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## FINITE GROUPS WHOSE MINIMAL SUBGROUPS ARE WEAKLY $\mathcal{H}^*$ -SUBGROUPS

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**ABSTRACT.** Let  $G$  be a finite group. A subgroup  $H$  of  $G$  is called an  $\mathcal{H}$ -subgroup in  $G$  if  $N_G(H) \cap H^g \leq H$  for all  $g \in G$ . A subgroup  $H$  of  $G$  is called a weakly  $\mathcal{H}^*$ -subgroup in  $G$  if there exists a subgroup  $K$  of  $G$  such that  $G = HK$  and  $H \cap K$  is an  $\mathcal{H}$ -subgroup in  $G$ . We investigate the structure of the finite group  $G$  under the assumption that every cyclic subgroup of  $G$  of prime order  $p$  or of order 4 (if  $p = 2$ ) is a weakly  $\mathcal{H}^*$ -subgroup in  $G$ . Our results improve and extend a series of recent results in the literature.

### 1. Introduction

Throughout only finite groups are considered. The notation and terminology used in this paper are standard and can be found in Doerk and Hawkes [7].

Recall that a subgroup  $H$  of a group  $G$  is called  $c$ -normal in  $G$  if there exists a normal subgroup  $K$  of  $G$  such that  $G = HK$  and  $H \cap K \leq H_G$ , where  $H_G = Core_G(H)$  is the largest normal subgroup of  $G$  contained in  $H$ . This concept was initiated by Wang [12] in 1996. Later on, in 2000, Ballester-Bolinchés et al. [3] extended  $c$ -normal subgroups of a group  $G$  to  $c$ -supplemented subgroups. They gave the following concept: A subgroup  $H$  of a group  $G$  is called  $c$ -supplemented in  $G$  if there exists a subgroup  $K$  of  $G$  such that  $G = HK$  and  $H \cap K \leq H_G$ . Also, in 2000, Bianchi et al. [4] introduced the concept of an  $\mathcal{H}$ -subgroup as follows: A subgroup  $H$  of a group  $G$  is called an  $\mathcal{H}$ -subgroup if  $N_G(H) \cap H^g \leq H$  for all  $g \in G$ . Recently, in 2012, Asaad, Heliel and Al-Shomrani [2] introduced a new concept, called a weakly  $\mathcal{H}$ -subgroup, as follows: A subgroup  $H$  of a group  $G$  is called a weakly  $\mathcal{H}$ -subgroup in  $G$  if there

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exists a normal subgroup  $K$  of  $G$  such that  $G = HK$  and  $H \cap K \in \mathcal{H}(G)$ , where  $\mathcal{H}(G)$  denotes the set of all  $\mathcal{H}$ -subgroups in  $G$ . This concept arises naturally as an extension of  $c$ -normality and  $\mathcal{H}$ -subgroup. All the above mentioned concepts are considered as nontrivial generalizations of normality.

Several authors have investigated the structure of a finite group  $G$  under the assumption that the minimal subgroups of  $G$  and the cyclic subgroups of order 4 satisfying one of the previously mentioned concepts (recall that a minimal subgroup of  $G$  is a subgroup of prime order). Starting with normality, in 1970, Buckley [5] proved that if  $G$  is a group of odd order whose minimal subgroups are normal, then  $G$  is supersolvable. Wang [12] relaxed the normality to the weaker condition  $c$ -normality and proved that if  $G$  is a group such that all of its cyclic subgroups of prime order or of order 4 are  $c$ -normal, then  $G$  is supersolvable. In 2004, Csörgö and Herzog [6] obtained the same previous result by replacing  $c$ -normality with  $\mathcal{H}$ -subgroup. By using the weakly  $\mathcal{H}$ -subgroup concept, in 2012, Al-Shomrani, Ramadan and Heliel [1] achieved new results through the theory of formation which extended and generalized the above mentioned results. One of these results is the following: Let  $\mathfrak{F}$  be a saturated formation containing the class of supersolvable groups  $\mathfrak{U}$  and let  $G$  be a group. Then  $G \in \mathfrak{F}$  if and only if  $G$  has a normal subgroup  $H$  such that  $G/H \in \mathfrak{F}$  and every cyclic subgroup of  $F^*(H)$  of prime order or of order 4 is a weakly  $\mathcal{H}$ -subgroup in  $G$ , where  $F^*(H)$  is the generalized Fitting subgroup of  $H$ . Also, in 2004, Wei et al. [16] used the  $c$ -supplement concept and obtained the same previous result. For more results in this direction, the reader is referred to [1], [3], [6] and [10]-[16].

In this paper, we introduce a new subgroup embedding property, namely, the weakly  $\mathcal{H}^*$ -subgroup property which may be viewed as a generalization of both  $c$ -supplement and weakly  $\mathcal{H}$ -subgroup (and of course  $c$ -normality and  $\mathcal{H}$ -subgroup) concepts as follows:

**Definition 1.1.** *A subgroup  $H$  of a group  $G$  is called a weakly  $\mathcal{H}^*$ -subgroup in  $G$  if there exists a subgroup  $K$  of  $G$  such that  $G = HK$  and  $H \cap K \in \mathcal{H}(G)$ .*

Let  $H$  be a subgroup of a group  $G$ . Suppose that  $H$  is  $c$ -supplemented in  $G$ . Then there exists a subgroup  $K$  of  $G$  such that  $G = HK$  and  $H \cap K \leq H_G$ . So, we have the subgroup  $H_G K$  in  $G$  such that  $G = H(H_G K)$  and  $H \cap H_G K = H_G(H \cap K) = H_G$ . Therefore  $H$  is a weakly  $\mathcal{H}^*$ -subgroup in  $G$ . Also, if  $H$  is a weakly  $\mathcal{H}$ -subgroup in  $G$ , then it is obvious that  $H$  is a weakly  $\mathcal{H}^*$ -subgroup in  $G$ . Thus  $c$ -supplemented subgroups and weakly  $\mathcal{H}$ -subgroups in  $G$  are weakly  $\mathcal{H}^*$ -subgroups in  $G$ . The following examples illustrate that the converse is not true in general and there is no inclusion relationship between the notions of  $c$ -supplement and weakly  $\mathcal{H}$ -subgroup.

**Example 1.2.** Consider the group  $G = S_3 \times S_3$ , where  $S_3$  is the symmetric group of degree 3. Set  $H = \langle ((123), (123)) \rangle$ . We will show that  $H$  is not a weakly  $\mathcal{H}$ -subgroup, but it is  $c$ -supplemented subgroup in  $G$ . Since  $((12), (1)) \notin N_G(H)$ , then  $H$  is not normal in  $G$ . Suppose that  $H$  is a weakly  $\mathcal{H}$ -subgroup in  $G$ . Then there exists a normal subgroup  $K$  of  $G$  such that  $G = HK$  and  $H \cap K \in \mathcal{H}(G)$ . Obviously, the order of  $K$  is greater than or equal to 12. If the order of  $K$  is 12, then  $G/K$  is abelian. Therefore  $G' = A_3 \times A_3$ , where  $G'$  is the derived subgroup of  $G$  and  $A_3$  is the alternating group of

degree 3, is contained in  $K$ , a contradiction. Hence the order of  $K$  is not 12. If the order of  $K$  is 18, then  $H \leq K$  as  $G$  has a normal Sylow 3-subgroup, a contradiction. So,  $K$  must equal to  $G$  and it follows that  $H \in \mathcal{H}(G)$ . Since  $H^{((12),(1))} \leq A_3 \times A_3 \leq N_G(H)$ , then  $H^{((12),(1))} = N_G(H) \cap H^{((12),(1))} \leq H$  as  $H \in \mathcal{H}(G)$  which is not true, a contradiction. Therefore  $H$  is not a weakly  $\mathcal{H}$ -subgroup in  $G$ . Now let  $P = \langle ((1), (12)), ((12), (1)) \rangle$  and let  $N = \langle ((123), (1)) \rangle$ . Clearly,  $N$  is normal in  $G$  and so  $R = PN$  is a subgroup of  $G$ . It is easy to see that  $G = HR$  and  $H \cap R = 1$  which implies that  $H$  is c-supplemented in  $G$ . Thus  $H$  is not a weakly  $\mathcal{H}$ -subgroup, but it is c-supplemented subgroup in  $G$ .

**Example 1.3.** Set  $G = A_5$ , the alternating group of degree 5. Let  $H = \langle (12)(45), (123) \rangle \cong S_3$ , the symmetric group of degree 3. We will show that  $H$  is a weakly  $\mathcal{H}$ -subgroup, but it is not c-supplemented subgroup in  $G$ . Since  $H$  is a maximal subgroup of  $G$  and  $G$  is simple, then  $N_G(H) = H$ , that is,  $H \in \mathcal{H}(G)$ . In particular,  $H$  is a weakly  $\mathcal{H}$ -subgroup in  $G$ . Now suppose that  $H$  is c-supplemented in  $G$ . Then there exists a subgroup  $K$  of  $G$  such that  $G = HK$  and  $H \cap K \leq H_G = 1$  as  $G$  is simple. Clearly, the order of  $K$  is greater than or equal to 10. Since  $G$  has no subgroups of orders 15, 20 and 30 and  $K \neq G$  as  $H_G = 1$ , then the only possible orders for  $K$  are 10 and 12. Assume that the order of  $K$  is 10. Since  $K$  is isomorphic to  $D_5$ , the dihedral group of order 10, it follows that  $K$  has 5 elements of order 2. Obviously, every element of  $K$  of order 2 is a product of two transpositions. If two distinct elements of  $K$  of order 2 have the same transposition, then their product is a 3-cycle, a contradiction. Therefore all the transpositions in the distinct elements of  $K$  of order 2 are different. This guarantees us 10 different transpositions appearing in the elements of  $K$  of order 2 as  $K$  has five of these elements. Since only 10 transpositions can be constructed from the set  $\{1, 2, 3, 4, 5\}$ , it follows that the transposition (45) must appear in some element of  $K$  of order 2. Therefore one of the elements (12)(45), (13)(45) or (23)(45) is in  $K$ . This implies that  $H \cap K \neq 1$ , a contradiction. Hence the order of  $K$  is not 10. So, we may assume that the order of  $K$  is 12. This means that  $|HK| = 72 > 60$  because  $H \cap K = 1$  which is a contradiction. Hence  $H$  is not c-supplemented in  $G$ . Thus  $H$  is a weakly  $\mathcal{H}$ -subgroup, but it is not c-supplemented subgroup in  $G$ .

**Example 1.4.** Theorem 3.1 in [3] states that a group  $G$  is solvable if and only if every Sylow subgroup of  $G$  is c-supplemented in  $G$ . So, every non-solvable group  $G$  must have a non c-supplemented Sylow subgroup, but every Sylow subgroup is an  $\mathcal{H}$ -subgroup. This gives us many examples of weakly  $\mathcal{H}$ -subgroups that are not c-supplemented.

In this paper, we unify and extend some recent results in the literature of c-supplement and weakly  $\mathcal{H}$ -subgroup (and of course c-normality and  $\mathcal{H}$ -subgroup) concepts by using the new concept weakly  $\mathcal{H}^*$ -subgroup. More precisely, we will investigate the structure of the group  $G$  under the assumption that every cyclic subgroup of  $G$  of prime order  $p$  or of order 4 (if  $p = 2$ ) is a weakly  $\mathcal{H}^*$ -subgroup in  $G$ .

## 2. Some basic definitions and preliminaries

In this section, we list some definitions and known results from the literature that will be used in proving our results. In addition, we give some properties of weakly  $\mathcal{H}^*$ -subgroups.

A class of groups  $\mathfrak{F}$  is a formation provided that the following two conditions are satisfied:

- (a) If  $G \in \mathfrak{F}$ , then  $G/H \in \mathfrak{F}$ , where  $H$  is any normal subgroup of  $G$ .
- (b) If  $M$  and  $N$  are normal subgroups of  $G$  such that  $G/M \in \mathfrak{F}$  and  $G/N \in \mathfrak{F}$ , then  $G/(M \cap N) \in \mathfrak{F}$ .

A formation  $\mathfrak{F}$  is called saturated if  $G/\Phi(G) \in \mathfrak{F}$  implies that  $G \in \mathfrak{F}$ . The  $\mathfrak{F}$ -residual, denoted by  $G^{\mathfrak{F}}$ , is the unique smallest normal subgroup of  $G$  such that  $G/G^{\mathfrak{F}} \in \mathfrak{F}$ . Throughout,  $\mathfrak{U}$  will denote the class of supersolvable groups and  $\mathfrak{N}$  will denote the class of nilpotent groups which are saturated formations, see [[8], p. 713, Satz 8.6 and p. 270, Satz 3.7].

A normal subgroup  $N$  of a group  $G$  is an  $\mathfrak{F}$ -hypercentral subgroup of  $G$  provided  $N$  possesses a chain of subgroups  $1 = N_0 \trianglelefteq N_1 \trianglelefteq \cdots \trianglelefteq N_s = N$  such that  $N_{i+1}/N_i$  is an  $\mathfrak{F}$ -central chief factor of  $G$ , see [[7], p. 387]. The product of all  $\mathfrak{F}$ -hypercentral subgroups of  $G$  is again an  $\mathfrak{F}$ -hypercentral subgroup, denoted by  $Z_{\mathfrak{F}}(G)$ , and called the  $\mathfrak{F}$ -hypercenter of  $G$ , see [[7], IV, 6.8]. For the class of nilpotent groups  $\mathfrak{N}$ , the  $\mathfrak{N}$ -hypercenter of a group  $G$  is simply the terminal member  $Z_{\infty}(G)$  of the ascending central series of  $G$ . For more details about saturated formations, see [[7], IV].

For any group  $G$ , the generalized Fitting subgroup  $F^*(G)$  is the set of all elements  $x$  of  $G$  which induce an inner automorphism on every chief factor of  $G$ .  $F^*(G)$  is an important characteristic subgroup of  $G$  and it is a natural generalization of  $F(G)$ . The basic properties of  $F^*(G)$  can be found in [[9], X 13].

**Lemma 2.1.** *Let  $G$  be a group and let  $H, K, N$  be subgroups of  $G$  satisfying  $H \in \mathcal{H}(G)$ ,  $H \leq K$  and  $N$  is normal in  $G$ . Then the following hold:*

- (a)  $H \in \mathcal{H}(K)$ .
- (b) If  $H$  is subnormal in  $K$ , then  $H$  is normal in  $K$ .
- (c) If  $N \leq H$ , then  $H \in \mathcal{H}(G)$  if and only if  $H/N \in \mathcal{H}(G/N)$ .
- (d) If  $H$  is a  $p$ -subgroup of  $G$  for some prime  $p$  and  $(|H|, |N|) = 1$ , then  $HN/N \in \mathcal{H}(G/N)$ .

*Proof.* For (a), (b) and (c), see [[4], Lemma 7(2), Theorem 6(2) and Lemma 2(1)], respectively. For (d), see [[6], Lemma 6]. □

**Lemma 2.2.** *Let  $H, M$  and  $N$  be subgroups of a group  $G$  satisfying  $H \leq M$  and  $N$  is normal in  $G$ . Then the following hold:*

- (a) If  $H$  is a weakly  $\mathcal{H}^*$ -subgroup in  $G$ , then  $H$  is a weakly  $\mathcal{H}^*$ -subgroup in  $M$ .
- (b) If  $N \leq H$  and  $H/N$  is a weakly  $\mathcal{H}^*$ -subgroup in  $G/N$ , then  $H$  is a weakly  $\mathcal{H}^*$ -subgroup in  $G$ .
- (c) If  $H$  is a  $p$ -subgroup of  $G$  for some prime  $p$  with  $(|H|, |N|) = 1$  and  $H$  is a weakly  $\mathcal{H}^*$ -subgroup in  $G$ , then  $HN/N$  is a weakly  $\mathcal{H}^*$ -subgroup in  $G/N$ .
- (d) If  $H \leq \Phi(M)$  and  $H$  is a weakly  $\mathcal{H}^*$ -subgroup in  $G$ , then  $H \in \mathcal{H}(G)$ .

*Proof.* (a) Since  $H$  is a weakly  $\mathcal{H}^*$ -subgroup in  $G$ , then there exists a subgroup  $K$  of  $G$  such that  $G = HK$  and  $H \cap K \in \mathcal{H}(G)$ . Obviously,  $M = M \cap HK = H(M \cap K)$ ,  $M \cap K$  is a subgroup of  $M$  and  $H \cap (M \cap K) = H \cap K \in \mathcal{H}(G)$ . Since  $H \cap K \leq H \leq M$ , it follows, by Lemma 2.1(a), that  $H \cap K \in \mathcal{H}(M)$ . Thus  $H$  is a weakly  $\mathcal{H}^*$ -subgroup in  $M$ .

(b) By hypothesis, there exists a subgroup  $K/N$  of  $G/N$  such that  $G/N = (H/N)(K/N)$  and  $(H/N) \cap (K/N) \in \mathcal{H}(G/N)$ . It is easy to see that  $G = HK$  and  $H \cap K \in \mathcal{H}(G)$  by Lemma 2.1(c). Thus  $H$  is a weakly  $\mathcal{H}^*$ -subgroup in  $G$ .

(c) Since  $H$  is a weakly  $\mathcal{H}^*$ -subgroup in  $G$ , then there exists a subgroup  $K$  of  $G$  such that  $G = HK$  and  $H \cap K \in \mathcal{H}(G)$ . Observe that  $|G : K|$  is a power of  $p$  and since  $(|H|, |N|) = 1$ , it follows that  $K$  contains at least one Sylow  $q$ -subgroup of  $N$  for each prime  $q$  divides the order of  $N$ , where  $q \neq p$ , and hence  $N \leq K$ . Clearly,  $G/N = (HN/N)(K/N)$  and  $(HN/N) \cap (K/N) = (H \cap K)N/N \in \mathcal{H}(G/N)$  by Lemma 2.1(d). Thus  $HN/N$  is a weakly  $\mathcal{H}^*$ -subgroup in  $G/N$ .

(d) By hypothesis, there exists a subgroup  $K$  of  $G$  such that  $G = HK$  and  $H \cap K \in \mathcal{H}(G)$ . Since  $M = H(M \cap K)$  and  $H \leq \Phi(M)$ , it follows that  $M = M \cap K$  and so  $K = G$ . Thus  $H \in \mathcal{H}(G)$ .  $\square$

**Lemma 2.3.** *Let  $G$  be a group. Then:*

- (a) *If  $N$  is a normal subgroup of  $G$ , then  $F^*(N) \leq F^*(G)$ .*
- (b) *If  $F^*(G)$  is solvable, then  $F^*(G) = F(G)$ .*

*Proof.* See [[9], X 13].  $\square$

**Lemma 2.4.** *Let  $\mathfrak{F}$  be a saturated formation containing the class of supersolvable groups  $\mathfrak{A}$ . Suppose that  $G$  is a group with a normal subgroup  $H$  such that  $G/H \in \mathfrak{F}$ . If every cyclic subgroup of  $F^*(H)$  of prime order or of order 4 is  $c$ -supplemented in  $G$ , then  $G \in \mathfrak{F}$ .*

*Proof.* See [[[13], Theorem 4.5] and [[16], Theorem 1.2]].  $\square$

**Lemma 2.5.** *Let  $\mathfrak{F}$  be a saturated formation containing the class of nilpotent groups  $\mathfrak{N}$  and let  $G$  be a group. Suppose that  $H$  is a normal subgroup of  $G$  such that  $G/H \in \mathfrak{F}$  and every cyclic subgroup of  $F^*(H)$  of order 4 is  $c$ -supplemented in  $G$ . Then  $G \in \mathfrak{F}$  if and only if every subgroup of  $F^*(H)$  of prime order lies in  $Z_{\mathfrak{F}}(G)$ .*

*Proof.* See [[13], Theorem 4.7].  $\square$

### 3. Results

We need the following lemma:

**Lemma 3.1.** *Let  $P$  be a normal  $p$ -subgroup of a group  $G$  and let  $L$  be a subgroup of  $P$ . Then  $L$  is  $c$ -supplemented in  $G$  if and only if  $L$  is a weakly  $\mathcal{H}^*$ -subgroup in  $G$ .*

*Proof.* If  $L$  is  $c$ -supplemented in  $G$ , then it follows easily that  $L$  is a weakly  $\mathcal{H}^*$ -subgroup in  $G$ . Conversely, assume that  $L$  is a weakly  $\mathcal{H}^*$ -subgroup in  $G$ . Then there exists a subgroup  $K$  of  $G$  such that  $G = LK$  and  $L \cap K \in \mathcal{H}(G)$ . It is clear that  $L \cap K$  is subnormal in  $G$ . Hence, by Lemma 2.1(b),  $L \cap K$  is normal in  $G$  and so  $L \cap K \leq L_G$ . Thus  $L$  is  $c$ -supplemented in  $G$ .  $\square$

Immediate consequence of Lemma 3.1 and Lemma 2.4, we have:

**Lemma 3.2.** *Let  $p$  be the smallest prime dividing the order of a group  $G$  and let  $P$  be a Sylow  $p$ -subgroup of  $G$ . If every cyclic subgroup of  $P$  of order  $p$  or of order 4 (if  $p = 2$ ) is a weakly  $\mathcal{H}^*$ -subgroup in  $G$ , then  $G$  is  $p$ -nilpotent.*

*Proof.* Assume that the result is false and let  $G$  be a counterexample of minimal order. Let  $S$  be a proper subgroup of  $G$ . By Lemma 2.2(a), every cyclic subgroup of  $S$  of order  $p$  or of order 4 (if  $p = 2$ ) is a weakly  $\mathcal{H}^*$ -subgroup in  $S$ . Thus  $S$  satisfies the hypothesis of the theorem and hence  $S$  is  $p$ -nilpotent by the minimal choice of  $G$ . So,  $G$  is a minimal non  $p$ -nilpotent group (non  $p$ -nilpotent group all of its proper subgroups are  $p$ -nilpotent). By [[8], p. 434, Satz 5.4 and p. 281, Satz 5.2],  $G$  is a minimal non-nilpotent group and so  $G = PQ$ , where  $P$  is a normal Sylow  $p$ -subgroup of  $G$  and  $Q$  is a non-normal cyclic Sylow  $q$ -subgroup of  $G$ , for some prime  $q \neq p$ . It is clear that  $G/P \cong Q$  is supersolvable. By hypothesis and Lemma 3.1, every cyclic subgroup of  $P = F^*(P)$  of order  $p$  or of order 4 (if  $p = 2$ ) is  $c$ -supplemented in  $G$ . Lemma 2.4 implies that  $G$  is supersolvable. Thus  $G$  is  $p$ -nilpotent as  $p$  is the smallest prime dividing the order of  $G$ , a contradiction completing the proof of the lemma.  $\square$

Now we prove:

**Theorem 3.3.** *Let  $G$  be a group such that every cyclic subgroup of  $G$  of prime order or of order 4 is a weakly  $\mathcal{H}^*$ -subgroup in  $G$ , then  $G$  is supersolvable.*

*Proof.* Assume that the result is false and let  $G$  be a counterexample of minimal order. By using Lemma 2.2(a) and repeated applications of Lemma 3.2,  $G$  has a Sylow tower of supersolvable type. Let  $p$  be the largest prime dividing the order of  $G$  and let  $P$  be a Sylow  $p$ -subgroup of  $G$ . Obviously,  $P$  is a normal subgroup of  $G$ . By Lemma 2.2(c), every cyclic subgroup of  $G/P$  of prime order or of order 4 is a weakly  $\mathcal{H}^*$ -subgroup in  $G/P$ . Therefore  $G/P$  is supersolvable by the minimal choice of  $G$ . By hypothesis and Lemma 3.1, every cyclic subgroup of  $P = F^*(P)$  of order  $p$  or of order 4 (if  $p = 2$ ) is  $c$ -supplemented in  $G$ . Thus  $G$  is supersolvable by Lemma 2.4, a contradiction completing the proof of the theorem.  $\square$

Now we can prove the following main theorem:

**Theorem 3.4.** *Let  $\mathfrak{F}$  be a saturated formation containing the class of supersolvable groups  $\mathfrak{A}$  and let  $G$  be a group. Then  $G \in \mathfrak{F}$  if and only if  $G$  has a normal subgroup  $H$  such that  $G/H \in \mathfrak{F}$  and every cyclic subgroup of  $F^*(H)$  of prime order or of order 4 is a weakly  $\mathcal{H}^*$ -subgroup in  $G$ .*

*Proof.* If  $G \in \mathfrak{F}$ , then we set  $H = 1$  and the theorem follows. Now we prove the converse. By Lemma 2.2(a), every cyclic subgroup of  $F^*(H)$  of prime order or of order 4 is a weakly  $\mathcal{H}^*$ -subgroup in  $F^*(H)$ . Theorem 3.3 implies that  $F^*(H)$  is supersolvable and it follows, by Lemma 2.3(b), that  $F^*(H) = F(H)$ . Note that every cyclic subgroup of  $F^*(H) = F(H)$  of prime order or of order 4 is c-supplemented in  $G$  by hypothesis and Lemma 3.1. Applying Lemma 2.4 yields  $G \in \mathfrak{F}$ . This completes the proof of the theorem.  $\square$

The following corollaries are immediate consequences of Theorem 3.4:

**Corollary 3.5.** *Let  $G$  be a group with a normal subgroup  $H$  such that  $G/H$  is supersolvable. If every cyclic subgroup of  $F^*(H)$  of prime order or of order 4 is a weakly  $\mathcal{H}^*$ -subgroup in  $G$ , then  $G$  is supersolvable.*

**Corollary 3.6.** *Let  $G$  be a group such that every cyclic subgroup of  $F^*(G)$  of prime order or of order 4 is a weakly  $\mathcal{H}^*$ -subgroup in  $G$ , then  $G$  is supersolvable.*

**Corollary 3.7.** *Let  $\mathfrak{F}$  be a saturated formation containing the class of supersolvable groups  $\mathfrak{U}$  and let  $G$  be a group. Then  $G \in \mathfrak{F}$  if and only if  $G$  has a solvable normal subgroup  $H$  such that  $G/H \in \mathfrak{F}$  and every cyclic subgroup of  $F(H)$  of prime order or of order 4 is a weakly  $\mathcal{H}^*$ -subgroup in  $G$ .*

**Corollary 3.8.** *Let  $\mathfrak{F}$  be a saturated formation containing the class of supersolvable groups  $\mathfrak{U}$  and let  $G$  be a group. Then  $G \in \mathfrak{F}$  if and only if  $G$  has a normal subgroup  $H$  such that  $G/H \in \mathfrak{F}$  and every cyclic subgroup of  $H$  of prime order or of order 4 is a weakly  $\mathcal{H}^*$ -subgroup in  $G$ .*

The following theorem is an improvement of [[10], Theorem 4(a)].

**Theorem 3.9.** *Let  $\mathfrak{F}$  be a saturated formation containing the class of supersolvable groups  $\mathfrak{U}$  and let  $G$  be a 2-nilpotent group. Then  $G \in \mathfrak{F}$  if and only if  $G$  has a solvable normal subgroup  $H$  such that  $G/H \in \mathfrak{F}$  and every subgroup of  $F(H)$  of odd prime order is a weakly  $\mathcal{H}^*$ -subgroup in  $G$ .*

*Proof.* If  $G \in \mathfrak{F}$ , then we set  $H = 1$  and the theorem follows. For the converse, assume that the result is false and let  $G$  be a counterexample of minimal order. Since  $G$  is 2-nilpotent, then  $H$  is 2-nilpotent. Therefore  $H = QN$ , where  $Q$  is a Sylow 2-subgroup of  $H$  and  $N$  is a normal Hall  $2'$ -subgroup of  $H$ . Clearly,  $H \neq 1$  and hence  $Q \neq 1$  or  $N \neq 1$ . Suppose that  $N \neq 1$ . Since  $N \text{ char } H$  and  $H$  is normal in  $G$ , it follows that  $N$  is normal in  $G$ . It is easy to see that  $G/N$  is 2-nilpotent,  $(G/N)/(H/N) \cong G/H \in \mathfrak{F}$  and  $F(H/N) = H/N \cong Q$  is a 2-group. Thus  $G/N$  satisfies the hypothesis of the theorem and hence  $G/N \in \mathfrak{F}$  by the minimal choice of  $G$ . Since  $F(N) \neq 1$  is a  $2'$ -group and  $F(N) \leq F(H)$ , then every subgroup of  $F(N)$  of prime order is a weakly  $\mathcal{H}^*$ -subgroup in  $G$  (note that  $F(N) \neq 1$  because  $N$  is of odd order and so it is solvable by the Odd Order Theorem). By Corollary 3.7, we have  $G \in \mathfrak{F}$ , a contradiction. Therefore  $N = 1$  and hence  $H = Q \neq 1$  is a 2-group. Since  $G$  is 2-nilpotent, then there exists a normal Hall  $2'$ -subgroup  $K$  of  $G$  such that  $G/K \cong P \in \mathfrak{U} \subseteq \mathfrak{F}$ , where  $P$  is a Sylow 2-subgroup of  $G$ . Thus  $G \cong G/(H \cap K) \in \mathfrak{F}$ , a contradiction completing the proof of the theorem.  $\square$

The next theorems are concerned with the class of nilpotent groups  $\mathfrak{N}$ . We start with the following lemma:

**Lemma 3.10.** *Let  $G$  be a group and let  $P$  be a Sylow  $p$ -subgroup of  $G$ , where  $p$  is a fixed prime number dividing the order of  $G$ . If every subgroup of  $P$  of order  $p$  lies in  $Z_\infty(G)$  and if  $p = 2$ , in addition, assume that every cyclic subgroup of  $P$  of order 4 is a weakly  $\mathcal{H}^*$ -subgroup in  $G$ , then  $G$  is  $p$ -nilpotent.*

*Proof.* Assume that the result is false and let  $G$  be a counterexample of minimal order. Let  $S$  be a proper subgroup of  $G$ . Obviously, every subgroup of  $S$  of order  $p$  lies in  $Z_\infty(S)$  as  $S \cap Z_\infty(G) \leq Z_\infty(S)$ . If  $p = 2$ , then every cyclic subgroup of  $S$  of order 4 is a weakly  $\mathcal{H}^*$ -subgroup in  $S$  by Lemma 2.2(a). Therefore,  $S$  satisfies the hypothesis of the theorem and so  $S$  is  $p$ -nilpotent by the minimal choice of  $G$ . Thus  $G$  is a minimal non  $p$ -nilpotent group and hence  $G = PQ$ , where  $P$  is a normal Sylow  $p$ -subgroup of  $G$  and  $Q$  is a non-normal cyclic Sylow  $q$ -subgroup of  $G$ , for some prime  $q \neq p$ . Note that  $G/P \cong Q$  is nilpotent. By hypothesis, every subgroup of  $P = F^*(P)$  of order  $p$  lies in  $Z_\infty(G)$ . Also, if  $p = 2$ , then every subgroup of  $P$  of order 4 is  $c$ -supplemented in  $G$  by Lemma 3.1. Thus  $G$  is nilpotent by Lemma 2.5, a contradiction completing the proof of the lemma.  $\square$

Immediate consequence of Lemma 3.10, we have:

**Corollary 3.11.** *Let  $G$  be a group such that every subgroup of  $G$  of prime order lies in  $Z_\infty(G)$  and every cyclic subgroup of  $G$  of order 4 is a weakly  $\mathcal{H}^*$ -subgroup in  $G$ , then  $G$  is nilpotent.*

Now we can prove:

**Theorem 3.12.** *Let  $\mathfrak{F}$  be a saturated formation containing the class of nilpotent groups  $\mathfrak{N}$  and let  $G$  be a group. Then  $G \in \mathfrak{F}$  if and only if  $G$  has a normal subgroup  $H$  such that  $G/H \in \mathfrak{F}$ , every subgroup of  $F^*(H)$  of prime order lies in  $Z_{\mathfrak{F}}(G)$  and every cyclic subgroup of  $F^*(H)$  of order 4 is a weakly  $\mathcal{H}^*$ -subgroup in  $G$ .*

*Proof.* If  $G \in \mathfrak{F}$ , then we set  $H = 1$  and the theorem follows. Now we prove the converse. It is clear that  $G^{\mathfrak{F}} \leq H$  and  $F^*(G^{\mathfrak{F}}) \leq F^*(H)$  by Lemma 2.3(a). Since  $[G^{\mathfrak{F}}, Z_{\mathfrak{F}}(G)] = 1$  by [[7], p. 390, Theorem 6.10], it follows that  $Z_{\mathfrak{F}}(G) \leq C_G(G^{\mathfrak{F}}) \leq C_G(F^*(G^{\mathfrak{F}}))$ . Let  $L$  be a subgroup of  $F^*(G^{\mathfrak{F}})$  of prime order. Then  $L \leq Z_{\mathfrak{F}}(G) \cap F^*(G^{\mathfrak{F}}) \leq C_G(F^*(G^{\mathfrak{F}})) \cap F^*(G^{\mathfrak{F}}) \leq Z(F^*(G^{\mathfrak{F}})) \leq Z_\infty(F^*(G^{\mathfrak{F}}))$ . So, every subgroup of  $F^*(G^{\mathfrak{F}})$  of prime order lies in  $Z_\infty(F^*(G^{\mathfrak{F}}))$ . By Lemma 2.2(a), every cyclic subgroup of  $F^*(G^{\mathfrak{F}})$  of order 4 is a weakly  $\mathcal{H}^*$ -subgroup in  $F^*(G^{\mathfrak{F}})$ . Thus  $F^*(G^{\mathfrak{F}})$  is nilpotent by Corollary 3.11. Therefore every cyclic subgroup of  $F^*(G^{\mathfrak{F}})$  of order 4 is  $c$ -supplemented in  $G$  by Lemma 3.1. Note that  $G/G^{\mathfrak{F}} \in \mathfrak{F}$  and every subgroup of  $F^*(G^{\mathfrak{F}})$  of prime order lies in  $Z_{\mathfrak{F}}(G)$ . Applying Lemma 2.5 yields  $G \in \mathfrak{F}$ . This completes the proof of the theorem.  $\square$

The following corollaries are immediate consequences of Theorem 3.12:

**Corollary 3.13.** *Let  $G$  be a group with a normal subgroup  $H$  such that  $G/H$  is nilpotent. If every subgroup of  $F^*(H)$  of prime order lies in  $Z_\infty(G)$  and every cyclic subgroup of  $F^*(H)$  of order 4 is a weakly  $\mathcal{H}^*$ -subgroup in  $G$ , then  $G$  is nilpotent.*



**Corollary 3.14.** *Let  $G$  be a group such that every subgroup of  $F^*(G)$  of prime order lies in  $Z_\infty(G)$  and every cyclic subgroup of  $F^*(G)$  of order 4 is a weakly  $\mathcal{H}^*$ -subgroup in  $G$ , then  $G$  is nilpotent.*

**Corollary 3.15.** *Let  $\mathfrak{F}$  be a saturated formation containing the class of nilpotent groups  $\mathfrak{N}$  and let  $G$  be a group. Then  $G \in \mathfrak{F}$  if and only if  $G$  has a solvable normal subgroup  $H$  such that  $G/H \in \mathfrak{F}$ , every subgroup of  $F(H)$  of prime order lies in  $Z_{\mathfrak{F}}(G)$  and every cyclic subgroup of  $F(H)$  of order 4 is a weakly  $\mathcal{H}^*$ -subgroup in  $G$ .*

**Corollary 3.16.** *Let  $\mathfrak{F}$  be a saturated formation containing the class of nilpotent groups  $\mathfrak{N}$  and let  $G$  be a group. Then  $G \in \mathfrak{F}$  if and only if  $G$  has a normal subgroup  $H$  such that  $G/H \in \mathfrak{F}$ , every subgroup of  $H$  of prime order lies in  $Z_{\mathfrak{F}}(G)$  and every cyclic subgroup of  $H$  of order 4 is a weakly  $\mathcal{H}^*$ -subgroup in  $G$ .*

#### 4. Some applications

Several theorems in the literature can be considered as special cases of our main results. In this section, we list some of these theorems.

**Corollary 4.1** ([11], Lemma 3.8). *Let  $p$  be the smallest prime dividing the order of a group  $G$  and let  $P$  be a Sylow  $p$ -subgroup of  $G$ . If every cyclic subgroup of  $P$  of order  $p$  or of order 4 (if  $p = 2$ ) is  $c$ -normal in  $G$ , then  $G$  is  $p$ -nilpotent.*

**Corollary 4.2** ([11], Theorem 3.9). *Let  $\mathfrak{F}$  be a saturated formation containing the class of supersolvable groups  $\mathfrak{U}$  and let  $G$  be a group. Then  $G \in \mathfrak{F}$  if and only if  $G$  has a normal subgroup  $H$  such that  $G/H \in \mathfrak{F}$  and every cyclic subgroup of  $H$  of prime order or of order 4 is  $c$ -normal in  $G$ .*

**Corollary 4.3** (Theorem 3 of [10] and Theorem 2 of [14]). *Let  $\mathfrak{F}$  be a saturated formation containing the class of supersolvable groups  $\mathfrak{U}$  and let  $G$  be a group. Then  $G \in \mathfrak{F}$  if and only if  $G$  has a solvable normal subgroup  $H$  such that  $G/H \in \mathfrak{F}$  and every cyclic subgroup of  $F(H)$  of prime order or of order 4 is  $c$ -normal in  $G$ .*

**Corollary 4.4** ([15], Theorem 3.2). *Let  $\mathfrak{F}$  be a saturated formation containing the class of supersolvable groups  $\mathfrak{U}$ . Suppose that  $G$  is a group with a normal subgroup  $H$  such that  $G/H \in \mathfrak{F}$  and every cyclic subgroup of  $F^*(H)$  of prime order or of order 4 is  $c$ -normal in  $G$ , then  $G \in \mathfrak{F}$ .*

**Corollary 4.5** ([10], Theorem 4(a)). *Let  $\mathfrak{F}$  be a saturated formation containing the class of supersolvable groups  $\mathfrak{U}$  and let  $G$  be a 2-nilpotent group. If  $G$  has a solvable normal subgroup  $H$  such that  $G/H \in \mathfrak{F}$  and every subgroup of  $F(H)$  of odd prime order is  $c$ -normal in  $G$ , then  $G \in \mathfrak{F}$ .*

**Corollary 4.6** ([10], Theorem 1). *Let  $\mathfrak{F}$  be a saturated formation containing the class of nilpotent groups  $\mathfrak{N}$  and let  $G$  be a group. Then  $G \in \mathfrak{F}$  if and only if  $G$  has a solvable normal subgroup  $H$  such that  $G/H \in \mathfrak{F}$ , every subgroup of  $F(H)$  of prime order lies in  $Z_{\mathfrak{F}}(G)$  and every cyclic subgroup of  $F(H)$  of order 4 is  $c$ -normal in  $G$ .*

**Corollary 4.7** ([1], Theorem 3.4). *Let  $p$  be the smallest prime dividing the order of a group  $G$  and let  $P$  be a Sylow  $p$ -subgroup of  $G$ . If every cyclic subgroup of  $P$  of order  $p$  or of order 4 (if  $p = 2$ ) is a weakly  $\mathcal{H}$ -subgroup in  $G$ , then  $G$  is  $p$ -nilpotent.*

**Corollary 4.8** ([1], Theorem 3.5). *Let  $\mathfrak{F}$  be a saturated formation containing the class of supersolvable groups  $\mathfrak{U}$  and let  $G$  be a group. Then  $G \in \mathfrak{F}$  if and only if  $G$  has a normal subgroup  $H$  such that  $G/H \in \mathfrak{F}$  and every cyclic subgroup of  $H$  of prime order or of order 4 is a weakly  $\mathcal{H}$ -subgroup in  $G$ .*

**Corollary 4.9** ([1], Corollary 3.10). *Let  $\mathfrak{F}$  be a saturated formation containing the class of supersolvable groups  $\mathfrak{U}$  and let  $G$  be a group. Then  $G \in \mathfrak{F}$  if and only if  $G$  has a solvable normal subgroup  $H$  such that  $G/H \in \mathfrak{F}$  and every cyclic subgroup of  $F(H)$  of prime order or of order 4 is a weakly  $\mathcal{H}$ -subgroup in  $G$ .*

**Corollary 4.10** ([1], Theorem 3.9). *Let  $\mathfrak{F}$  be a saturated formation containing the class of supersolvable groups  $\mathfrak{U}$  and let  $G$  be a group. Then  $G \in \mathfrak{F}$  if and only if  $G$  has a normal subgroup  $H$  such that  $G/H \in \mathfrak{F}$  and every cyclic subgroup of  $F^*(H)$  of prime order or of order 4 is a weakly  $\mathcal{H}$ -subgroup in  $G$ .*

Based on the new concept “weakly  $\mathcal{H}^*$ -subgroup” and the results that have been studied in this paper, the following questions arise:

**Question 1.** *Let  $P$  be a Sylow  $p$ -subgroup of a group  $G$ , where  $p$  is the smallest prime dividing  $|G|$ . Assume that all maximal subgroups of  $P$  are weakly  $\mathcal{H}^*$ -subgroups in  $G$ . Is  $G$   $p$ -nilpotent?*

**Question 2.** *Assume that all maximal subgroups of every Sylow subgroup of a group  $G$  are weakly  $\mathcal{H}^*$ -subgroups in  $G$ . Is  $G$  supersolvable?*

**Question 3.** *Let  $\mathfrak{F}$  be a saturated formation containing the class of supersolvable groups  $\mathfrak{U}$  and  $H$  be a normal subgroup of  $G$  such that  $G/H \in \mathfrak{F}$ . Assume that every non-cyclic Sylow subgroup  $P$  of  $H$  has a subgroup  $D$  with  $1 < |D| < |P|$  such that every subgroup of  $P$  of order  $|D|$  (and 4 if  $|D| = 2$ ) is a weakly  $\mathcal{H}^*$ -subgroup in  $G$ . Is  $G \in \mathfrak{F}$ ?*

**Question 4.** *Let  $\mathfrak{F}$  be a saturated formation containing the class of supersolvable groups  $\mathfrak{U}$  and  $H$  be a normal subgroup of  $G$  such that  $G/H \in \mathfrak{F}$ . Assume that every non-cyclic Sylow subgroup  $P$  of  $F^*(H)$  has a subgroup  $D$  with  $1 < |D| < |P|$  such that every subgroup of  $P$  of order  $|D|$  (and 4 if  $|D| = 2$ ) is a weakly  $\mathcal{H}^*$ -subgroup in  $G$ . Is  $G \in \mathfrak{F}$ ?*

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