International Journal of Group Theory
ISSN (print): 2251-7650, ISSN (on-line): 2251-7669
Vol. 3 No. 2 (2014), pp. 57-75.

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www.ui.ac.ir

CONJECTURES ON THE NORMAL COVERING NUMBER OF THE FINITE SYMMETRIC AND ALTERNATING GROUPS

DANIELA BUBBOLONI, CHERYL E. PRAEGER* AND PABLO SPIGA

Communicated by Robert Turner Curtis

Abstract. Let $\gamma(S_n)$ be the minimum number of proper subgroups $H_i,\ i = 1, \dots, l$ of the symmetric group S_n such that each element in S_n lies in some conjugate of one of the H_i. In this paper we conjecture that

$$\gamma(S_n) = \frac{n}{2}\left(1 - \frac{1}{p_1}\right)\left(1 - \frac{1}{p_2}\right) + 2,$$

where p_1, p_2 are the two smallest primes in the factorization of $n \in \mathbb{N}$ and n is neither a prime power nor a product of two primes. Support for the conjecture is given by a previous result for $n = p_1^{\alpha_1} p_2^{\alpha_2}$, with $(\alpha_1, \alpha_2) \neq (1, 1)$. We give further evidence by confirming the conjecture for integers of the form $n = 15q$ for an infinite set of primes q, and by reporting on a `Magma` computation. We make a similar conjecture for $\gamma(A_n)$, when n is even, and provide a similar amount of evidence.

1. Introduction

Let G be the symmetric group S_n or the alternating group A_n of degree $n \in \mathbb{N}$, acting naturally on the set $\Omega = \{1, \dots, n\}$. In this paper we study the optimum way of covering the elements of G by conjugacy classes of proper subgroups. If $H_1, \dots, H_l$, with $l \in \mathbb{N}, l \geq 2$ are pairwise non-conjugate proper subgroups of G such that

$$G = \bigcup_{i=1}^{l} \bigcup_{g \in G} H_i^g,$$

we say that $\Delta = \{H_i^g \,|\, 1 \leq i \leq l, g \in G\}$ is a *normal covering* of G and that $\delta = \{H_1, \dots, H_l\}$ is a *basic set* for G which generates Δ. We call the elements of Δ the *components* and the elements of δ

MSC(2010): Primary: 20B30; Secondary: 20B35, 11P83.
Keywords: Covering, symmetric group, alternating group.
Received: 11 October 2013, Accepted: 15 November 2013.
*Corresponding author.

the *basic components* of the normal covering. The minimum cardinality $|\delta|$ of a basic set δ for G is called the *normal covering number* of G, denoted $\gamma(G)$ and a basic set of size $\gamma(G)$ is called a *minimal basic set*. Note that the parameters $\gamma(S_n)$, $\gamma(A_n)$ are defined only for $n \geq 3$ and $n \geq 4$, respectively. More generally $\gamma(G)$ is defined for each non-cyclic group G, since cyclic groups cannot be covered by proper subgroups.

The first two authors introduced the parameter $\gamma(G)$ in [6], and proved there that $a\varphi(n) \leq \gamma(G) \leq bn$, where $\varphi(n)$ is the Euler totient function and a, b are positive real constants depending on whether G is alternating or symmetric and whether n is even or odd (see [6, Theorems A and B]).

Recently, in [7, Theorem 1.1], we established a lower bound linear in n, given by $cn \leq \gamma(G) \leq \frac{2}{3}n$, where c is a positive real constant, and in [7, Remark 6.5], which describes a general method for obtaining explicit values for c, it was shown that in the case where $G = S_n$ with n even, the constant c can be taken as 0.025 for $n > 792000$. This settled the question as to whether $\gamma(G)$ grows linearly in n, rather than growing more slowly as a constant times $\varphi(n)$. Proof of the improved lower bound in [7] relies on certain number theoretic results (see [5]), and the value of c obtained is unrealistically small because of the many approximations needed first to obtain and next to apply those results. Thus, despite the innovative methods used to achieve the definitive result that $\gamma(S_n)$ and $\gamma(A_n)$ grow linearly with n, we still do not know, and we wish to know, optimum values for the constant c, both for $\gamma(S_n)$ and $\gamma(A_n)$.

We know quite a lot (but not everything) about these parameters in the case where n is a prime power, and we summarise these results in Remark 1.3(c).

Our interest in this paper is in the case where n has at least two distinct prime divisors, that is to say, n is of the form

$$n = p_1^{\alpha_1} \cdots p_r^{\alpha_r} \tag{1.1}$$

where $r \geq 2$, the p_i are primes, $\alpha_i \in \mathbb{N}$, and $p_i < p_j$ for $i < j$. We make first two conjectures about the normal covering numbers of S_n and A_n for such values of n. We believe that the value of $\gamma(S_n)$, for all such n, as well as the value of $\gamma(A_n)$ when n is even, is strongly connected to the following quantity:

$$g(n) := \frac{n}{2}\left(1 - \frac{1}{p_1}\right)\left(1 - \frac{1}{p_2}\right) + 2 \tag{1.2}$$

where n, p_1, p_2 are as in (1.1). Note that $g(n)$ is an integer since $g(n) - 2$ is equal to the number of natural numbers less than $n/2$ and not divisible by either p_1 or p_2(see Proposition 2.4 iii)). In the case where $r = 2$ in (1.1), a connection between $\gamma(G)$ and $g(n)$ emerged from results in [7], which are summarised in Remark 1.3 (d). These results, and other evidence, motivate the following conjectures, the first of which will also appear in the forthcoming edition of the Kourovka Notebook [10, Problem 18.23].

Conjecture 1. *Let n be as in* (1.1) *with $r \geq 2$ and $n \neq p_1p_2$. Then $\gamma(S_n) = g(n)$.*

Conjecture 2. *Let n be as in* (1.1) *with n even, $r \geq 2$, and $n \neq 2p_2$ or 12. Then $\gamma(A_n) = g(n)$.*

As we note in Remark 1.3 (d), if $r = 2$ then Conjecture 2 is true, and also Conjecture 1 is true for odd n (but open for even n). Moreover, extensive computation performed with `Magma` [1] shows that both conjectures hold for the first several hundred values of n.

That $g(n)$ is a common upper bound for $\gamma(S_n)$ and $\gamma(A_n)$ for all $n \in \mathbb{N}$, with $r \geq 2$, is not difficult to prove (see Proposition 3.1). The issue is to show that $g(n)$ is also a lower bound. We provide in Theorem 1.1 further evidence for the truth of Conjectures 1 and 2, for infinite families of integers n with three distinct prime divisors (see also Corollary 5.4).

Theorem 1.1. (a) *Let $n = 15q$, where q is an odd prime such that $q \equiv 2 \pmod{15}$ and $q \not\equiv 12 \pmod{13}$. Then $\gamma(S_n) = g(n) = 4q + 2$.*

(b) *Let $n = 6q$, where q is a prime, $q \geq 11$. Then $\gamma(A_n) = g(n) = q + 2$.*

The role of the function $g(n)$ in the study of normal coverings of finite groups seems to go beyond the symmetric and alternating case: Britnell and Maróti in [3] have recently shown that, if n is as in (1.1), with $r \geq 2$, and q is a prime power, then for all linear groups G with $SL_n(q) \leq G \leq GL_n(q)$, the upper bound $\gamma(G) \leq g(n)$ holds, with equality for $r = 2$ and for other infinite families of n-dimensional classical groups with $r \geq 3$.

We make now several more remarks about Conjectures 1 and 2.

Remark 1.2. (a) Observe that $\frac{1}{2}\left(1 - \frac{1}{p_1}\right)\left(1 - \frac{1}{p_2}\right) \geq \left(1 - \frac{1}{2}\right)\left(1 - \frac{1}{3}\right) = \frac{1}{6}$. Thus, for all $n \in \mathbb{N}$ with $r \geq 2$ and $n \neq p_1p_2$, Conjecture 1 implies $\gamma(S_n) \geq \frac{n}{6} + 2$, and thus $\gamma(S_n) \geq cn$ with $c = \frac{1}{6}$, a value much larger than the value 0.025 given in [7, Remark 6.5]. Moreover, if Conjecture 1 is true, then $c = \frac{1}{6}$ is the largest constant such that $\gamma(S_n) \geq cn$ for all $n \in \mathbb{N}$. Namely Conjecture 1 would imply that $\gamma(S_n) = g(n) = \frac{n}{6} + 2$ for all multiples n of 6, greater than 6.

(b) The case $n = 12$ is a genuine exception in Conjecture 2 because $\gamma(S_{12}) = 4 = g(12)$, while $\gamma(A_{12}) = 3$ (see Lemma 6.1). We note that Lemma 6.1 corrects [6, Proposition 7.8(c)] which is incorrect just in the case $n = 12$; this error is repeated in [6, Corollary 7.10] (which asserts that $\gamma(A_{12}) = 4$). We give therefore, in Table 1, a correct version of [6, Table 3].

n	3	4	5	6	7	8	9	10	11	12
$\gamma(S_n)$	2	2	2	2	3	3	4	3	5	4
$\gamma(A_n)$	-	2	2	2	2	2	3	3	4	3

Table 1. Values of $\gamma(S_n)$ and $\gamma(A_n)$ for small n.

(c) Conjecture 2 does not address the case of $G = A_n$ with n odd. For n an odd prime, we know (see Remark 1.3 (c)) that the difference between $\gamma(S_n)$ and $\gamma(A_n)$ is unbounded. If this were the case also for odd non-prime powers n, then these parameters could not be both equal to $g(n)$. Although our computer experiments indicate that $\gamma(A_n) \neq g(n)$ for some odd integers n, we do not have a sufficiently good understanding to predict the general behaviour of $\gamma(A_n)$ for odd n.

Our third conjecture relates to the form of a minimal basic set. We state it in Section 2.2 as Conjecture 3, since it uses notation introduced in that section. In particular we show, in Proposition 3.2, that Conjecture 3 implies both Conjecture 1 for n odd, and Conjecture 2. Finally, in this introductory section, we make a few general comments about the normal covering number.

Remark 1.3. (a) For the purpose of computing $\gamma(G)$ we can always assume that the basic components in δ are maximal subgroups of G.

(b) If $\gamma(S_n)$ is realized by a basic set δ not involving A_n, then $\gamma(A_n) \leq \gamma(S_n)$, since in this case the set of subgroups $H \cap A_n$, for $H \in \delta$, forms a basic set for A_n. It is not known whether the inequality $\gamma(A_n) \leq \gamma(S_n)$ holds for all n, and this remains an open question.

(c) Suppose that $n = p^a$ for a prime p and a positive integer a. For $a = 1$ we know that, for p odd, $\gamma(S_p) = \frac{p-1}{2}$ (see [6, Proposition 7.1]) while $\gamma(A_p)$ lies between $\lceil \frac{p-1}{4} \rceil$ and $\lfloor \frac{p+3}{3} \rfloor$ (see [6, Remark 7.2]). In particular we have $\gamma(S_p) - \gamma(A_p) \geq \frac{p-9}{6}$.

Now consider $a \geq 2$. Then $\gamma(S_n) = \frac{n}{2}(1 - \frac{1}{p}) + 1$ if p is odd, and if $p = 2$ then $\gamma(S_n)$ lies between $\frac{n+8}{12}$ and $\frac{n+4}{4}$ (see [6, Proposition 7.5]). For alternating groups we have $\gamma(A_n) = \frac{n+4}{4}$ if $p = 2$ and $n \neq 8$ and when p is odd we only know that $\gamma(A_n)$ lies between $\frac{n}{4}(1 - \frac{1}{p})$ and $\frac{n}{2}(1 - \frac{1}{p}) + 1$.

(d) Suppose now that n is as in (1.1) with $r = 2$. If $\alpha_1 + \alpha_2 = 2$, that is to say, if $n = p_1 p_2$, then by [6, Proposition 7.6], $\gamma(S_n) = g(n) - 1$ if n is odd, while $\gamma(A_n) = g(n) - 1$ if n is even, with $g(n)$ as in (1.2). Moreover, if $\alpha_1 + \alpha_2 \geq 3$, then it was shown in [6, Proposition 7.8] that $\gamma(S_n) = g(n)$ if n is odd, while $\gamma(A_n) = g(n)$ if n is even. In particular, these results show that, when $r = 2$, Conjecture 2 is true, and also Conjecture 1 is true for odd n.

In the light of Remark 1.3(b)–(d) we pose the following problems.

Problem 1. Determine whether or not the inequality $\gamma(A_n) \leq \gamma(S_n)$ holds for all $n \geq 4$. Moreover is it true that for all even n, up to a finite number of exceptions, we have $\gamma(A_n) = \gamma(S_n)$?

Problem 2. Determine the values of $\gamma(S_n)$ for $n = 2^a \geq 16$, and of $\gamma(A_n)$ for all odd prime powers n.

Problem 3. Determine the values of $\gamma(S_n)$ for $n = 2^{\alpha_1} p_2^{\alpha_2}$, and of $\gamma(A_n)$ for odd integers $n = p_1^{\alpha_1} p_2^{\alpha_2}$, where $\alpha_1, \alpha_2 \in \mathbb{N}$.

2. **Notation and basic facts**

2.1. **Arithmetic.** In the study of the normal coverings of symmetric and alternating groups we often face some arithmetic questions. Thus it is natural to begin with a purely arithmetic section. For $a, b \in \mathbb{R}$, with $a \leq b$, we use the usual notation $(a, b), [a, b), (a, b], [a, b]$ to denote the intervals.

Throughout this section, for $n \in \mathbb{N}$, assume the notation (1.1), with $r \in \mathbb{N}$, that is write $n = p_1^{\alpha_1} \cdots p_r^{\alpha_r}$, with p_i primes, $\alpha_i \in \mathbb{N}$, and $p_i < p_j$ for $i < j$. Note that we include here the possibility that n is a prime power.

Definition 2.1. *For each subset K of $R := \{1, \dots, r\}$, define the following subsets of $\mathbb{N}$:*

$$P_K := \{x \in \mathbb{N} : 1 \leq x \leq n,\ p_k \mid x \textit{ for all } k \in K\}$$

$$P^K := \{x \in \mathbb{N} : 1 \leq x \leq n,\ p_k \nmid x \textit{ for all } k \in K\}$$

and the rational numbers:

$$p_K := \prod_{k \in K} \frac{1}{p_k}, \qquad p^K := \prod_{k \in K} \left(1 - \frac{1}{p_k}\right).$$

For $I, J \subseteq R$ with $I \cap J = \varnothing$, set

$$P_I^J := P_I \cap P^J.$$

Note that $P_I = P_I^{\varnothing}$ and $P^J = P_{\varnothing}^J$ and, as usual, we consider a product over the empty set to be equal to 1. From now on we use the notation introduced in Definition 2.1, without further reference. For the purpose of this paper it is important to obtain the order of $P_I^J \cap [1, n/2)$, because that allows us to give an interpretation of the quantity $g(n)$ as well as to count, in many circumstances, various sets of partitions or sets of intransitive maximal subgroups of S_n. We start by finding the cardinality of P_I^J.

Lemma 2.2. *Let $n, a, b, c \in \mathbb{N}$, with $n = abc$ and $\gcd(a, b) = 1$. If $d \mid a$, then*

$$|\{x \in \mathbb{N} : 1 \leq x \leq n,\ d \mid x,\ \gcd(x, b) = 1\}| = \frac{n\, \varphi(b)}{bd}.$$

In particular, for all $I, J \subseteq R$ with $I \cap J = \varnothing$, we have $|P_I^J| = n\, p_I p^J$.

Proof. Let $n, a, b, c, d \in \mathbb{N}$, with $n = abc$, $\gcd(a, b) = 1$ and $d \mid a$. We set

$$X = \{x \in \mathbb{N} : 1 \leq x \leq n,\ d \mid x,\ \gcd(x, b) = 1\}$$

and

$$Y = \{x \in \mathbb{N} : \text{there exists } u \in \mathbb{N}, u \leq n/d \text{ with } x = du,\ \gcd(u, b) = 1\}.$$

Since $d \mid a$ and $\gcd(a, b) = 1$, we have that $\gcd(d, b) = 1$. It follows that $X = Y$. In particular $|X|$ is equal to the number of positive integers less than or equal to $n/d = b\left(c\,\dfrac{a}{d}\right) \in \mathbb{N}$ which are coprime to b. But, given any $k \in \mathbb{N}$, the number of positive integers coprime to b and contained in the interval $[1, bk]$, is given by $k\varphi(b)$. Thus $|X| = c\,\dfrac{a}{d}\varphi(b) = \dfrac{n\,\varphi(b)}{bd}$.

Let now $I, J \subseteq R$ with $I \cap J = \varnothing$ and consider $a = \prod_{i \in I} p_i^{\alpha_i}, d = \prod_{i \in I} p_i, b = \prod_{j \in J} p_j^{\alpha_j}, c = \frac{n}{ab}$. Since $I \cap J = \varnothing$, we have that $\gcd(a, b) = 1$ and thus the above result applies giving

$$|P_I^J| = \frac{nb \prod_{j \in J}(1 - \frac{1}{p_j})}{b \prod_{i \in I} p_i} = n\, p_I p^J.$$

$\square$

Next we decide, for n even, when the integer $n/2$ lies in P_I^J.

Lemma 2.3. *Let $n \in \mathbb{N}$ be even (that is, $p_1 = 2$), and let $I, J \subseteq R$ with $I \cap J = \varnothing$.*

i) *Suppose $\alpha_1 = 1$. Then $n/2 \in P_I^J$ if and only if $I \subseteq \{2, \dots, r\}$ and either $J = \{1\}$ or $J = \varnothing$.*

ii) *Suppose* $\alpha_1 \geq 2$. *Then* $n/2 \in P_I^J$ *if and only if* $J = \varnothing$.

Proof. i) Since $\alpha_1 = 1$, we have that $n/2 = p_2^{\alpha_2} \cdots p_r^{\alpha_r}$ is odd. Suppose that $n/2 \in P_I^J$. Then $1 \notin I$ so $I \subseteq \{2, \dots, r\}$. Moreover, if $j \in J$ then $p_j \nmid n/2 = p_2^{\alpha_2} \cdots p_r^{\alpha_r}$ and thus the only possibility is $j = 1$, so that $J \subseteq \{1\}$. It follows that $J = \{1\}$ or $J = \varnothing$. Conversely, it is clear that, if $I \subseteq \{2, \dots, r\}$ and either $J = \{1\}$ or $J = \varnothing$, then $n/2 \in P_I^J$

ii) Since $\alpha_1 \geq 2$, we have $n/2 = 2^{\alpha_1 - 1} \cdots p_r^{\alpha_r}$ even and divisible by the same prime factors as n. Thus if $n/2 \in P_I^J$, we necessarily have $J = \varnothing$. Conversely if $J = \varnothing$, then $n/2 \in P_I^{\varnothing} = P_I$ holds for all $I \subseteq R$. □

Proposition 2.4. *Let* $n \in \mathbb{N}$ *and let* $I, J \subseteq R$ *with* $I \cap J = \varnothing$.

i) *If* n *is odd, then* $|P_I^J \cap [1, n/2)| = \lfloor \frac{n}{2} p_I p^J \rfloor$.

ii) *If* n *is even (that is,* $p_1 = 2$*), then we have the following:*

a) *Suppose* $\alpha_1 = 1$. *If* $J = \varnothing$ *and* $I \subseteq \{2, \dots, r\}$, *then* $|P_I^J \cap [1, n/2)| = \frac{n p_I p^J}{2} - 1$, *and otherwise* $|P_I^J \cap [1, n/2)| = \lfloor \frac{n}{2} p_I p^J \rfloor$.

b) *Suppose* $\alpha_1 \geq 2$. *If* $J = \varnothing$, *then* $|P_I^J \cap [1, n/2)| = \frac{n p_I p^J}{2} - 1$, *and otherwise* $|P_I^J \cap [1, n/2)| = \frac{n}{2} p_I p^J$.

iii) *If* $r \geq 2$, *and* $J = \{j_1, j_2\} \subseteq R$ *with* $j_1 \neq j_2$, *then*

$$|P^J \cap [1, n/2)| = \frac{n}{2}\left(1 - \frac{1}{p_{j_1}}\right)\left(1 - \frac{1}{p_{j_2}}\right) = g(n) - 2.$$

In particular, for all $n \in \mathbb{N}$ *with* $r \geq 2$, $g(n)$ *is a positive integer.*

iv) *If* $n > 2$ *then* $|P^R \cap [1, n/2)| = \frac{\varphi(n)}{2}$.

Proof. Let $A = P_I^J \cap (0, n/2)$ and $B = P_I^J \cap (n/2, n)$. Since, trivially, $A = P_I^J \cap [1, n/2)$, our aim is to compute $|A|$. Consider the bijective map $f : \mathbb{Z} \to \mathbb{Z}$ given by $f(x) = n - x$, for all $x \in \mathbb{Z}$. Note that, since for all $i \in R$, we have $p_i \mid x$ if and only if $p_i \mid n - x$, then $f(A) = B$. In particular $|A| = |B|$. Moreover, we have

$$A \cup B \subseteq P_I^J \subseteq A \cup B \cup \{n/2, n\}$$

and thus, by Lemma 2.2, observing that $A \cap B = \varnothing$,

$$\frac{n p_I p^J}{2} - 1 \leq |A| \leq \frac{n p_I p^J}{2},$$

where the bounds are not necessarily integers.

To decide the exact value of $|A|$ we must take in account when $n, n/2 \in P_I^J$. It is easy to see that $n \in P_I^J$ if and only if $J = \varnothing$. For the case of $n/2$ we invoke Lemma 2.3, and we distinguish several cases.

Assume first that n is odd. If $J = \varnothing$, we have $n \in P_I^J = P_I$ while $n/2$ is not an integer. Hence in this case $2|A| + 1 = |P_I| = n p_I$ which implies that p_I is odd and $|A| = \lfloor \frac{n p_I p^J}{2} \rfloor$, since $p^J = 1$. If $J \neq \varnothing$, then $n, n/2 \notin P_I^J$ and thus $2|A| = |P_I^J| = n p_I p^J$, which implies that $p_I p^J$ is even and $|A| = \frac{n p_I p^J}{2} = \lfloor \frac{n p_I p^J}{2} \rfloor$. In particular we have now proved part i), and we have also proved parts iii) and iv) for n odd.

Next assume that n is even but not divisible by 4. If $J = \varnothing$ and $I \subseteq \{2, \ldots, r\}$, then both $n, n/2 \in P_I^J$, so that $2|A| + 2 = np_I p^J$ and hence $np_I p^J$ is even and $|A| = \frac{np_I p^J}{2} - 1$. If $J = \{1\}$, then $n/2 \in P_I^J$ but $n \notin P_I^J$, and so $2|A| + 1 = np_I p^J$ which implies that $np_I p^J$ is odd and $|A| = \lfloor \frac{np_I p^J}{2} \rfloor$. In the remaining cases either (1) $J = \varnothing$ and $1 \in I$, or (2) J is neither $\varnothing$ nor $\{1\}$. In both of these cases $n/2 \notin P_I^J$; in case (1) we have $n \in P_I^J$ so that $2|A| + 1 = |P_I^J|$ and in case (2) we have $n \notin P_I^J$ so that $2|A| = |P_I^J|$. Thus in both cases $|A| = \lfloor \frac{np_I p^J}{2} \rfloor$ and part ii)a) is proved as well as part iv) with $n \equiv 2 \pmod 4$: note that the condition $n > 2$ in iv) guarantees that $\frac{np^R}{2}$ is an integer. The particular situation with $I = \varnothing$ and $|J| = 2$ arises in case (2) and here we have $|A| = \frac{np_I p^J}{2}$, which proves part iii) for $n \equiv 2 \pmod 4$.

Finally assume that n is divisible by 4. If $J = \varnothing$, then both $n, n/2 \in P_I^J$ and $2|A| + 2 = np_I p^J$; so $np_I p^J$ is even and $|A| = \frac{np_I p^J}{2} - 1$. If $J \neq \varnothing$, then neither n nor $n/2$ lies in P_I^J and hence $2|A| = np_I p^J$, so that $np_I p^J$ is even and $|A| = \frac{np_I p^J}{2}$. Thus part ii)b) is proved as well as parts iii) and iv) for n divisible by 4. □

2.2. **Maximal subgroups.** Let $G = S_n, A_n$. As mentioned in Remark 1.3 (a), in order to determine $\gamma(G)$ we may assume that the basic components of a normal covering for G are maximal subgroups of G. These subgroups may be intransitive, primitive or imprimitive. Each maximal subgroup of A_n which is intransitive or imprimitive is obtained as $M \cap A_n$, where M is maximal in S_n and respectively intransitive or imprimitive. Thus to give an overview of the maximal subgroups of G which are intransitive or imprimitive, it is enough to describe them for $G = S_n$.

The set of maximal subgroups of S_n which are intransitive is given, up to conjugacy, by

$$\mathcal{P} := \{\ P_x = S_x \times S_{n-x}\ : 1 \leq x < n/2\}. \tag{2.1}$$

If $\mathfrak{X} \subseteq \mathbb{N} \cap [1, n/2)$ we use the notation

$$\mathcal{P}_{\mathfrak{X}} := \{\ P_x \in \mathcal{P}\ :\ x \in \mathfrak{X}\}. \tag{2.2}$$

The set of imprimitive maximal subgroups of S_n is given, up to conjugacy, by

$$\mathcal{W} := \{\ S_b \wr S_m\ : 2 \leq b \leq n/2, b \mid n, m = n/b\}.$$

Recall that, for n even, the intransitive subgroup $P_{n/2} = S_{n/2} \times S_{n/2}$ is not maximal in S_n because it is contained in the imprimitive subgroup $S_{n/2} \wr S_2$.

Intransitive subgroups play a major role in normal coverings of the symmetric and alternating group and we conjecture that, apart from at most two components, each minimal basic set consists of intransitive components. To be more precise we make the following conjecture.

Conjecture 3. *Let n as in (1.1), with $r \geq 2$. Then, for each minimal basic set δ of S_n consisting of maximal subgroups, the subset of intransitive subgroups in δ is precisely*

$$\mathcal{P}_{\min}(S_n) := \{P_x \in \mathcal{P}\ :\ \gcd(x, p_1 p_2) = 1\}. \tag{2.3}$$

If n is even then, for each minimal basic set δ of A_n consisting of maximal subgroups, the subset of intransitive subgroups in δ is precisely

$$\mathcal{P}_{\min}(A_n) := \{P_x \cap A_n \ : P_x \in \mathcal{P}_{\min}(S_n)\}. \tag{2.4}$$

Note that in Conjecture 3 we do not exclude $n = p_1p_2$. We will see in Proposition 3.1 that, when $r \geq 2$, also two imprimitive subgroups play a role in the normal coverings of S_n and A_n.

2.3. **Partitions.** Let $n, k \in \mathbb{N}$, with $k \leq n$. A *k-partition* of n is an unordered k-tuple $T = [x_1, \dots, x_k]$, with $x_i \in \mathbb{N}$ for each $i \in \{1, \dots, k\}$, such that $n = \sum_{i=1}^{k} x_i$. We sometimes simply refer to a k-partition as a partition. The x_i are called the *terms* of the k-partition. Let $\sigma \in S_n$ and let $X_1, \dots, X_k$, with $k \in \mathbb{N}$, be the orbits of $\langle \sigma \rangle$. Let $x_i = |X_i| \in \mathbb{N}$. Note that the fixed points of σ correspond to the $x_i = 1$, while the lengths of the cycles in which σ splits are given by the $x_i \geq 2$. Then $\sum_{i=1}^{k} x_i = n$ and we say that $T = [x_1, \dots, x_k]$ is the *partition associated* to σ or the *type* of σ. For instance if $\sigma = (123) \in S_4$, then $X_1 = \{1, 2, 3\}, X_2 = \{4\}$ and the type of σ is $[1, 3]$. Since permutations in S_n are conjugate if and only if they have the same type, we can identify the conjugacy classes of permutations of S_n with the partitions of n (for more details see Section 1.1 in [7]). When a subgroup H of S_n contains a permutation of type T we say that 'T belongs to H' and we write $T \in H$.

These concepts are crucial for our research: for a set δ of subgroups of S_n is a basic set if and only if, for each partition T of n, there exists $H \in \delta$ such that T belongs to H.

In particular the following set of 2-partitions will be important for our work:

$$\mathcal{T} := \{T_x = [x, n - x] : 1 \leq x < n/2\}. \tag{2.5}$$

Note that, when n is even, we exclude the partition $[n/2, n/2]$. The partitions in $\mathcal{T}$ correspond to the simplest types of permutations in S_n apart from n-cycles, and allow us to identify most of the intransitive components in a basic set. The counting problems arising are very easily managed by the following remark.

Remark 2.5. The maps $f : \mathbb{N} \cap [1, n/2) \to \mathcal{T},\ f(x) = T_x$ and $F : \mathcal{T} \to \mathcal{P},\ F(T_x) = P_x$, with $\mathcal{P}$ and $\mathcal{T}$ as in (2.1) and (2.5), are bijections and, for each x, P_x is the only subgroup in $\mathcal{P}$ which contains a permutation of type $T_x \in \mathcal{T}$. In particular $|\mathcal{T}| = |\mathcal{P}| = \lfloor \frac{n-1}{2} \rfloor$, and for each $\mathfrak{X} \subseteq \mathbb{N} \cap [1, n/2)$ we have $|\mathcal{P}_{\mathfrak{X}}| = |\mathfrak{X}|$, with $\mathcal{P}_{\mathfrak{X}}$ as in (2.2).

The subset of $\mathcal{T}$ given by

$$\mathcal{A} := \{T_x \in \mathcal{T} \ : x \in \mathfrak{A}\}, \tag{2.6}$$

where

$$\mathfrak{A} := \{x \in \mathbb{N} : 1 \leq x < n/2,\ \gcd(x, n) = 1\} \tag{2.7}$$

and the corresponding set of intransitive components $\mathcal{P}_{\mathfrak{A}}$ play an important role in this paper. Note that, by Proposition 2.4 iv) and Remark 2.5, we have

$$|\mathcal{A}| = |\mathfrak{A}| = |\mathcal{P}_{\mathfrak{A}}| = \frac{\varphi(n)}{2}. \tag{2.8}$$

3. **An upper bound**

We produce a normal covering of S_n with a basic set of size $g(n)$ involving only intransitive and imprimitive subgroups, for any non-prime-power n (even or odd).

Proposition 3.1. *Let n be as in (1.1) with $r \geq 2$. Then $\gamma(S_n) \leq g(n)$ and $\gamma(A_n) \leq g(n)$.*

Proof. We claim that the set

$$\delta = \{P_x \in \mathcal{P} \ : \gcd(x, p_1p_2) = 1\} \cup \{S_{p_1} \wr S_{n/p_1},\ S_{p_2} \wr S_{n/p_2}\}$$

does not involve A_n and is a basic set for S_n of size $g(n)$. The argument is somewhat inspired by the proof of [6, Proposition 7.8]. Note that $p_1 \geq 2$ and $p_2 \geq 3$. Consider an arbitrary type $T = [x_1, \dots, x_k]$ of permutations in S_n, with each $x_i \in \mathbb{N}$, $k \geq 1$, and $\sum_{i=1}^{k} x_i = n$. If each x_i is divisible by p_1 then $T \in S_{p_1} \wr S_{n/p_1}$ in δ, and if each x_i is divisible by p_2 then $T \in S_{p_2} \wr S_{n/p_2}$ in δ. In particular δ covers the n-cycles. So we may assume that $k \geq 2$ and that both $K_1 = \{i \in \{1, \dots, k\} : p_1 \nmid x_i\}$ and $K_2 = \{i \in \{1, \dots, k\} : p_2 \nmid x_i\}$ are nonempty subsets of $K = \{1, \dots, k\}$.

If $K_1 \cap K_2 \neq \varnothing$, then there exists $i \in K$ such that $p_1, p_2 \nmid x_i$. In particular we have $x_i \neq n/2$, as otherwise p_2, which divides n, would divide x_i. Let x be the unique natural number in $\{x_i, n - x_i\}$ which is less than $n/2$. Note that, since $p_1, p_2 \mid n$, we have $p_1, p_2 \nmid x$. Thus $T \in P_x$ with $P_x \in \delta$.

Assume next that $K_1 \cap K_2 = \varnothing$: then for all $i \in K$ we have that if $p_1 \nmid x_i$ then $p_2 \mid x_i$ and that if $p_2 \nmid x_i$ then $p_1 \mid x_i$. Let $i \in K_1$ and $j \in K_2$. Then we have $i \neq j$, and $p_1 \nmid x_i,\ p_2 \mid x_i$, and also $p_2 \nmid x_j,\ p_1 \mid x_j$. This implies that $p_1 \nmid x_i + x_j$ and $p_2 \nmid x_i + x_j$, and it follows that $x_i + x_j \neq n, n/2$. Let x be the unique natural number in $\{x_i + x_j, n - (x_i + x_j)\}$ which is less than $n/2$: then $p_1, p_2 \nmid x$ and so $T \in P_x$ with $P_x \in \delta$. Now, to get the result for A_n, use Remark 1.3b). □

We recall that for $n = p_1p_2$, the inequalities in Proposition 3.1 are strict since, by [6, Proposition 7.6], for n odd we have $\gamma(S_{p_1p_2}) = g(p_1p_2) - 1$ and for n even we have $\gamma(A_{p_1p_2}) = g(p_1p_2) - 1$. Note that the basic set in the proof of Proposition 3.1 admits as intransitive components precisely those belonging to the set $\{P_x \in \mathcal{P} \ : \ \gcd(x, p_1p_2) = 1\}$, in line with Conjecture 3. We can also observe that Conjecture 3 is stronger of both Conjecture 1 and Conjecture 2 and that it implies a characterization of the minimal basic sets for S_n when n is odd, and for A_n when n is even. Recall the definitions of $\mathcal{P}_{\min}(S_n)$ and $\mathcal{P}_{\min}(A_n)$ given in (2.3) and (2.4).

Proposition 3.2. *Assume that Conjecture 3 holds. Let*

$$\mathcal{W}_i := \{S_{p_i} \wr S_{n/p_i}, S_{n/p_i} \wr S_{p_i}\}$$

for $i = 1, 2$. Then:

i) *Conjecture 1 and Conjecture 2 hold.*
ii) *If n is odd, with $r \geq 2$ and $n \neq p_1p_2$, then the only minimal basic sets of S_n consisting of maximal subgroups are $\delta = \mathcal{P}_{\min}(S_n) \cup \{I_{p_1}, I_{p_2}\}$, where $I_{p_i} \in \mathcal{W}_i$, for $i = 1, 2$.*
iii) *If n is even, with $r \geq 2$ and $n \neq 2p_2$, then the only minimal basic sets of A_n consisting of maximal subgroups are $\delta = \mathcal{P}_{\min}(A_n) \cup \{I_{p_1} \cap A_n, I_{p_2} \cap A_n\}$, where $I_{p_i} \in \mathcal{W}_i$, for $i = 1, 2$.*

Proof. Let $n \in \mathbb{N}$ with $r \geq 2$ and $n \neq p_1p_2$. Let δ be a minimal basic set for G consisting of maximal subgroups. Since Conjecture 3 holds by assumption, the intransitive components in δ are precisely the $g(n)-2$ subgroups in $\mathcal{P}_{\min}(G)$. Moreover, by Proposition 3.1, we know that $|\delta| \leq g(n)$. Consider the types $T_i = [p_i, n-p_i]$ for $i = 1, 2$. Observe that, by assumption, we have $n > p_1p_2$ and therefore also $p_i < n - p_i$ for $i = 1, 2$. Note that the types T_i belong to A_n if and only if n is even, and that they do not belong to any subgroup in $\mathcal{P}_{\min}(G)$. Let first $G = S_n$, with n odd or $G = A_n$, with n even. Thus there exists a transitive subgroup H_1 maximal in G, with $H_1 \in \delta$ and H_1 containing T_1. Note that $H_1 \neq A_n$ because $G = S_n$ is considered only for n odd. Assume that H_1 is primitive and examine the list of the primitive subgroups of S_n containing a permutation of type $[k, n-k]$ with $k = p_i < n/2$, in Theorem 3.3 of [11]. Since n is not a proper power, $n \neq p_1p_2$ and k is a prime, there is no such primitive subgroup H_1. It follows that H_1 is imprimitive and the only possibility is $H_1 = I_{p_1} \cap G$, for some $I_{p_1} \in \mathcal{W}_1$. Since $T_2 = [p_2, n-p_2]$ does not belong to any subgroup in $\mathcal{W}_1$, the same argument shows that δ contains also a component $H_2 = I_{p_2} \cap G$, for some $I_{p_2} \in \mathcal{W}_2$. In particular $\gamma(G) = g(n)$. So we have proved ii) and iii) as well as i) except for S_n with n even. In this last case we consider separately $A_n \notin \delta$ and $A_n \in \delta$. If $A_n \notin \delta$ then, as above, we need two further components to cover T_1 and T_2, and we find $\gamma(S_n) = g(n)$. If $A_n \in \delta$ we consider the type $T = [2, 2, n-4]$: since T does not belong to A_n or to any subgroup in $\mathcal{P}_{\min}(S_n)$, we conclude that δ has a further component containing T and thus, again, $\gamma(S_n) = g(n)$. $\square$

In other words if Conjecture 3 is true, we have a complete description of those minimal normal coverings consisting of maximal subgroups, both for S_n (n odd) and A_n (n even), with n not a prime power or the product of two primes. In particular, in these cases, this rules out any role for primitive subgroups in normal coverings.

4. Imprimitive subgroups containing permutations with at most four cycles of globally coprime lengths

Let $n \in \mathbb{N}$ be composite, say $n = bm$ where $b \in \mathbb{N}$ with $2 \leq b \leq n/2$ and $m = n/b$. Let $W = S_b \wr S_m \in \mathcal{W}$, be a maximal imprimitive subgroup of S_n stabilising a block system $\mathcal{B}$ consisting of m blocks of size b. Let $\sigma \in S_n$ such that the partition $T = [x_1, \ldots, x_k]$ associated with σ satisfies $k \leq 4$ and $\gcd(x_1, \ldots, x_k) = 1$, that is to say, the x_i are 'globally coprime'. Note that, if $k = 2$, then these partitions are the 2-partitions in the set $\mathcal{A}$ defined in (2.6). Let $X_1, \ldots, X_k$ be the corresponding $\langle\sigma\rangle$-orbits, and let $\sigma_i = \sigma|_{X_i}$ for each i. In this section we describe how to check if W contains a conjugate of σ, that is, a permutation of type T, echoing some ideas in the proof of [7, Lemma 4.2].

For $i \in \{1, \ldots, k\}$, let $\mathcal{B}_i = \{B \in \mathcal{B} \mid B \cap X_i \neq \varnothing\}$. Observe that $\langle\sigma_i\rangle$ acts transitively on $\mathcal{B}_i$ and that the action of σ_i on $\mathcal{B}_i$ is equivalent to the action of σ on $\mathcal{B}_i$. It follows that $d_i = |B \cap X_i|$ is independent of the choice of $B \in \mathcal{B}_i$: in particular $d_i \mid x_i = |X_i|$ and, of course, $1 \leq d_i \leq b$. Moreover, since the orbits X_i form a partition of $\Omega = \{1, \ldots, n\}$, we also have $\mathcal{B}_i \cap \mathcal{B}_j = \varnothing$ or $\mathcal{B}_i = \mathcal{B}_j$, for distinct $i, j \in \{1, \ldots, k\}$, and $\cup_{i=1}^{k} \mathcal{B}_i = \mathcal{B}$.

In addition, the assumption $\gcd(x_1, \dots, x_k) = 1$ leads to more restrictions on $\mathcal{B}$. For example, we cannot have $\mathcal{B}_i = \mathcal{B}$ for some $i \in \{1, \dots, k\}$, since this would imply that $\mathcal{B} = \mathcal{B}_i$ for all i and hence that $|\mathcal{B}| = m$ divides $\gcd(x_1, \dots, x_k)$. Similarly we cannot have $\mathcal{B}_i \cap \mathcal{B}_j = \varnothing$ for all distinct $i, j \in \{1, \dots, k\}$, since this would imply that $|B| = b$ divides $\gcd(x_1, \dots, x_k)$.

4.1. **The case $k = 2$.** As a first consequence of the comments above, we obtain that there exists no maximal imprimitive subgroup of S_n containing a permutation of type $T = [x, n - x] \in \mathcal{A}$: for we showed above that we must have either $\mathcal{B}_1 \cap \mathcal{B}_2 = \varnothing$ or $\mathcal{B}_1 = \mathcal{B}_2 = \mathcal{B}$, and the previous paragraph argues that neither of these is possible.

4.2. **The case $k = 3$.** Let now $k = 3$, that is, $T = [x_1, x_2, x_3]$ with $\gcd(x_1, x_2, x_3) = 1$. From the previous paragraphs, up to relabeling, we may assume that $\mathcal{B}_1 = \mathcal{B}_2$ and that $\mathcal{B}_3 \cap \mathcal{B}_1 = \varnothing$. Thus there is just one pattern to examine:

$$\begin{array}{|c|c|}\hline & x_1 \\ \cline{2-2} x_3 & \\ & x_2 \\ \hline \end{array} \tag{4.1}$$

Here X_3 is a union of blocks while X_1, X_2 share the same set of blocks. In particular $b = d_3 = d_1 + d_2$ divides $\gcd(x_3, n)$. The number of blocks in X_3 is x_3/b and the remaining $m - x_3/b$ blocks meet both X_1 and X_2 so that $\frac{n-x_3}{b} \mid \gcd(x_1, x_2)$.

The two conditions $b \mid \gcd(x_3, n)$ and $\frac{n-x_3}{b} \mid \gcd(x_1, x_2)$ are quite strong and can often be used to prove that T does not belong to W. The following examples play a role in the proof of Theorem 1.1.

Example 4.1. *No permutation of type $Z = [3, q, 14q - 3]$ or $X = [10, 4q, 11q - 10]$ belongs to an imprimitive subgroup of S_{15q}, for $q \geq 7$ prime.*

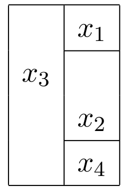
Proof. Clearly the terms in Z are pairwise coprime. In particular they are globally coprime. Due to the relabeling we made in our argument above, we must examine the unique possible pattern (4.1) for each of the three choices of x_3 in $\{3, q, 14q - 3\}$. For each choice, we get $\frac{15q-x_3}{b} \mid \gcd(x_1, x_2) = 1$ so that $b = 15q - x_3$ must divide n, which is not possible for any of the choices for x_3.

Now consider the partition X. If $x_3 = 10, 4q$, then $b \mid \gcd(x_3, 15q)$ implies that $b = 5, q$, against the fact that $\frac{n-x_3}{b} = 3q - 2, 11$ does not divide $\gcd(x_1, x_2) = 1, 10$, respectively. Hence $x_3 = 11q - 10$ which implies that $\frac{n-x_3}{b} = \frac{4q+10}{b}$ divides $\gcd(x_1, x_2) = 2$; so $b = 2q + 5$ or $4q + 10$, neither of which divides $n = 15q$. $\square$

4.3. **The case $k = 4$.** Let us explore finally $k = 4$, that is $T = [x_1, x_2, x_3, x_4]$ with $\gcd(x_1, \dots, x_4) = 1$. From the previous paragraphs, up to relabeling, we may assume that $\mathcal{B}_1 = \mathcal{B}_2$ and that $\mathcal{B}_3 \cap \mathcal{B}_1 = \mathcal{B}_3 \cap \mathcal{B}_2 = \varnothing$. Thus there are three cases (a), (b), (c) to examine, each characterized by some additional conditions and giving rise to a different pattern.

(a) $\mathcal{B}_3 \cap \mathcal{B}_4 = \varnothing, \mathcal{B}_1 = \mathcal{B}_4$.

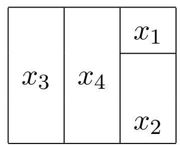
(4.2)



Here X_3 is a union of blocks while X_1, X_2, X_4 share the same blocks. In particular $b = d_3 = d_1 + d_2 + d_4 \mid \gcd(x_3, n)$. The number of blocks in X_3 is x_3/b and the remaining $m - x_3/b$ blocks meet each of X_1, X_2, X_4 so that $|\mathcal{B}_1| = \frac{n-x_3}{b} \mid \gcd(x_1, x_2, x_4)$.

(b) $\mathcal{B}_3 \cap \mathcal{B}_4 = \varnothing, \mathcal{B}_4 \cap \mathcal{B}_1 = \varnothing, \mathcal{B}_4 \cap \mathcal{B}_2 = \varnothing$.

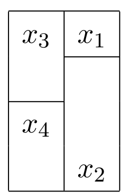
(4.3)



Here X_3, X_4 are disjoint unions of blocks while X_1, X_2 share the same blocks. In particular $b = d_3 = d_4 = d_1 + d_2 \mid \gcd(x_3, x_4, n)$. The number of blocks contained in $X_3 \cup X_4$ is given by $(x_3 + x_4)/b$ and the remaining $m - (x_3 + x_4)/b$ blocks meet both X_1, X_2. Therefore $\frac{n-x_3-x_4}{b} \mid \gcd(x_1, x_2)$.

(c) $\mathcal{B}_3 = \mathcal{B}_4, \mathcal{B}_4 \cap \mathcal{B}_1 = \varnothing, \mathcal{B}_4 \cap \mathcal{B}_2 = \varnothing$.

(4.4)



Here X_3, X_4 share the same blocks and X_1, X_2 share the same remaining blocks. In particular $b = d_3 + d_4 = d_1 + d_2 \mid \gcd(x_3 + x_4, n)$. The number of blocks contained in $X_3 \cup X_4$ is $|\mathcal{B}_3| = \frac{x_3+x_4}{b}$ and divides $\gcd(x_3, x_4)$. Moreover $\frac{n-x_3-x_4}{b}$ divides $\gcd(x_1, x_2)$, the quotient giving the number of blocks contained in $X_1 \cup X_2$.

The arithmetic conditions deduced in these cases are also often sufficiently strong to prove that T does not belong to W. Anyway, if necessary, we can strengthen these conditions excluding some congruences. The two following examples play a role in the proof of Theorem 1.1.

Example 4.2. *Let $q \geq 7$ be a prime. Then:*

i) *no permutation of type $U = [5, q-5, 10q+5, 4q-5]$ belongs to an imprimitive subgroup of S_{15q};*

ii) *if $q \geq 11$ and $q \not\equiv 12 \pmod{13}$, then no permutation of type*

$$V = [q-7, q+7, 6q-7, 7q+7]$$

belongs to an imprimitive subgroup of S_{15q}.

Proof. To begin with note that for both U and V the greatest common divisor of their terms is 1 so that these partitions are within the category we are exploring in this section. Let $W = S_b \wr S_m$, with $2 \leq b \leq n/2$ a proper divisor of n and $m = n/b$. Thus $b, m \in A = \{3, 5, 15, q, 3q, 5q\}$.

i) Assume first that U belongs to W. We show that each of the three possible cases (a),(b),(c), corresponding respectively to the patterns (4.2),(4.3),(4.4), leads to a contradiction.

Case (a). We must examine each of the possibilities for

$$x_3 \in \{5, q-5, 10q+5, 4q-5\}.$$

Recall that in this context we have $b \mid \gcd(x_3, n)$ and $\frac{15q - x_3}{b} \mid \gcd(x_1, x_2, x_4)$. For each choice of x_3 we have $\gcd(x_1, x_2, x_4) = 1$, and so the second condition implies that $b = 15q - x_3$. However, by the first condition $b \leq x_3$, and it follows that $2x_3 \geq 15q$, whence $x_3 = 10q + 5$. Thus $b = 5q - 5$, which does not divide $15q$.

Case (b). We must examine each possibility for (x_3, x_4) in

$$\{(5, q-5), (5, 4q-5), (5, 10q+5), (q-5, 4q-5), (q-5, 10q+5), (4q-5, 10q+5)\}.$$

Recall that now we have $b \mid \gcd(x_3, x_4, 15q)$ and $\frac{15q - x_3 - x_4}{b} \mid \gcd(x_1, x_2)$.

The first condition implies that $\gcd(x_3, x_4) > 1$ and is divisible by a prime $p \in \{3, 5, q\}$. This implies that $(x_3, x_4, p) = (5, 10q+5, 5)$ or $(q-5, 4q-5, 3)$, and in these cases we find that $b = 5$ or 3 respectively. The second condition then requires that $q - 2$ divides $\gcd(q-5, 4q-5) = \gcd(q-5, 3)$ or $\frac{10(q+1)}{3}$ divides $\gcd(5, 10q+5) = 5$, respectively, neither of which is possible.

Case (c). Due to the symmetric role played by $\mathcal{B}_3, \mathcal{B}_4$ with respect to $\mathcal{B}_1, \mathcal{B}_2$, we need to examine only the possibilities for

$$(x_3, x_4) \in \{(5, q-5), (5, 4q-5), (5, 10q+5)\}.$$

Recall that here we have $b \mid \gcd(x_3 + x_4, n)$ and the number of blocks contained in $X_3 \cup X_4$ divides $\gcd(x_3, x_4)$; moreover $\frac{n - x_3 - x_4}{b} \mid \gcd(x_1, x_2)$.

For the first two possibilities for (x_3, x_4), the condition $b \mid \gcd(x_3 + x_4, n)$ implies that $b = q$. In the first case the second condition gives $14 \mid 4q - 5$, which is impossible since $4q - 5$ is odd. If $(x_3, x_4) = (5, 4q-5)$, then $\gcd(5, 4q-5) = 1$, which implies that $|\mathcal{B}_3| = 1$ and so $b = x_3 + x_4 = 4q$, a contradiction. Thus $(x_3, x_4) = (5, 10q+5)$, and we find that $|\mathcal{B}_3|$ divides $\gcd(5, 10q+5) = 5$ and cannot be 1, as otherwise $b = x_3 + x_4 = 10(q+1) \notin A$. Hence $|\mathcal{B}_3| = 5$ and $b = 2(q+1) \notin A$.

ii) The proof follows with a case-by-case argument similar to the proof of part i). The peculiarity with respect to i) is in case (c) where, to conclude, we need the assumption $q \not\equiv 12 \pmod{13}$. We treat in detail this case, having in mind the corresponding pattern (4.4).

Case (c). Due to the symmetric role played by $\mathcal{B}_3, \mathcal{B}_4$ with respect to $\mathcal{B}_1, \mathcal{B}_2$, we need to examine only the possibilities for

$$(x_3, x_4) \in \{(q-7, q+7), (q-7, 6q-7), (q-7, 7q+7)\}.$$

Here we have $b \mid \gcd(x_3+x_4,n)$ and $\frac{n-x_3-x_4}{b} \mid \gcd(x_1,x_2)$. In the first and third cases, the condition $b \mid \gcd(x_3+x_4,n)$ implies that $b=q$. The second condition then implies that 13 divides $\gcd(6q-7,7q+7)=\gcd(q+1,13)$, or that 7 divides $\gcd(q+7,6q-7)=1$, respectively, neither of which is possible, because of our assumption $q \not\equiv 12 \pmod{13}$. Thus $(x_3,x_4)=(q-7,6q-7)$, and the first condition implies that $b \mid \gcd(7q-14,15q)=\gcd(q-2,15)$ and so $b\in\{3,5,15\}$. On the other hand we have $\gcd(q-7,6q-7)\mid 5$ and thus $|\mathcal{B}_3|\in\{1,5\}$; if $|\mathcal{B}_3|=1$ then $b=7q\notin A$ and thus $|\mathcal{B}_3|=5, b=\frac{7q-14}{5}$, which is incompatible with $b\in\{3,5,15\}$. □

5. Degrees which are products of three odd primes

In this section we deal with degrees of the form $n=15q$, where $q\geq 7$ is a prime, and prove Theorem 1.1 (a). We will also see that Conjecture 1 holds for an infinite family of these degrees. We begin this section adapting to our purposes some deep classical and recent results about primitive permutation groups. In particular, Theorem 3.3 in [11] plays an important role. It lists, for each degree n, which types $[x,n-x]\in\mathcal{T}$ of permutations may belong to some primitive subgroup of S_n other than S_n and A_n. We invoke that theorem several times to exclude the presence of primitive components in the normal coverings.

Lemma 5.1. *Let $q\geq 7$ be a prime. Then no primitive proper subgroup of S_{15q} contains a permutation with type belonging to $\mathcal{T}$.*

Proof. We simply examine the lists in [11, Theorem 3.3] having in mind that n is odd and not a proper power, and that no type in $\mathcal{T}$ can belong to A_{15q}. It is easily checked that no case arises. □

Lemma 5.2. *Let $q\geq 7$ be a prime and let K be a primitive subgroup of S_{15q}, with $K\ngeq A_{15q}$. Then the number of fixed points of each nontrivial permutation in K is less than $9q$.*

Proof. By [9, Corollary 1], since $n=15q$ is not a proper power, we find that the number of fixed points of each permutation in K is less than or equal to $\frac{4}{7}(15q)$, and thus, in particular, is less than $9q$. □

Lemma 5.3. ([12, Theorem 13.8], [8, Theorem 4.11]) *A primitive group of degree n, which contains a permutation of type $[m,1,\ldots,1]$ where $2\leq m\leq n-5$, contains A_n.*

Proof of Theorem 1.1 (a) Let $n=15q$, with q an odd prime satisfying $q\equiv 2 \pmod{15}$ and $q\not\equiv 12 \pmod{13}$, so that $q\geq 17$. Let δ be a minimal basic set for S_n. We may assume that each component in δ is a maximal subgroup of S_n. By Proposition 3.1, $|\delta|\leq 4q+2$. We show that also $|\delta|\geq 4q+2$.

We are interested in the set $\mathcal{A}$ defined in (2.6) and also in the following subsets of $\mathcal{T}$, corresponding to subsets of $\mathfrak{T}:=\mathbb{N}\cap[1,n/2)$ characterized by a suitable coprimality condition.

$$\begin{aligned}
\mathcal{B}&=\{T_x\in\mathcal{T} \ : \ x\in\mathfrak{B}\}, &\quad \mathfrak{B}&=\{x\in\mathfrak{T}: \ \gcd(x,n)=3\},\\
\mathcal{C}&=\{T_x\in\mathcal{T} \ : \ x\in\mathfrak{C}\}, &\quad \mathfrak{C}&=\{x\in\mathfrak{T} \ : \ \gcd(x,n)=5\},\\
\mathcal{D}&=\{T_x\in\mathcal{T} \ : \ x\in\mathfrak{D}\}, &\quad \mathfrak{D}&=\{x\in\mathfrak{T} \ : \ \gcd(x,n)=q\}.
\end{aligned}$$

Their orders are immediate and coincide with that of the corresponding set of intransitive maximal subgroups containing them. By equality (2.8), Proposition 2.4 i) and Remark 2.5 we have:

$$|\mathcal{A}| = |\mathfrak{A}| = |\mathcal{P}_{\mathfrak{A}}| = 4(q-1), \qquad |\mathcal{B}| = |\mathfrak{B}| = |\mathcal{P}_{\mathfrak{B}}| = 2(q-1),$$

$$|\mathcal{C}| = |\mathfrak{C}| = |\mathcal{P}_{\mathfrak{C}}| = q-1, \qquad |\mathcal{D}| = |\mathfrak{D}| = |\mathcal{P}_{\mathfrak{D}}| = 4.$$

Note that since n is odd, no permutation of type belonging to $\mathcal{T}$ lies in A_n. Thus by Lemma 5.1, no permutation having type in $\mathcal{T}$ belongs to a primitive subgroup in δ. In particular this holds for all the types in $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}$. By Section 4.1, it follows that the permutations having type in $\mathcal{A}$ belong only to intransitive components and so $\delta \supseteq \mathcal{P}_{\mathfrak{A}}$, which gives $|\delta| \geq 4(q-1)$. This means that we need to force only 6 further components to finish the proof.

Consider now the permutations having type belonging to $\mathcal{B}$. Since these do not belong to any of the subgroups in $\mathcal{P}_{\mathfrak{A}}$, we need other components in δ to contain them and those components must be intransitive or imprimitive. If both $S_3 \wr S_{5q} \notin \delta$ and $S_{5q} \wr S_3 \notin \delta$, then we would need a further $2(q-1) > 6$ intransitive components and thus $|\delta| > 4q+2$, a contradiction. So one of $S_3 \wr S_{5q}$ and $S_{5q} \wr S_3$, belongs to δ. Let I_3 denote this component of δ, and note that, since now $|\delta| \geq 4q-3$, we need to force only 5 further components.

As $|\mathcal{C}| = q-1 > 6$, looking to the permutations having type belonging to $\mathcal{C}$, and observing that they do not belong to any of the subgroups in $\mathcal{P}_{\mathfrak{A}}$ or to I_3, we see that δ must contain $S_5 \wr S_{3q}$ or $S_{3q} \wr S_5$. Let denote with I_5 this additional subgroup in δ.

At this point we know that $\delta \supseteq \mathcal{P}_{\mathfrak{A}} \cup \{I_3, I_5\}$ and we need to force just 4 components. If neither $S_q \wr S_{15}$ nor $S_{15} \wr S_q$, belongs to δ, to cover the permutations of type belonging to $\mathcal{D}$ we need exactly 4 additional intransitive components and the proof is complete. So we may assume that $I_q \in \delta$, where I_q is one of $S_q \wr S_{15}$ or $S_{15} \wr S_q$.

Next suppose that $A_{15q} \in \delta$. We consider the type $U = [5, q-5, 10q+5, 4q-5]$. By Example 4.2 no imprimitive subgroup contains a permutation of type U. Moreover $U \notin P_x$ for all $x \in \mathfrak{A}$ because no term and no sum of two terms in the partition U is coprime to $n = 15q$: this follows from the assumption $q \equiv 2 \pmod{15}$, which implies both $q \equiv 2 \pmod 3$ and $q \equiv 2 \pmod 5$ so that $3 \mid q-5, 4q-5, q+7, 7q+7$ and $5 \mid q-7, 6q-7$. Also $U \notin A_{15q}$. This means that we need a further component, say K, to cover U, and K is either intransitive or primitive. We now have that δ contains the subset

$$\delta' := \{P_x, I_3, I_5, I_q, A_{15q}, K \ : \ x \in \mathfrak{A}\}$$

of size $4q+1$. Suppose, for a contradiction, that $\delta = \delta'$, and let $\sigma \in K$ have type U. Suppose first that K is primitive: then $\mu = \sigma^{10q+5} \neq id$, because $3 \mid q-5$, due to $q \equiv 2 \pmod 3$, but $3 \nmid 10q+5$. Moreover the number of fixed points of μ is at least $10q+10 > 9q$, contradicting Lemma 5.2. Thus K is intransitive. To be more precise

$$K \in P_U := \{P_5, P_q, P_{q-5}, P_{4q-5}, P_{5q-10}, P_{4q}, P_{5q-5}\}.$$

Consider now the type $V = [q-7, q+7, 6q-7, 7q+7]$. By Example 4.2 no imprimitive subgroup contains a permutation of type V and, arguing as for U, it is immediately checked that $V \notin P_x$ for all $x \in \mathfrak{A}$. On the other hand for each choice of K in P_U, we see that $V \notin K$. In other words it is not possible to cover V by the components in δ. Thus if δ contains A_{15q}, then $|\delta| = 4q+2$.

Finally suppose that $A_{15q} \notin \delta$. We consider the types $Z = [3, q, 14q-3]$ and $X = [10, 4q, 11q-10]$. Since each term and the sum of each pair of terms in Z, X is divisible by 3 or by 5 or by q, we have that both Z and X do not belong to any intransitive component of $\mathcal{P}_{\mathfrak{A}}$. On the other hand, by Example 4.1 they do not belong to any imprimitive subgroup. Suppose that δ contains a primitive component $H < S_{15q}$ containing either Z or X. If H contains an element η of type Z, then $\eta^{q(14q-3)} \in H$ is a 3-cycle, which is impossible by Lemma 5.3, because $H \neq A_{15q}$. Thus H contains an element θ of type X; but then θ^{40q} is a $(11q-10)$-cycle and, since $H \neq A_{15q}$, this is impossible again by Lemma 5.3. Hence δ contains intransitive maximal subgroups containing X, Z. Now Z, X belong, respectively, only to the intransitive subgroups in $\mathcal{P}_Z$ and in $\mathcal{P}_X$, where

$$\mathcal{P}_Z := \{P \in \mathcal{P} : Z \in P\} = \{P_3, P_q, P_{q+3}\}$$

and

$$\mathcal{P}_X := \{P \in \mathcal{P} : X \in P\} = \{P_{10}, P_{4q}, P_{4q+10}\}.$$

Since $\mathcal{P}_Z \cap \mathcal{P}_X = \varnothing$, we need two additional intransitive components K_1, K_2 in δ to cover both Z and X, where $K_1 \in \mathcal{P}_Z, K_2 \in \mathcal{P}_X$. We now have that δ contains the subset

$$\delta' := \{P_x, I_3, I_5, I_q, K_1, K_2 \ : \ x \in \mathfrak{A}\}$$

of size $4q+1$. Assume, for a contradiction, that $\delta = \delta'$. To reach a final contradiction, consider again the type U and recall that $U \notin P$ for all $P \in \mathcal{P}_{\mathfrak{A}} \cup \{I_3, I_5, I_q\}$, and it is also immediately checked that $U \notin P$ for all $P \in \mathcal{P}_Z \cup \mathcal{P}_X$. □

We show now that there are infinitely many primes q satisfying the conditions of Theorem 1.1, thus confirming Conjecture 1 for a new infinite family of odd integers.

Corollary 5.4. *There exist infinitely many primes q such that $\gamma(S_{15q}) = 4q+2 = g(15q)$.*

Proof. It is enough to observe that, by the previous result, we have $\gamma(S_{15q}) = 4q+2$ for all q primes with $q \equiv 2 \pmod{195}$, where $195 = 15 \cdot 13$. Since 2 and 195 are coprime, the famous Theorem of Dirichlet on primes in arithmetic progressions assures that there are infinitely many primes of the form $q = 2 + 195\,k$, with $k \in \mathbb{N}$. □

6. Even degrees

In this section we discuss Conjecture 2. First we justify the exclusion of $n = 12$.

Proposition 6.1. $\gamma(A_{12}) = 3$*, and* $\gamma(S_{12}) = g(12) = 4$.

Proof. We see that $\delta = \{M_{12}, [S_3 \wr S_4] \cap A_{12}, [S_5 \times S_7] \cap A_{12}\}$ is a basic set for A_{12} by checking that all the types of permutations in A_{12} are contained in one of the components. Since $\gamma(A_{12}) \neq 2$, by [4, Theorem 3.9] (see also [2, Theorem, Sec. 4]), we deduce that $\gamma(A_{12}) = 3$.

For S_{12}, it follows from [4, Theorem 4.2] (see also [2, Proposition 3.4]) that $\gamma(S_{12}) \geq 3$, and from Proposition 3.1 that $\gamma(S_{12}) \leq g(12) = 4$. Suppose for a contradiction that δ is a basic set for S_{12} consisting of maximal subgroups and that $|\delta| = 3$. The proof of [6, Corollary 7.10] correctly shows that $A_{12} \notin \delta$, and then by [6, Lemma 5.2] we deduce that $S_5 \times S_7 \in \delta$. Moreover, inspection of the lists in [11, Theorem 3.3], shows that no permutation of type $[2, 10], [4, 8]$, or $[3, 9]$ belongs to a proper primitive subgroup of S_{12}. Thus we conclude that neither of the other two subgroups H, K of δ is primitive, and neither H nor K fixes a point. This implies that no subgroup in δ contains the type $[1, 11]$. $\square$

Now we prove Theorem 1.1 (b), giving extra confirmation for Conjecture 2.

Proof of Theorem 1.1 (b). Let $n = 6q$, with $q \geq 11$ a prime. If $U \leq S_n$, we write for simplicity, $\overline{U} = U \cap A_n$. For each $\mathfrak{X} \subseteq \mathbb{N} \cap [1, n/2)$, use the notation $\overline{\mathcal{P}}_{\mathfrak{X}} = \{\overline{P} : P \in \mathcal{P}_{\mathfrak{X}}\}$. Let δ be a minimal basic set for A_n with maximal components. By Proposition 3.1 we know that $|\delta| \leq q + 2$. Our aim is to show that $|\delta| \geq q + 2$.

To do that we first consider the set $\mathfrak{A}' = \mathfrak{A} \setminus \{1\}$ (see (2.7)) and the corresponding set of partitions $\mathcal{A}' = \mathcal{A} \setminus \{[1, n-1]\}$ and intransitive components $\overline{\mathcal{P}}_{\mathfrak{A}'} = \overline{\mathcal{P}}_{\mathfrak{A}} \setminus \{\overline{P}_1\}$. Moreover, we consider the following subsets of $\mathcal{T}$, corresponding to subsets of $\mathfrak{T} := \mathbb{N} \cap [1, n/2)$ characterized by a suitable coprimality condition:

$$\begin{aligned} \mathcal{E} &= \{T_x \in \mathcal{T} : x \in \mathfrak{E}\}, & \mathfrak{E} &= \{x \in \mathfrak{T} \ : \ \gcd(x, n) = 2\}, \\ \mathcal{F} &= \{T_x \in \mathcal{T} : x \in \mathfrak{F}\}, & \mathfrak{F} &= \{x \in \mathfrak{T} \ : \ \gcd(x, n) = 3\}, \\ \mathcal{G} &= \{T_x \in \mathcal{T} : x \in \mathfrak{G}\}, & \mathfrak{G} &= \{x \in \mathfrak{T} \ : \ \gcd(x, n) = q\}. \end{aligned}$$

It follows from Proposition 2.4 ii)a) and Remark 2.5 that:

$$\begin{aligned} |\mathcal{A}'| &= |\overline{\mathcal{P}}_{\mathfrak{A}'}| = q - 2, & |\mathcal{E}| &= |\mathfrak{E}| = |\overline{\mathcal{P}}_{\mathfrak{E}}| = q - 1, \\ |\mathcal{F}| &= |\mathfrak{F}| = |\overline{\mathcal{P}}_{\mathfrak{F}}| = \tfrac{q-1}{2}, & |\mathcal{G}| &= |\mathfrak{G}| = |\overline{\mathcal{P}}_{\mathfrak{G}}| = 2. \end{aligned}$$

By [11, Theorem 3.3], when $n = 6q$ with q prime, no permutation having type in $\mathcal{T} \setminus \{[1, n-1]\}$ belongs to a proper primitive subgroup. In particular this holds for all the types in $\mathcal{A}', \mathcal{E}, \mathcal{F}, \mathcal{G}$. Moreover, by Section 4.1, it follows immediately that the permutations in $\mathcal{A}'$ cannot belong to an imprimitive component and so $\delta \supseteq \mathcal{P}_{\mathfrak{A}'}$, which gives $|\delta| \geq q - 2$. This means that we need to force only 4 further components to conclude. Consider permutations having type in $\mathcal{E}$. Since these do not belong to any subgroup in $\mathcal{P}_{\mathfrak{A}'}$, we need additional components to contain them and those components must be intransitive or imprimitive. If δ contains neither $\overline{S_2 \wr S_{3q}}$ nor $\overline{S_{3q} \wr S_2}$, then we need $q - 1 > 4$ additional intransitive components and thus $|\delta| > q + 2$, a contradiction. So one of $\overline{S_2 \wr S_{3q}}$ and $\overline{S_{3q} \wr S_2}$, belongs to δ. Denote this component by I_2 and note that we then need to force 3 further components.

Similarly permutations having type in $\mathcal{F}$ do not belong to any of the subgroups in $\mathcal{P}_{\mathfrak{A}'}$ or to I_2. If δ contains neither $\overline{S_3 \wr S_{2q}}$ nor $\overline{S_{2q} \wr S_3}$, then we need at least $\frac{q-1}{2} > 3$ (since $q \geq 11$) additional intransitive components to cover these permutations, which is a contradiction. Thus one of $\overline{S_3 \wr S_{2q}}$ or $\overline{S_{2q} \wr S_3}$, denoted I_3, lies in δ, and we need to force just 2 more components.

Permutations of type in $\mathcal{G}$ do not belong to I_2, I_3, or any subgroup in $\mathcal{P}_{\mathfrak{A}'}$. If δ contains neither $\overline{S_q \wr S_6}$ nor $\overline{S_6 \wr S_q}$, then we need two additional intransitive components to cover these permutations and the proof is complete. Thus we may assume that δ contains one of $\overline{S_q \wr S_{15}}$ or $\overline{S_{15} \wr S_q}$, denoted I_q. Finally, since no component in the subset $\mathcal{P}_{\mathfrak{A}'} \cup \{I_2, I_3, I_q\}$ of δ contains the type $[1, n-1]$, it follows that we need an additional component for it, and hence $|\delta| \geq q+2$. □

6.1. **The symmetric group.** Our tools for the symmetric group S_n with n even seem too weak. We have no general argument even to show that at least the intransitive subgroups of $\mathcal{P}_{\mathfrak{A}}$ belong to any minimal basic set of maximal subgroups for S_n, even though computational evidence with `Magma` suggests that this is the case. In particular our computations gave no counterexamples to Conjecture 1. For n even, Conjectures 1 and 2 are linked: if Conjecture 2 is true, then either also Conjecture 1 is true, or A_n belongs to each minimal basic set of S_n. However, in our experience A_n does not play a significant role in the minimal basic sets of S_n. As an example, a detailed argument of more than a page is needed to confirm that $\gamma(A_{30}) = \gamma(S_{30}) = g(30) = 7$, and for space reasons we have not included it here.

Acknowledgements

We thank Florian Luca for helpful advice on the arithmetic section of the paper. This paper forms part of Australian Research Council grant DP130100106 of the second author.

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Daniela Bubboloni

Dipartimento di Scienze per l'Economia e l'Impresa, University of Firenze, Via delle Pandette 9-D6, 50127 Firenze, Italy

`Email: daniela.bubboloni@unifi.it`

Cheryl E. Praeger

Centre for Mathematics of Symmetry and Computation, School of Mathematics and Statistics, The University of Western Australia, Crawley, WA 6009, Australia

Also affiliated with King Abdulaziz University, Jeddah, Saudi Arabia

`Email: Cheryl.Praeger@uwa.edu.au`

Pablo Spiga

Dipartimento di Matematica e Applicazioni, University of Milano-Bicocca, Via Cozzi 55, 20125 Milano, Italy

`Email: pablo.spiga@unimib.it`