of a solvable group must lie in the Fitting subgroup of $G$. Recall $x$ in $G$ is non-vanishing if $\chi(x) \neq 0$ for all $\chi \in \text{Irr}(G)$. A key result of [4] shows that the square of a non-vanishing non-identity element of a solvable group is in $F(G)$. We offer a few modest results towards this conjecture. E.G. This conjecture is true for solvable $G$ except possibly when $|G|$ is divisible by 2 and a Mersenne prime.

The question at hand is related to a question about regular orbits of $F(G)$. In the above situation, $F(G)$ does not have a regular orbit on $V$. We determine those $G$ that act primitively on a vector space with this property.
space in which \( F(G) \) has no regular orbit in Theorem \( 2.2 \). Here \( |V| = 5^2, 3^4, 3^8, \) or \( q^2 \) for a Mersenne prime \( q \). We do not require solvability for Theorems \( 2.1 \) and \( 2.2 \). Recall that \( V \) is a quasi-primitive \( G \)-module if \( V_N \) is homogeneous for all normal subgroups \( N \) of \( G \). Via Clifford’s Theorem, primitive implies quasi-primitive.

If \( V \) is a finite vector space of order \( q^m \) over a field \( F \) of order \( q \), then the elements of \( V \) may be labeled by \( GF(q^m) \) consistently with the action of \( F \). The set \( \Gamma(q^m) = \{ x \rightarrow ax^\sigma | a \in GF(q^m) - \{0\}, \sigma \in \text{Gal}(GF(q^m)/F) \} \) is a group of linear transformations of \( V \) with a normal cyclic group \( \Gamma_0 \) “of multiplications” of order \( q^m - 1 \). Also \( \Gamma/\Gamma_0 \cong \text{Gal}(GF(q^m)/F) \) is cyclic of order \( m \) and a generator of this group induces the automorphism \( a \rightarrow a^q \) on \( \Gamma_0 \). In particular, \( \Gamma(q^2) \) has a cyclic normal subgroup \( \Gamma_0 \) of index two and center of order \( q - 1 \).

2. Results

**Theorem 2.1.** Suppose that \( V \) is a faithful quasi-primitive \( G \)-module and there exists \( 1 \neq x \in F(G) \) such that each element of \( V \) is centralized by a \( G \)-conjugate of \( x \). Then

(a) \( |V| = q^2 \) for a Mersenne prime \( q \) and \( G = F(G) = P \times S \subseteq \Gamma(V) \) where \( S \) is cyclic of odd order and \( P \) is the Sylow-2-subgroup of \( \Gamma(V) \) that is semi-dihedral of order \( 4(q + 1) \);
(b) \( |V| = 5^2 \) and \( F(G) = QZ \) for normal subgroups \( Q \cong Q_8 \) and \( Z \cong Z_4 \) of \( G \) with \( Q \cap Z = Z(Q) \) while \( G/F(G) \cong Z_3 \) or \( S_3 \);
(c) \( |V| = 3^4 \) and \( F(G) \) is isomorphic to a central product \( Q_8 Y D_8 \) and \( G/F(G) \) is isomorphic to \( Z_5, D_{10}, \) the Frobenius group \( F_{20}, A_5, \) or \( S_5 \).

In all cases, \( C_{F(G)}(v) = \langle x \rangle \) for an involution \( x \) whenever \( v \in V^# = V - \{0\} \).

Each of these conclusions does exist. Indeed \( G \) acts transitively on the non-zero vectors of \( V \), save possibly in (a) if \( G \) is proper in \( \Gamma \). Hence the semi-direct product \( VG \) is a 2-transitive permutation group. Huppert [11] classified and constructed the solvable 2-transitive groups. The three solvable groups in (c) are known as Bucht groups. Note that \( Q_8 Y D_8 \) has a unique faithful irreducible module of order \( 3^4 \) and the transitivity of the Bucht group of order \( 5 \cdot 32 \) shows the centralizer in \( Q_8 Y D_8 \) of a non-zero vector has order 2. Note that in (b), that the 10 non-central involutions of \( F(G) \) are conjugate in \( G \). In all cases, \( |C_{F(G)}(v)| = 2 \) for each non-zero vector \( v \).

The question at hand is related to a question about regular orbits of \( F(G) \). In Theorem \( 2.1 \) observe that \( F(G) \) does not have a regular orbit on \( V \). We determine those \( G \) that act primitively on a vector space in which \( F(G) \) has no regular orbit in Theorem \( 2.2 \). Here \( |V| = 5^2, 3^4, 3^8, \) or \( q^2 \) for a Mersenne prime \( q \).

**Theorem 2.2.** If \( V \) is a quasi-primitive \( G \)-module and \( F(G) \) has no regular orbit on \( V \); then

(a) \( |V| = q^2 \) for a Mersenne prime \( q \) and \( G = F(G) = P \times S \subseteq \Gamma(V) \) where \( S \) is cyclic of odd order and \( P \in \text{Syl}_2(F(G)) \) is either the Sylow-2-subgroup of \( \Gamma(V) \) that is semi-dihedral of order \( 4(q + 1) \) or \( P \) is dihedral of order \( 2(q + 1) \); or
(b) \(|V| = 5^2\) and \(F(G) = QZ\) for normal subgroups \(Q \cong Q_8\) and \(Z \cong Z_4\) of \(G\) with \(Q \cap Z = Z(G)\) while \(G/F(G) \cong Z_3 \) or \(S_3\); or

(c) \(|V| = 3^4\) and \(F(G)\) is isomorphic to a central product \(Q_8YD_8\) and \(G/F(G)\) is isomorphic to \(Z_5, D_{10}, F_{20}, A_5, \) or \(S_5\); or

(d) \(|V| = 3^4\) and \(F(G)\) is a central product \(QYS\) of normal subgroups \(Q \cong Q_8\) and \(S\) of \(G\) with \(S\) semi-dihedral of order 16 or dihedral of order 8. Also \(G/F(G) \cong Z_3 \) or \(S_3\); or

(e) \(|V| = 3^8\) and \(F(G)\) is a central product \(EYS\) of an extra-special group \(E\) of order 32 and a semi-dihedral group \(S\) of order 16. The only element of \(F\) that centralizes an element of each \(G\)-orbit in \(V\) is the identity. Also, if \(G\) is solvable, then \(F_2(G)/F(G) \cong Z_5, Z_3, \) or \(Z_3 \times Z_3\).

We next derive Theorem 2.1 from Theorem 2.2.

**Proof of Theorem 2.1**. The final statement of Theorem 2.1 follows from comments in the paragraph following the Theorem’s statement. We assume that \(V\) is a quasi-primitive \(G\)-module and there exists \(1 \neq x \in F(G)\) such that each \(v \in V\) is centralized by a \(G\)-conjugate of \(x\). No \(v \in V\) is in a regular \(F(G)\) orbit and so Theorem 2.2 applies. So \(V\) and \(G\) satisfy one of the conclusions (a)-(d), but not (e), of Theorem 2.2.

If \(|V| = q^2\) for a Mersenne prime \(q\), then the centralizer in \(\Gamma(V)\) of a non-zero vector of \(V\) has order two. Now \(\Gamma(V)\) is nilpotent and its Sylow-2-subgroup is semi-dihedral of order \(4(q+1)\). Hence \(G\) and its Sylow-2-subgroup \(P\) must contain all \(q+1\) non-central involutions and they must be conjugate in \(P\). Hence \(P\) is a Sylow-2-subgroup of \(\Gamma(V)\). Here \(V\) and \(G\) are as in conclusion (a) of Theorem 2.1. Conclusions (b) and (c) of Theorem 2.2 are precisely those of Theorem 2.1.

Finally, we may assume that \(|V| = 3^4\) and that \(F(G)\) is a central product \(QYS\) of normal subgroups \(Q \cong Q_8\) and \(S\) of \(G\) with \(S\) semi-dihedral of order 16. Also \(G/F(G) \cong Z_3 \) or \(S_3\). There exists \(Q_1 \subseteq S\) with \(Q_1\) normal in \(G\) and \(Q_1 \cong Q_8\). By Theorem 2.10 below, \(QQ_1\) has a regular orbit on \(V\). Since \(QQ_1\) is normal in \(G\); the \(G\)-conjugacy class of \(x\) lies in \(F(G) - QQ_1\). By Proposition 2.8 below; \(QQ_1\) contains 20 of the 24 involutions of \(QS = F(G)\). Thus \(c_{Q_1}(x)\) has at most 4 elements. But every \(v \in V\) is centralized by a conjugate of \(x\) and hence \(4|C_{V}(x)| \geq |V|\). But \(|C_{V}(x)| \leq |V|^{1/2}\) by Lemma 2.7 below. This implies \(|V| \leq 16\), a contradiction; completing the proof of Theorem 2.1. \(\square\)

**Corollary 2.3.** Suppose \(V\) is a faithful irreducible \(G\)-module and there exists \(1 \neq x \in F(G)\) such that each \(v \in V\) is centralized by a \(G\)-conjugate of \(x\). Then there exists a faithful irreducible \(H\)-module \(W\) and a transitive permutation group \(S\) of degree \(n\) such that \(G\) is isomorphic to a subgroup of \(H\) wreath \(S\) in its action on \(V = W^n\). Here \(H\) is a factor group of a subgroup of \(G\). Furthermore, we have that

(i) \(|W| = q^2\); \(H = F(H) \subseteq \Gamma(W)\); and the Sylow-2-subgroup of \(H\) is semi-dihedral of order \(4(q+1)\);

(ii) \(|W| = 5^2\); \(F(H) \cong Q_8YZ_4\); and \(H/F(H) \cong Z_3\) or \(S_3\); or

(iii) \(|W| = 3^4\); \(F(H) \cong Q_8YD_8\); with \(H/F(H) \cong Z_5, D_{10}, F_{20}, A_5\) or \(S_5\).

Note \(\text{char}(V)\) is a Mersenne prime or \(\text{char}(V) = 5\) and 3 divides \(|G|\).
Proof. By Theorem 2.1 we may assume $V$ is not a quasi-primitive $G$-module and choose $M$ maximal such that $M$ is normal in $G$ and $V_M$ is not homogeneous. Write $V_M = V_1 \oplus \ldots \oplus V_k$ for subspaces $V_i$ of $V$ that are permuted faithfully and transitively by $S = G/M$. Set $H_i = N_G(V_i)$ and $C_i = C_G(V_i)$. We claim there is some $1 \neq z \in F(H_1/C_1)$ that centralizes an element of each $H_1$-orbit in $V_1$. If the claim is valid, the conclusion of this corollary holds since $G$ is isomorphic to a subgroup of $H_1/C_1$ wreath $S$ and since $H_1/C_1$ satisfies the conclusion of this Corollary (via induction).

We first show $x \in H_i$ for each $i$. Let $L$ be the smallest normal subgroup of $G$ containing $x$ and note $L \subseteq F(G)$. We may assume that $x \in L - (L \cap H_i)$. Since $L \cap H_i$ is normal in $H_i$ and $L \cap H_i$ is properly contained in the nilpotent group $L$; it follows that $N_G(L \cap H_i) > H_i$. By choice of $M$, $H_i$ is a maximal subgroup of $G$ and so $L \cap H_i$ is normal in $G$. Let $0 \neq w \in V_i$ and choose a conjugate $x^g$ of $x$ that centralizes $w$. Note $x^g$ must lie in $L \cap H_i$ and so does $x$ by normality of $L \cap H_i$. Hence $x \in H_i$ for all $i$.

Set $H = H_1/C_1$. Now $G$ is isomorphic to a subgroup of $G^* = H \wr S$ in its action on $V$ and certainly $x$ centralizes an element in each $G^*$-orbit of $V$. Since $x \in H_i$ for each $i$ and $x \in F(G)$, we have that $x \in F(H_1/C_1)$ for all $i$ and thus $x \in F(G^*)$. To complete the claim, we may assume that $G = G^* = H \wr S$. Write $x = (x_1, \ldots, x_k)$. Since $x \neq 1$, we may assume without loss of generality that $x_1 \neq 1$. Set $z = x_1 \in F(H_1/C_1)$. Let $w \in V_i$. Then $(w, 0, \ldots, 0) \in V$ is centralized by $x^g$ for some $g \in G$. In particular $(w, 0, \ldots, 0)$ is centralized by $(z^g, 0, \ldots, 0)$ and $w$ is centralized by $z^g$. But $\{(z^g, 0, \ldots, 0) | g \in G\} = \{(z^h, 0, \ldots, 0) | h \in H\}$. Hence, $w$ is centralized by some $H$-conjugate of $z$, proving the claim and the Corollary.

In [4], it was proven that a non-vanishing element $x$ of a solvable group satisfies $x^2 \in F(G)$ and conjectured that $x \in F(G)$. Recall $x$ is a non-vanishing of $G$ if $\chi(x) \neq 0$ for all $\chi \in \text{Irr}(G)$.

**Corollary 2.4.** If $G$ is solvable and no Mersenne prime divides $|G|$, then every non-vanishing element of $G$ is in $F(G)$.

**Proof.** Assume $x \in G$ is non-vanishing. Note that $Nx$ is a non-vanishing element of $G/N$ whenever $N > 1$ is normal in $G$. By induction, $Nx \in F(G/N)$ and so $x$ centralizes every chief factor of $G/N$. If $1 \neq M$ is normal in $G$ and $N \cap M = 1$, then $x$ also centralizes all chief factors of $G/M$ and those of $G$; whence $x \in F(G)$. So we may assume $G$ has a unique minimal normal subgroup.

Gashutz’s Theorem (1.12 of [5]) states that $F(G)/\Phi(G) = F(G/\Phi(G))$ is a completely reducible and faithful $G/F(G)$-module whenever $G$ is solvable. By the inductive hypothesis and the last paragraph, $\Phi(G) = 1$ while $F(G)$ is the unique minimal normal subgroup of $G$. Then $F(G)$ and $\text{Irr}(F(G))$ are faithful irreducible $G/F(G)$-modules.

Let $\lambda \in \text{Irr}(F(G))$. Since $\chi(x) \neq 0$ for all $\chi \in \text{Irr}(G|\lambda)$; we have that $\psi^G(x) \neq 0$ for some $\psi \in \text{Irr}(I_G(\lambda))$. This implies that $x^g \in I_G(\lambda)$ for some $g \in G$. By induction $x \in F_2(G)$. So $x \in F(G)$ or the hypotheses of Corollary 2.3 are satisfied with respect to the action of $G/F(G)$ on $\text{Irr}(F(G))$. Since $|G|$ is not divisible by a Mersenne prime, the conclusion of Corollary 2.3 is not satisfied. So $x \in F(G)$. \qed
Corollary 2.5. Suppose $G$ is solvable and $F = F(G)$ is a minimal normal subgroup of $G$. If $F$ is a primitive $G/F$-module, then every non-vanishing element of $G$ lies in $F$.

Proof. Note that by Gaschütz’s theorem, $F$ is indeed a faithful irreducible $G/F$-module (and $F$ even splits over $F$). If $G^* = G/F$ has a regular orbit on $V = \text{Irr}(F)$; then there exists $\lambda \in \text{Irr}(F)$ with $I_G(\lambda) = F$ and $\lambda^G \in \text{Irr}(G)$ vanishes off $F$ and the conclusion holds. We may assume that $G^*$ and $V$ satisfy the conclusion of Theorem 2.2.

Let $F_i = F_i(G)$. Observe that in all cases of the conclusion of Theorem 2.2 with $G$ solvable and for all $i > 0$, there exists $F_i \subseteq A_i \subseteq F_{i+1}$ such that $A_i$ is normal in $G$ with $A_i/F$ abelian and there exists $\theta \in \text{Irr}(F_{i+1}/F_i)$ that vanishes on $F_{i+1} - A_i$. By normality, every $G$-conjugate of $\theta$ vanishes on $F_{i+1} - A_i$. By Clifford’s Theorem, every $\chi \in \text{Irr}(G|\theta)$ vanishes on $F_{i+1} - A_i$. Since $A_i/F$ is abelian, there exists $\lambda \in \text{Irr}(F_i)$ in a regular orbit of $A_i/F$, Since $A_i$ is normal in $G$, it follows that every $\beta \in \text{Irr}(G|\lambda)$ vanishes on $A_i - F_i$. Thus for all $i > 0$, every non-vanishing element of $G$ lying in $F_{i+1}$ indeed lies in $F_i$. As $G$ is solvable, every non-vanishing element of $G$ lies in $F$. □

What remains is to prove Theorem 2.2.

If $V$ is a completely reducible and faithful $N$-module for a nilpotent group $N$, then a key result of [6, Theorem 3.1] says that $N$ has a regular orbit on $V \oplus V$. Applying this and Clifford’s theorem, we have the following:

Lemma 2.6. Suppose $V$ is an irreducible faithful $G$-module and $N$ is a normal nilpotent subgroup of $G$. If $N$ has no regular orbit on $V$, then $V_N = V_1 \oplus \ldots \oplus V_k$ for non-isomorphic irreducible $N$-modules $V_i$, $i = 1, \ldots, k$. In particular, if $V$ is quasi-primitive and $N$ has no regular orbit, then $V_N$ is irreducible.

Lemma 2.7. If $V$ is a faithful quasi-primitive $G$-module and $P$ is a normal $p$-subgroup of $G$; then

(i) $P = EZ$ where $Z = Z(P)$ and $E$ is an extra-special $p$-group (allowing $|E| = p$). Note $P \cap Z = Z(E)$. Also exp$(E) = p$ if $p$ is odd. or

(ii) $p = 2, P = ET$ where $T$ is quaternion, dihedral or semi-dihedral with $|T| > 8$. Also $E$ is extra-special (allowing $|E| = 2$) and $E \cap T = Z(E)$. Also $T$ has a unique cyclic subgroup $U$ of index two and both $U$ and $EU$ are characteristic in $P$.

For a non-central element $x$ of $P$ of order $p$, we have $|C_V(x)| \leq |V|^{1/p}$. Also, if $V_0$ is an irreducible $EU$-submodule of $V$, then $(V_0)_U \cong eW$ for an irreducible $U$-submodule $W$ where $e^2 = |E/Z(E)|$.

Proof. That (i) or (ii) holds follows from Theorem 1.2 of [5]. Note, for $p = 2$, this is slightly different from 1.9 or 1.10 of [5], which require solvability of $G$. In particular, we do not assume $E$ or $T$ is normal in $G$ (in (ii)) and $E$ maybe different from that of Theorem 1.9 of [5]. Let $x$ be a non-central element of $P$ of order $p$. If $x \in EU$ which is normal in $G$, then the Brauer character afforded $< x >$ is a multiple of the regular $\rho_{<x>}$ and so $|C_V(x)| = |V|^{1/p}$. So we may assume that $p = 2$ and $x \notin U$. But $< Ux >$ is normal in $P$ and $x$ is conjugate to some $ux$ for $1 \notin u \in U$. Since $C_V(u) = 0$, it follows that $C_V(x) \cap C_V(ux) = 0$. Since $|C_V(x)| = |C_V(ux)|$; it follows that $|C_V(x)| \leq |V|^{1/2}$. □
If \( p \) is odd in the above Lemma, then all elements of order \( p \) lie in \( E \). If \( p = 2 \) and we are in part (i) above, then all elements of order 2 lie in \( EZ_0 \) where \( Z_0 \) has order 4. It is a bit more complicated in (ii). We do a count, which above lets us know how many involutions are in certain subgroups of \( P \). For a subset \( X \) of a group, we let \( \sigma_2(X) = \{|x \in X|x^2 = 1\} \) and \( \tau_4(X) \) denote the number of elements of \( X \) of order 4.

**Proposition 2.8.**  
(i) If \( E \) is extra-special of order \( 2^{2n+1} \); then \( \sigma_2(E) = 2^{2n} \pm 2^n \) and \( \tau_4(E) = 2^{2n} \mp 2^n \) (respectively);  
(ii) If \( EU \) is a non-trivial central product of an extra-special group \( E \) of order \( 2^{2n+1} \) and a cyclic 2-group \( U \) with \(|U| \) at least 4; then \( \sigma_2(EU) = 2^{2n+1} \); and  
(iii) If \( ET \) is a non-trivial central product of an extra-special 2-group \( E \) and a 2-group \( T \) that is dihedral, quaternion, or semi-dihedral; then \( \sigma_2(ET) = 2^{2n+1} + |U|2^{2n-1} \) if \( T \) semi-dihedral and \( \sigma_2(ET) = 2^{2n+1} + |U|(2^{2n-1} \pm 2^{n-1}) \) otherwise where \( U \) is a cyclic subgroup of \( T \) on index 2.

**Proof.** First suppose that \( A \) and \( B \) are 2-groups with unique central involutions, say \( a \) and \( b \) respectively. Also assume that \( x^2 = a \) (respectively \( y^2 = b \)) whenever \( x \in A \) has order 4 (respectively \( y \in B \) has order 4). If \( G \) is the non-trivial central product of \( A \) and \( B \); i.e. \( G = (AB)/Z \) where \( Z = \langle ab \rangle \); then

\[
\sigma_2(G) = \frac{1}{2}(\tau_4(A)\tau_4(B) + \sigma_2(A)\sigma_2(B)).
\]

Note that equation (2.1) follows from the following two observations: If \( w = (x,y) \) is in \( A \times B \), then \( w^2 \) is in \( Z \) if and only if both \( x^2 = a \) and \( y^2 = b \) or both \( x^2 = 1 \) and \( y^2 = 1 \). Also, \( w^2 \) is in \( Z \) if and only if \( (zw)^2 \) is in \( Z \).

Since an extra-special two-group is a central product of extra-special two-groups of order 8 (i.e. \( D_8 \) and \( Q_8 \)) and since \( \{\sigma_2(E), \tau_4(E)\} = \{2,6\} \) when \(|E| = 8 \), part (i) can be proven by induction using equation (2.1). Note the two equalities in (i) are equivalent as \(|E| = 2^{2n+1} \) and \( \exp(E) = 4 \). Parts (ii) and (iii) follow from part (i) and equation (2.1). For (iii), we use that every one of the \(|U| \) elements of \( T - U \) has order 2 or 4 and the number of elements of order 2 in \( T - U \) is \(|U| \) (when \( T \) is dihedral), 0 (when \( T \) is quaternion) or \(|U|/2 \) (if \( T \) semi-dihedral).

**Corollary 2.9.** If \( EU \) is a non-trivial central product of an extra-special group \( E \) of order \( 2^{2n+1} \) and a cyclic 2-group \( U \) with \(|U| \) at least 4; then the number of (elementary abelian) subgroups of order 4 that intersects \( U \) trivially is \((2^{2n+1} - 2) \cdot (2^{2n-1} - 2)/3\).

**Proof.** Let \( x \) be a non-central involution. Then \( CEU(x) \) has index two in \( EU \) and is a union of \( U \)-cosets. But each \( U \)-coset has two elements of order 4 and two of order 2 or 1. Thus \( CEU(x) \) has \(|EU|/4 = 2^n \) elements of order two or one including 1 and \( x \). For each involution \( y \neq x \) that commutes with \( x \), there is a unique subgroup of order 4 containing \( x \) and \( y \) (and \( xy \)). Exactly one of these elementary abelian subgroups intersects \( U \) non-trivially. Thus there are exactly \((2^{2n} - 2)/2 - 1 \) elementary abelian subgroups of order 4 of \( EU \) containing \( x \) that “avoid” \( U \). As \( EU \) has \( 2^{2n+1} - 2 \) non-central involutions, the Corollary follows. \( \square \)
Theorem 2.10. Suppose that V is a faithful irreducible P-module and P = EZ for Z = Z(P) and an extra-special 2-group E. If P has no regular orbit on V, then

(i) |V| = 9 and P = E ≅ D8;
(ii) |V| = 5^2; |E| = 8, |Z| = 4, and |P| = 16; or
(iii) |V| = 3^4 and P = E is a central product of Q_8 and D_8.

In all cases, C_E(v) = < t > for an involution t whenever v ∈ V#.

Proof. First consider the case P = E has order 8, so that P ≅ Q_8 or D_8. The only involution in Q_8 is central and so every Q_8-orbit on V# is regular. Assume E ≅ D_8 and note that |V| = q^2 and each of the four non-central involutions centralizes q − 1 non-zero vectors of V with no overlap. Then E has a regular orbit unless q^2 − 1 = 4(q − 1); i.e. q = 3. In the case q = 3, C_E(v) = < t > for an involution t for all v ∈ V#.

The result holds when P = E has order 8.

Still assume that |E| = 8. Now P = Q × S for a cyclic 2' group S and a 2-group Q with Q = EZ_0 where Z_0 = Z(Q). Every S-orbit on V# is regular and (|S|, |Q|) = 1. So P has no regular orbit if and only if Q has no regular orbit. If |Q| = 8, then |V| = 9 by the last paragraph, whence S = 1 and P = EZ = E and conclusion (i) holds. So we may assume that 4 divides |Z_0|. By Proposition 2.8, Q has exactly 6 non-central involutions, each of which is contained in the subgroup Q_0 of Q of order 16 that contains E. So Q_0 has no regular orbit and each v ∈ V# is centralized by one and only one involution. Now |C_V(t)| = |V|^{1/2} for each non-central involution (Lemma 2.7); and so |V| − 1 = 6(|V|^{1/2} − 1). Then |V| = 5^2; S = 1 and |Z| = 4; as in conclusion (ii).

When |V| = 3^4 and E is a central product of Q_8 and D_8 (here E has 10 non-central involutions); then E = F(B) for a Bucht group B ⊆ GL(V) of order 2^5 · 5. Since B acts transitively on V#; whence C_B(v) = C_E(v) has order two for all v ∈ V#. Note that Z = C_P(E) must have order 2 and conclusion (iii) holds.

As above, P = Q × S for a cyclic 2'-group S and a 2-group Q ⊇ E with S acting Frobeniously on V. Since P has no regular orbit on V, each v ∈ V# is centralized by an involution in Q. Hence V = ⋃ C_V(x) where the union is over all non-central involutions of Q. By Lemma 2.7, |C_V(x)| ≤ |V|^{1/2} for all involutions in Q, which is a central product of E and Z(Q). If t is the number of non-central involutions of Q, then t ≥ |V|^{1/2}.

Now |P : Z| = |Q : Z(Q)| = |E : Z(E)| = e^2 where e = 2^n for an integer n > 0. By the second paragraph, we may assume that n > 1 and e ≥ 4. Now V is an irreducible P-module by Lemma 2.6 and V_Z ≅ W_e for a faithful irreducible Z-module W. Note |Z| divides q − 1 where q = |W|. Now t ≥ q^{e/2} by the last paragraph.

By Proposition 2.8, we see 2e^2 = 2^{2n+1} ≥ t ≥ q^{e/2}. Since 2e^2 ≥ q^{e/2} and e ≥ 4; we have that e = 8 and q = 3 or that e = 4 and q is 3 or 5. Note that |Z| divides q − 1. When e = 8 and q = 3, then P = Q = E and Proposition 2.8 shows that 72 = e^2 + e ≥ t ≥ 3^4, a contradiction. Similarly, we have a contradiction when P = Q = E, e = 4, and q = 5. Hence we may assume that e = 4 and with q = 3
or 5. When \( q = 5 \), then \( |Z| = 4 \). When \( q = 3 \), \( |Z| = 2 \) and \( P = E \cong Q_8 \langle Y Q_8 \cong D_8 \rangle Y D_8 \) by the third paragraph. Applying Proposition 2.8, note that \( t = 30 \) when \( q = 5 \) and \( t = 18 \) when \( q = 3 \).

For \( v \in V^\# \), \( C_P(v) \) is isomorphic to a subgroup of \( P/Z \) as \( C_Z(v) = 1 \). So \( C_P(v) \) is elementary abelian of order at most 4. Hence \( C_P(v) \) has 0, 1, or 3 involutions. Since \( P \) has no regular orbit; \( C_P(v) \) has exactly 1 or 3 involutions for \( v \in V^\# \). Let \( a_i \) denote the number of \( v \) in \( V \) centralized by exactly \( i \) involutions in \( P \). Since each non-central involution centralizes \( q^2 - 1 \) elements of \( V^\# \); it follows that 
\[
a_1 + a_3 = q^4 - 1 \quad \text{and} \quad a_1 + 3a_3 = t(q^2 - 1) .
\]
For \( q = 3 \), we then have \( a_1 = 48 \) and \( a_3 = 32 \). For \( q = 5 \), then \( a_1 = 576 \) and \( a_3 = 48 \).

If \( B \) is any elementary abelian subgroup of \( P \) of order 4 that avoids \( Z \) (i.e. \( B \cap Z = 1 \)), then the Brauer character afforded \( B \) by \( V \) is the regular character \( \rho_B \). In this case, \( C_V(B) \) is a one dimensional and there are \( q - 1 \) \( v \) in \( V^\# \) centralized by \( B \). In fact, \( B = C_P(v) \) for these \( q - 1 \) vectors. No non-zero vector is centralized by two such \( B \). Thus there exist at most \( a_3/(q - 1) \) (i.e. 16 or 12) elementary abelian subgroups of \( P \) that avoid \( Z \). If \( |Z| = 4 \), then Corollary 2.9 shows that there are 60 elementary subgroups of order 4 that avoid \( Z \), a contradiction. Thus \( |Z| = 2, q = 3 \), and \( P = E \cong Q_8 \langle Y Q_8 \cong D_8 \rangle Y D_8 \). Now \( P \) is a normal subgroup of a group \( G \subseteq GL(V) \) with \( G/P \cong Z_3 \times Z_3 \) (here \( G \) is a central product of two copies of \( SL(2,3) \)). Now \( G/P \) has a regular orbit on \( P/Z(P) \). So there is an \( x \in P - Z(P) \) whose centralizer in \( G \) is in \( P \) and hence the \( G \)-conjugacy class of \( x \) has 18 elements. Since \( P \) has 18 non-central involutions (and 12 elements of order 4), the involutions in \( P \) are \( G \)-conjugate. If \( O \) is the set of vectors in \( V \) fixed by exactly one involution, it follows that \( t \) divides \( |O| \), a contradiction as \( t = 18 \) and \( |O| = 48 \). This completes the proof.

Alex Turull has an inclusion-exclusion counting arguments that determines the exact number of regular orbits above, although the argument is a bit different for when \( |U| = 2 \) or not and they are a bit longer than current arguments.

**Proof of Theorem 2.3** We assume that \( V \) is a faithful irreducible and quasi-primitive \( G \)-module and that \( F = F(G) \) has no regular orbit on \( V \). We must show \( V \) and \( G \) satisfy one of the conclusions (a)-(e) of Theorem 2.2. In particular, we must show \( |V| = 5^2, 3^4, 3^8 \) or \( q^2 \) for a Mersenne prime.

Applying Lemma 2.7, \( F = ET \) where the Sylow-subgroups of \( E \) are extra-special (or of prime order). Also \( T = C_F(E) \) is cyclic or there exists \( U \) normal in \( G \) with \( |T : U| = 2 \), \( U \) cyclic, and 8 dividing \( |U| \). In all cases \( T \cap E = Z(E) \). Since \( F \) has no regular orbit, \( V \) is an irreducible \( F \)-module by Lemma 2.6. Also \( V_T \cong eW \) for a faithful irreducible \( T \)-module \( W \) and integer \( e \) where \( e^2 = |E : Z(E)| = |F : T| \).

Note that \( |U| \) divides \( |W| - 1 \).

First suppose that \( F = T \); i.e. that \( e = 1 \). Since \( U \) acts fixed-point-freely on \( V \) and \( F \) has no regular orbit on \( V \), each \( v \in V^\# \) is fixed by a unique involution in \( T - U \). Indeed each \( v \in V^\# \) is fixed by a unique involution \( T_0 - U_0 \) where \( T_0 \) and \( U_0 \) are the Sylow-2-subgroups of \( T \) and \( U \) respectively. It follows from Lemma 2.7 that \( T_0 \) is dihedral or semi-dihedral and so the number of involutions in \( T_0 - U_0 \) is either \( |U_0| \) or \( |U_0|/2 \). Now \( |C_V(x)| = |V|^{1/2} \) for all involutions \( x \) in \( T_0 - U_0 \). By uniqueness, \( |V| - 1 = t(|V|^{1/2} - 1) \) where \( t \) is the number of involutions in \( T_0 - U_0 \). Thus \( |V|^{1/2} + 1 \) is a power of 2. Since \( |V|^{1/2} \) is a power of a prime, it follows that \( |V| = q^2 \) for a Mersenne prime \( q \). Also \( t = q + 1 \).
and \( |T_0| = 2|U_0| \) is \( 2t \) or \( 4t \). If \( W_0 \) is an irreducible \( U \)-submodule of \( V \) and \( 0 \neq w \in W_0 \), then \( w \) is
centralized by an involution in \( T - U \); whence \( W_0 \) is invariant under \( < U, x > = T = F \). So \( V \) is an
irreducible \( U \)-module and since \( U \) is normal and abelian, it follows that \( G \subseteq \Gamma(V) \) (see \[5\] Theorem 2.1]). Here \( V \) and \( G \) are as in conclusion (a) of Theorem 2.2. So we assume that \( F > T \) and \( e > 1 \).

Since \( F \) has no regular orbit on \( V \), we have that \( V = \bigcup C_V(x) \) where the union is taken over all
non-central elements \( x \) of prime order in \( F \). Now \( |C_V(x)| \leq |V|^{1/p} \) if \( x \in F \) has order \( p \), a prime by
Lemma 2.7. Let \( s \) denote the number of non-central subgroups of odd prime order of \( F \) and let \( t \) be
the number of non-central involutions in \( F \). Then
\[
(2.2) \quad s \cdot |V|^{1/3} + t \cdot |V|^{1/2} \geq |V| \text{ and } s + t \geq |V|^{1/2}.
\]
In particular, \( |F(G)| > s + t \geq |V|^{1/2} \). Also, \( |V| = |W|^e \). On the other hand, \( |F| = e^2|U| |T : U| \)
with \( |T : U| \leq 2 \). Also \( 8 \) divides \( |U| \) if \( |T : U| = 2 \). Also \( |U| \) divides \( |W| - 1 \). In particular,
\( 2e^2|W| > |F| > s + t \geq |V|^{1/2} \geq |W|^{e/2} \) and \( |W| < (2e^2)^2/(e^2 - 2) \). As \( |W| \geq 3 \), it follows that \( e < 13 \).

Since each prime divisor of \( e \) divides \( q - 1 \); we further have that \( e \leq 4; e = 5 \) with \( |W| = 11 \); \( e = 6 \)
with \( |W| = 7; e = 8 \) with \( |W| = 3 \) or \( 5 \); or \( e = 9 \) with \( |W| = 4 \).

In all the cases where \( e > 4 \), note that \( 8 \) does not divide \( |U| \) and so \( T \) is cyclic. In the case where
\( e = 8 \), now Theorem 2.10 shows that \( F \) does have a regular orbit, a contradiction. When \( e \) is 5, 6, or
9; note that
\[
|F| = e^2|U| \leq e^2(|W| - 1) \leq |W|^{e/2} \leq |V|^{1/2} \leq s + t,
\]
a contradiction. Hence we may assume \( e \) is 2, 3, or 4.

When \( e = 3 \), the Hall-2'-subgroup \( S \) of \( F \) is a central product of an extra-special group of order 27
and exponent 3 with \( Z(S) \). Hence \( s = 12 \). If \( t = 0 \), then equation (2.2) yields \( 12 \geq |V|^{2/3} \geq |W|^2 \),
a contradiction as \( 3 \) divides \( |W| - 1 \). So \( t > 0 \) and the Sylow-2-subgroup \( P \) of \( F \) is dihedral or
semi-dihedral. Note that \( P \) has a cyclic subgroup of order \( |P|/2 \) and \( 3|P|/2 \) divides \( |W| - 1 \). Also
\( t \leq |P|/2 \leq (|W| - 1)/3 \leq |V|^{1/3}/3 \). By equation (2.2),
\[
12|V|^{1/3} + |V|^{5/6}/3 > |V|.
\]
This implies \( |V| < 64 \) and so \( |W| = |V|^{1/3} < 7 \), a contradiction as \( 6 \) divides \( |W| - 1 \). Hence \( e = 2 \) or
4.

Since \( e = 2 \) or 4, the Hall-2'-subgroup \( S \) of \( F \) is cyclic, \( s = 0 \), and \( S \) acts Frobeniusly on \( V \). Thus
the Sylow-2-subgroup \( P \) of \( F \) has no regular orbit on \( V \) and so \( V_P \) is irreducible by Lemma 2.6. In
particular, \( C_G(P) \) is cyclic (see \[5\], Lemma 2.9) and \( C_G(P) \subseteq F \). If \( T \) is cyclic, then Theorem 2.10
applies to \( F \) and we have that one of the following:

(i) \( |V| = 9 \) and \( F = E \cong D_8 \);
(ii) \( |V| = 5^2; |E| = 8, |Z| = 4, \) and \( |F| = 16 \); or
(iii) \( |V| = 3^4 \) and \( F \) is a central product of \( Q_8 \) and \( D_8 \).

In (i), \( \text{Aut}(F) \) is a 2-group; so \( G \) is a 2-group and \( G \cong D_8 \). Here \( V \) is not quasi-primitive, although
\( G \subseteq \Gamma(V) = \Gamma(3^2) \) and conclusion (a) is still satisfied. In (ii), it is easy to see that \( G/C_G(P) \) is
solvable and applying Theorem 1.9 and Corollary 1.10 of \[5\] shows that \( F \) is a central product of \( Z \).
and a normal subgroup $Q \cong Q_8$ and also that $G/F \cong A_3$ or $S_3$. Conclusion (b) holds in (ii). Assume that $|V| = 3^4$ and $F$ is a central product of $Q_8$ and $D_8$. Note that $G/F$ acts faithfully on $F/Z$ where $Z = Z(F)$ and $G/F$ permutes the 10 non-central involutions of $F$. For $x$ a non-central element of $F$, the $F$-conjugacy class of $x$ is $\{x, xz\}$ where $Z = < z >$. It follows that $G/F$ faithfully permutes the 5 conjugacy classes of non-central involutions of $F$. Hence $G/F$ is isomorphic to a subgroup of $S_5$. If $G$ is a faithful $F/Z$-module. Therefore, $|G/F| = 5$; then $G/F$ is dihedral, quaternion, or semi-dihedral and $|T_0| \geq 16$. Now $\dim(W)$ is even as $W$ is an irreducible $T_0$-module.

Suppose that $e = 2$ and $|W| = 9$. Now $S = 1$ and $F = P$. Then $T = T_0 \in \text{Syl}_2(GL(2, 3))$ is semi-dihedral of order 16. Now $V$ and $F$ are as in conclusion (e) of Theorem 2.2. We still need to show that $G/F$ is isomorphic to $A_3$ or $S_3$. Since $U$ and $EU$ are characteristic in $F$, it follows that $\text{Aut}(P)$ is a $\{2, 3\}$-group. Since $V_F$ is irreducible, $C_P(G) \subseteq F(G) = P$ and $G$ is a $\{2, 3\}$-group, hence $G$ is solvable. We may apply Theorem 1.9 of [5] to assume (with no loss of generality) that $E$ and $T$ are normal in $G$, that $E \cong Q_8$ and there exists $F \subseteq A \subseteq G$ such that $A/F$ acts faithfully and completely reducible on the chief factor $F/T$ and $|G/A| \leq 2$. As $F$ is a 2-group, it follows that $G/F \cong A_3$ or $S_3$. Conclusion (d) is satisfied here.

Assume that $e = 2$. Since $\dim(W)$ is even, we may assume from the last paragraph that $|W|$ is at least 25. By Proposition 2.8, $t \leq 6 + 3|U_0|$. If $(|W| - 1)/|U_0| \geq 5$, then

$$t \leq 6 + 3(|W| - 1)/5 < |W| \leq |V|^{1/2};$$

contradicting equation (2.2). So $(|W| - 1)/|U_0| \leq 4$ and so $|W| - 1$ is a power of two or three times a power of 2. Since $|W|$ is a prime power but not a prime nor 9, elementary arguments [5] Propositions 3.1 and 3.2] show that $|W| = 5^2$ or $7^2$. Also $|U_0| = (|W| - 1)/3 = 8$ or 16 (respectively) and $|T_0| = 16$ or 32 (respectively). Now $W$ is an irreducible $U_0$-module with $U_0$ normal and abelian in $T$, we have $T \subseteq \Gamma(W)$ and $T_0 \in \text{Syl}(\Gamma(W))$. Now $T_0$ induces the automorphism $a \rightarrow a^q$ (for $q = 3$ or 5 (resp.) of $U_0$; we see that $T_0$ is semi-dihedral. Now Proposition 2.8 shows that

$$t = 6 + 2|U_0| < |W| \leq |V|^{1/2}$$

contradicting equation (2.2). Hence we may assume that $e \neq 2$.

We now have that $e = 4$. We have from Proposition 2.8 (note $n = 2$) that

$$30 + 10|U| \leq t \leq |W|^2 \leq |V|^{1/2}.$$
and $E$, an extra-special of order 32. The first statement of conclusion (e) of Theorem 2.2 is satisfied. We verify the last two statements.

Now $EU$ is characteristic in $F$ and so $EU$ is normal in $G$. By Theorem 2.10, $EU$ has a regular orbit on $V$. Suppose $1 \neq x \in F$ centralizes an element of each $G$-orbit on $V$. So does every power of $x$. Since $EU$ has a regular orbit on $V$ and $|F : EU| = 2$, we see that $<x> \cap EU = 1$ and $x$ is an involution. Every element of $V$ is centralized by a conjugate of $x$ and so $V = \cup_{g \in G} C_V(x^g)$.

Thus $|cl_G(x)||C_V(x)| \geq |V|$ and so $|cl_G(x)| \geq |V|^{1/2} = 81$. By Proposition 2.8 ((ii),(iii)); we see that $|cl_G(x)| \leq$ the number of involution in $F - EU = 64$. By this contradiction, no such $x$ exists.

Finally assume that $G$ is solvable. Applying Corollary 1.10 of [5], we may assume that $E$ and $T$ are normal in $G$ and that $E/Z(E)$ is a faithful $G/F$-module that is irreducible (of order 16) or the direct sum of two irreducible modules of order 4. Since $G/F$ preserves a symplectic form on $E/Z(E)$, then $G/F$ has no element of order 15 (see [3, Lemma 9.2.4]). Hence $F_2(G)/F$ is isomorphic to $Z_5, Z_3,$ or $Z_3 \times Z_3$ and conclusion (e) of the Theorem is satisfied. □

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References


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