THE COPRIME GRAPH OF A GROUP

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ABSTRACT. The coprime graph Γ_G with a finite group G as follows: Take G as the vertex set of Γ_G and join two distinct vertices u and v if $(|u|, |v|) = 1$. In the paper, we explore how the graph theoretical properties of Γ_G can effect on the group theoretical properties of G.

1. Introduction and results

Study of algebraic structures by graphs associated with them gives rise to many recent and interesting results in the literature. This field is relatively new, and over the years different types of graphs of a group were defined. For example, prime graph [6] and the non-commuting graph [1], and of course Cayley graphs, which have a long history. For more graphs, see [2, 3, 4, 7]

Let G be a finite group. One can associate a graph to G in many different ways. Since the order of an element is one of the most basic concepts of group theory, we associate a graph Γ_G with G (called the coprime graph of G) as follows: Take G as the vertices of Γ_G and two distinct vertices x and y are adjacent if and only if $(|x|, |y|) = 1$. For example, Fig. 1 is the coprime graph of Z_6, and Fig.2 is Γ_{S_3}. It is easy to see that the coprime graph on G is simple.

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Fig. 1: Γ_{Z_6}
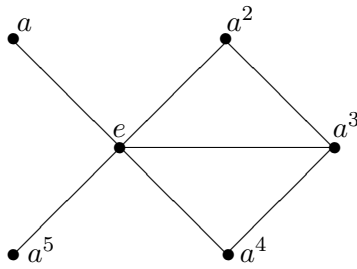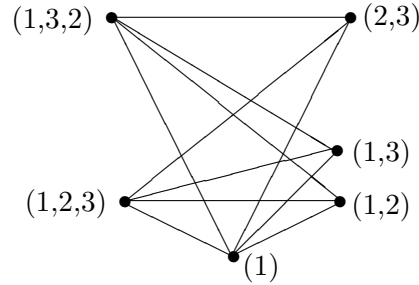

Fig. 2: $\Gamma_{S_3} \cong K_{1,2,3}$

In this paper, we consider *simple graphs* which are undirected, with no loops or multiple edges. Let Γ be a graph, $V(\Gamma)$ and $E(\Gamma)$ denote the sets of vertices and edges of Γ, respectively. Γ is respectively called *empty* and *complete* if $V(\Gamma)$ is empty and every two distinct vertices in $V(\Gamma)$ are adjacent. A complete graph of order n is denoted by K_n. The *degree* of a vertex v in Γ, denoted by $deg_\Gamma(v)$, is the number of edges which are incident to v. A subset Ω of $V(\Gamma)$ is called a *clique* if the induced subgraph of Ω is complete. The order of the largest clique in Γ is its *clique number*, which is denoted by $\omega(\Gamma)$. If $u, v \in V(\Gamma)$, the $d(u,v)$ denotes the length of the shortest path between u and v. The largest distance between all pairs of $V(\Gamma)$ is called the *diameter* of Γ and denoted by $diam(\Gamma)$. A set S of vertices of Γ is a *dominating set* of Γ if every vertex in $V(\Gamma) \setminus S$ is adjacent to some vertex in S, the cardinality of a minimum dominating set is called the *domination number* of Γ and is denoted by $\gamma(\Gamma)$. Γ is a *bipartite* graph means that $V(\Gamma)$ can be partitioned into two subsets U and W, called partite sets, such that every edge of Γ joins a vertex of U and a vertex of W. If every vertex of U is adjacent to every vertex of W, Γ is called a *complete bipartite graph*, where U and W are independent. A complete bipartite graph with $|U| = s$ and $|W| = t$ is denoted by $K_{s,t}$. Similarly, we can define a *complete k-partite graph*. For more information about this concept of graph theory the reader can refer to [5].

All groups considered are finite. The number of elements of G is called its *order* and is denoted by $|G|$. The order of an element x of G is the smallest positive integer n such that $x^n = e$. The order of an element x is denoted by $|x|$. Now we introduce three commonly used theorems in group theory. Lagrange's theorem: If H is a subgroup of G, then the order of H divides the order of G. In particular, the order of x divides the order of G for every element x of G. Sylow's theorem: For any prime factor p with multiplicity n of the order of G, there exists a Sylow p-subgroup of G, of order p^n. Cauchy's theorem: If p is a prime number dividing the order of G, then G contains an element of order p. These theorems are frequently used in the following sections. For more notations and terminologies in group theory, please refer to [8].

The outline of this paper is as follows. In Section 2, we give some properties of coprime graph on diameter, planarity, partition, clique number, etc. We also characterize some groups whose coprime graphs are complete, planar, a star, or regular and so on. In Section 3 we classify the groups whose coprime graphs have end-vertices. In Section 4, we give some results on automorphism groups of coprime graphs. Particularly, we obtain that $Aut(G) = Aut(\Gamma_G)$ if and only if G is isomorphic to Z_3

or the Klein group $Z_2 \times Z_2$. In Section 5, we prove some general graph theoretical properties which hold for the coprime graphs of the dihedral groups, such as degree, traversability, planarity etc.

2. **Some properties of coprime graph**

Proposition 2.1. *Let G be any group. Then $diam(\Gamma_G) \leq 2$. In particularly, Γ_G is connected and the girth of Γ_G equals 3 or ∞.*

Proof. Let u and v be two distinct vertices of Γ_G. If $(|u|, |v|) = 1$, then u is adjacent to v, and hence $d(u, v) = 1$. Consequently, we may assume that u and v are non-identity elements of G and $(|u|, |v|) \neq 1$. Note that $(|u|, |e|) = 1$ and $(|v|, |e|) = 1$, then the vertex e is adjacent to both u and v and we obtain $d(u, v) = 2$. This means that Γ_G is connected and $diam(\Gamma_G) \leq 2$. If there exist $x \neq e, y \neq e \in G$ such that x and y are joined by some edge, then $\{x, y, e\}$ is a cycle of order 3 of Γ_G and so the girth of Γ_G is 3. Otherwise every two vertices(non-identity) of Γ_G are not adjacent, that is, Γ_G is a tree, which implies that the girth of Γ_G is equal to ∞ □

Proposition 2.2. *Let G be a group with order greater than 2. Then $\{e\}$ is a unique dominating set of size 1 of Γ_G. In particular, $\gamma(\Gamma_G) = 1$ and $deg_{\Gamma_G}(e) = |G| - 1$.*

Proof. By the definition of the coprime graph, it is easy to see that $\{e\}$ is a dominating set of Γ_G. Now we prove uniqueness. If $\{x\}$ is also a dominating set of Γ_G, where $x \neq e$. Then x is adjacent to every element of G. Let $|x| = m$, then we claim that x belongs to the center $Z(G)$ of G. If not, then there exists an element $g \in G$ such that $g^{-1}xg$ and x are conjugate and $g^{-1}xg \neq x$. It follows that $|g^{-1}xg| = |x|$. Hence, in this case, $g^{-1}xg$ and x are non-adjacent, a contradiction and the proof of the claim is finished. Let y be an element such that $y \neq 1$ and $y \neq x$. Note that $yx = xy$, then xy is an element of G and $|xy| = |x||y|$. It mans that $(|xy|, |x|) = |x|$. Consequently, x and xy are not adjacent, this is a contradiction as $\{x\}$ is also a dominating set. The proof of the proposition is now complete. □

Proposition 2.3. *Let G be a group. Then $diam(\Gamma_G) = 1$ if and only if G is isomorphic to cyclic group Z_2 with order 2.*

Proof. Suppose that $diam(\Gamma_G) = 1$. If $|G| \geq 3$, then there exist at least two non-identity elements x and y of G such that x and y are adjacent. Thus $(|x|, |y|) = 1$. Note that $|x| \neq 1$ and $|y| \neq 1$, this follows that either $|x| > 2$ or $|y| > 2$. If $|x| = n > 2$, then we have $\langle x \rangle$ is a cyclic subgroup of order n of G. It is obvious that $\varphi(n) \geq 2$ for $n > 2$, where $\varphi(n)$ is Euler's totient function on n. It means that $\varphi(n) \geq 2$ contains at least two elements u and v with order n. Thereby u is not adjacent to v since their orders are equal. That is say Γ_G is not complete, this is a contradiction. Similarly the case $|y| > 2$ also gives a contradiction. Consequently $|G| \leq 2$, that is, Γ_G is isomorphic to the cyclic group of order 2.

The converse is clear. □

Corollary 2.4. *Let G be a group. Then Γ_G is regular if and only if G is isomorphic to Z_2.*

Corollary 2.5. *Let G be a group with order greater than 2. Then Γ_G is not complete.*

Proposition 2.6. *Let G be a group. Then $\Gamma_G \cong K_{1,|G|-1}$ if and only if G is a p-group for some prime integer p.*

Proof. Assume that $\Gamma_G \cong K_{1,|G|-1}$. By the definite of coprime graph, it is easy to see that $V(\Gamma_G)$ can be partitioned into two independent subsets $\{e\}$ and $V(\Gamma_G) \setminus \{e\}$. If G is not a p-group for some prime integer p, then $|G|$ contains at least two prime divisors p_1 and p_2. By Cauchy's Theorem, G exists two elements x and y such that their orders are p_1 and p_2 respectively. This follows that x and y are adjacent, note that $|x| \neq 1$ and $y \neq 1$, a contradiction.

For the converse, let G is a $p-$group for some prime integer p. Since the order of every non-identity element of G equals a power of p, every two non-identity elements are not adjacent. Notice that every non-identity element and e of G are joined by an edge. Thus $\Gamma_G \cong K_{1,|G|-1}$. $\qquad\square$

Proposition 2.7. *Let G be a group. Then, G is not a p-group if and only if Γ_G is not bipartite.*

Proof. If G is not a p-group, then we can know there exist at least two non-identity elements x, y such that $(|x|, |y|) = 1$. Then x is adjacent to y. Assume that Γ_G is bipartite and U and V are classes of this partition. Since $deg_{\Gamma_G}(e) = |V(\Gamma_G)| - 1$, we have $\{e\} = U$ or $\{e\} = V$. If $\{e\} = U$, then $x, y \in V$. However, x and y are joined by an edge, this is a contradiction. A similar argument with $\{e\} = V$ also shows a contradiction. Thus Γ_G is not bipartite.

Note that Γ_G is a star for a p-group G and star is bipartite, then the converse is true. Now the proof is complete. $\qquad\square$

Proposition 2.8. *Let p and q be two distinct prime numbers and G be a non-cyclic group with order pq. Then Γ_G is a complete 3-partite graph.*

Proof. Assume that G is non-cyclic and $|G| = pq$. Then, by Lagrange's theorem and Sylow's theorem, it is easy to see that the order of every element of G is a divisor of pq and there exists at least an element such that its order equals $1, p$ or q. Let $U = \{x \in G \mid |x| = p\}$ and $V = \{x \in G \mid |x| = q\}$. Clearly, every element in U is adjacent to every element in V. Note that U and V are independent sets. Thus U, V and $\{e\}$ are partite sets of Γ_G, and hence Γ_G is a complete 3-partite graph. $\qquad\square$

Proposition 2.9. *Let G be a group and $\pi(G)$ the set of prime divisors of $|G|$. Then $\omega(\Gamma_G) = |\pi(G)| + 1$.*

Proof. Assume that $|\pi(G)| = n$ and $|G| = p_1^{r_1} p_2^{r_2} \cdots p_n^{r_n}$, where $p_1, p_2, \ldots, p_n$ are distinct prime integers and $r_i \geq 1$ for all $i \in \{1, 2, \ldots, n\}$. By Cauchy's theorem, G contains an element of order p_i for every i. Let g_i be the element of order p_i. Clearly, the subgraph of Γ_G induced by the set $\{e, g_1, g_2, \ldots, g_n\}$ is complete. That is, $\omega(\Gamma_G) \geq |\pi(G)| + 1$. For any element $g(\neq e)$ in G, by Lagrange's theorem, we have that $|g| \mid |G|$. Thus there exists at least a prime integer p_i such that

$(|g|, |p_i|) = p_i$. It follows that the subgraph of Γ_G induced by the set $\{e, g_1, g_2, \ldots, g_n, g\}$ is not complete. Namely, the subgraph induced by $\{e, g_1, g_2, \ldots, g_n, g\}$ is not a clique of Γ_G. Consequently, it is easy to see that $\omega(\Gamma_G) = |\pi(G)| + 1$. $\qquad \square$

Proposition 2.10. *Let G be a group. If G is planar, then G is a p-group, or $|G| = 2^t p^k$ and G contains the only element of order 2, where p is an odd prime, t and k are two non-negative integers.*

Proof. Suppose that Γ_G is planar. Since the complete graph of order 5 is non-planar, we have $\omega(\Gamma_G) \leq 4$. That is, $|\pi(G)| \leq 3$(see Proposition 2.9). If $|\pi(G)| = 3$, then by Cauchy's theorem, we can see that there exist at least three distinct elements x, y and z such that $|x| = q_1 \geq 2$, $|y| = q_2 \geq 3$ and $|z| = q_3 \geq 5$, where q_1, q_2 and q_3 are distinct prime integers. Thereby, G has at least $1, 2$ and 4 elements of order q_1, q_2 and q_3, respectively. Note that the identity e, then Γ_G has a subgraph Γ which is isomorphic to $K_{1,1,2,4}$. Clearly, $|V(\Gamma)| = 8$ and $|E(\Gamma)| = 21$. Since $|E(\Gamma)| > 3|V(\Gamma)| - 6$, it means that $K_{1,1,2,4}$ is non-planar by Theorem 9.2 of [5], a contradiction. If $|\pi(G)| = 2$ and $2 \notin \pi(G)$, then, similarly, we can know that Γ_G has a subgraph which is isomorphic to $K_{1,2,4}$. It is easy to prove that $K_{1,2,4}$ is not planar, this is also a contradiction. Now we assume that $|\pi(G)| = 2$ and $2 \in \pi(G)$. If the number of elements of order 2 of G is greater than 1, then G contains at least 3 elements of order 2 since $|G|$ is even. A similar argument shows that Γ_G has a non-planar subgraph which is isomorphic to $K_{1,2,3}$. Thus, the element of G of order 2 is unique. Finally, if $|\pi(G)| = 1$, then G is a $p-$group. Namely Γ_G is planar. $\qquad \square$

Proposition 2.11. *Let G be a group. If G is cyclic with order $2p$ for some odd prime p, then Γ_G is planar.*

Proof. For some odd prime number p, if G is cyclic and $|G| = 2p$, then the number of elements of order $2p$ of G is $p - 1$. By Lagrange's theorem, these elements of order $2p$ are only adjacent to e. Clearly, G contains $p - 1$ elements of order $p - 1$ and an elements of order 2. Thus, it is easy to see that the coprime graph of G is planar. $\qquad \square$

Remark 2.12. *Let $G = S_3$. Since the number of elements of G of order 2 is 3, Γ_G is not planar. Let $G = Z_4 \times Z_3 \times Z_3$. Clearly, G has a unique element of order 2, and it is easy to see that Γ_G is planar.*

Proposition 2.13. *Let G_1 and G_2 be two groups. If $G_1 \cong G_2$, then $\Gamma_{G_1} \cong \Gamma_{G_2}$.*

Proof. Let ϕ be an isomorphism from G_1 to G_2. Obviously, ϕ is a one-to-one correspondence between Γ_{G_1} and Γ_{G_2}. Let x and y be two vertices of Γ_{G_1}. Since $|g| = |g^\phi|$ for all $g \in G_1$, we can see that $xy \in E(\Gamma_{G_1})$ if and only if $x^\phi y^\phi \in E(\Gamma_{G_2})$. Thus, ϕ is a graph automorphism from Γ_{G_1} to Γ_{G_2}. Namely $\Gamma_{G_1} \cong \Gamma_{G_2}$. $\qquad \square$

Remark 2.14. *The converse of Proposition 2.13 is not true in general. Let $G_1 = D_8$ and let $G_2 = Z_8$. We see that G_1 and G_2 are 2-groups. Clearly $\Gamma_{G_1} \cong \Gamma_{G_2}$, but $G_1 \not\cong G_2$.*

3. **Groups whose coprime graphs have end-vertices**

Theorem 3.1. *Let G be a group of order $p_1^{r_1} p_2^{r_2} \cdots p_n^{r_n}$, where p_i is a prime for every $i \in \{1, 2, \ldots, n\}$ and r_i is a non-negative integer for every $i \in \{1, 2, \ldots, n\}$. Then Γ_G has no end-vertex if and only if G has no elements of order $p_1^{k_1} p_2^{k_2} \cdots p_n^{k_n}$, where $1 \le k_i \le r_i$.*

Proof. It is straightforward.   □

Theorem 3.2. *Let G be a group. Then G contains precisely a non-identity element x which is an end-vertex in Γ_G if and only if G is isomorphic to Z_2.*

Proof. Suppose that x is a unique end-vertex in Γ_G, where $x \in G$ and $x \ne e$. If the order of G equals 2, then $G \cong Z_2$. Thus we may assume that $|G| > 2$. Now we consider two cases:

Case 1. The order of x is 2. Since x is a unique end-point, there exist $y, z \in V(\Gamma_G)$ such that yz is an edge of Γ_G and $y, z \in G \setminus \{e\}$. Clearly, y and z are not adjacent to x. It is easy to see that $(|x|, |y|) \ne 1$ and $(|x|, |z|) \ne 1$. Thus we have $2 \mid |y|$ and $2 \mid |z|$. That is, $(|x|, |y|) \ne 1$, this is a contradiction since x and y are adjacent.

Case 2. Assume that the order of x is greater than 2. Notice that $\langle x \rangle$ is a cyclic subgroup of order greater that 2 of G. Thereby there exists at least an element w in G such that $|x| = |w|$. Since the end-point is unique in Γ_G, there is an element $u(\ne e)$ such that u is adjacent to w. Furthermore, $(|u|, |w|) \ne 1$, namely $(|u|, |w|) \ne 1$. It follows that u is also adjacent to w, a contradiction.

Thus G is only isomorphic to Z_2. On the other hand, the converse is obvious and the proof is complete.   □

Proposition 3.3. *Let G be a group of order n, where $n \ge 3$. If G is cyclic, then Γ_G contains some end-vertex. Particularly, the number of end-vertices of Γ_G is greater than or equal to $\varphi(n)$.*

Proof. Since G is cyclic, there exists at least an element x of G such that $|x| = n$. By Lagrange's theorem, it is easy to see that $|x|$ is divided exactly by $|g|$ for all $g \in G$, that is, $(|x|, |g|) = |g|$. It follows that x and g are adjacent if and only if g is identity element of G. Namely, x is a end-vertex of Γ_G. It is well known that the number of the generators of cyclic group G is $\varphi(n)$. Thus, the number of end-vertices of Γ_G is greater than or equal to $\varphi(n)$.   □

Remark 3.4. *Let $G \cong Z_{12}$. Then the number of end-vertices of $\Gamma_{Z_{12}}$ is greater than $\varphi(12)$ as these elements of order 6 of Z_{12} are also the end-vertices of $\Gamma_{Z_{12}}$. But, if $G \cong Z_6$, then the number of end-vertices of Γ_{Z_6} is equal to $\varphi(6)$. In general, the converse of Proposition 3.3 is false. Such as the Klein 4-group $K_2 \times K_2$ or the dihedral group D_8, they are non-cyclic. However, $\Gamma_{Z_2 \times Z_2}$ and Γ_{D_8} have 3 end-vertices and 7 end-vertices, respectively. More specifically that every $p-$group of non-cyclic is a counter-example.*

Theorem 3.5. *Let G be a group with order greater than 2. Then Γ_G contains precisely two end-vertices if and only if G is isomorphic to Z_3 or Z_6, or a non-cyclic group G satisfying the following conditions:*

(1) $\pi(G) = \{2, 3\}$;

(2) G contains two elements x and y, such that $|x| = |y| = 6$ and $y = x^{-1}$;

(3) $|g| < 6$ for every $g \in G$, where $g \neq x, y$.

Proof. Assume that Γ_G contains only two end-vertices x and y. It is obvious that x and y are all non-identity element. Now we claim the following conclusions and prove the necessity of the proposition.

Step 1. $|x| = |y|$.

Assume, to the contrary, that this is not the case. Then we have $(|x|, |y|) \geq 2$ and hence there is at least an element of $\{x, y\}$ such that its order is greater than 3. Without loss of generality, we may suppose that $|x| \geq 4$. Therefore, there exists an element $a(\neq x)$ of $\langle x \rangle$ such that $|a| = |x|$. If $a \neq y$, then, for some $g \in G$, g is adjacent to a in Γ_G. It follows that $(|g|, |a|) = 1$. Hence $(|g|, |x|) = 1$, and so g and x are adjacent in Γ_G, a contradiction. Thus we have $a = y$. This means that $|x| = |y|$, again, a contradiction.

Step 2. $y = x^{-1}$ and $|x| \geq 3$.

If $y \neq x^{-1}$, then we have $|x| = |y| = 2$. Otherwise, x^{-1} is an end-point of Γ_G since $|x^{-1}| = |x|$, it contradicts with the hypothesis. Since x and y are end-vertices of Γ_G, the order of G is a power of 2. Thus we may suppose that $|G| = 2^k$ for some positive integer k. That is, Γ_G is a star by Proposition 2.6. Consequently, $2^k - 1 = 2$, namely $2^k = 3$, this is impossible.

Step 3. $|x| = |y| = 3, 4$ or 6.

Let $|x| = n$. If $\varphi(n) \neq 2$, then $\varphi(n) \geq 3$. Thus, there exist at least 3 distinct elements of order n. A similar argument above means that that is impossible. It follows that $\varphi(n) = 2$. Thereby, $|x| = |y| = n = 3, 4$ or 6.

Step 4. Finishing the proof.

If $|x| = |y| = 3$, then we have G is a $3-$group. Let $|G| = 3^k$ for some positive integer k. Then we have $3^k - 1 = 2$. That is, $|G| = 3$. Thus G is isomorphic to Z_3. If $|x| = |y| = 4$, then G is a $2-$group and $|G| \geq 4$. By Proposition 2.6, the number of end-vertices of Γ_G is greater than 3, a contradiction. Finally, we assume that $|x| = |y| = 6$. Obviously, $\pi(G) = \{2, 3\}$. If G is cyclic, then G is isomorphic to Z_6 by Proposition 3.3. If not, then $|g| < 6$ for every $g \in G$, where $g \neq x, y$.

On the other hand, the sufficiency of the proposition is clear.   □

Remark 3.6. *Let $G = D_{12}$, the dihedral group with order 12. Clearly, G is not cyclic. It is easy to see that G satisfies three conditions of Theorem 3.5. Thus Γ_G contains precisely two end-vertices, moreover, their orders equal 6 in G.*

We end this section with the following question.

Question 3.7. *Is it possible to characterize all finite groups G whose coprime graph contains precisely three end-vertices?*

4. **The automorphism groups of coprime graphs**

Proposition 4.1. *Let G be a group. If G is a p-group for some prime integer p, then $Aut(\Gamma_G)$ is isomorphic to the symmetric group $S_{|G|-1}$.*

Proof. It follows from Proposition 2.6. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Theorem 4.2. *Let G be a group. Then $Aut(\Gamma_G) = Aut(G)$ if and only if G is isomorphic to Z_3 or the Klein group $Z_2 \times Z_2$.*

Proof. Suppose that $Aut(\Gamma_G) = Aut(G)$ for group G. If $|G| \leq 3$, then $G \cong Z_2$ or $G \cong Z_3$. It is well known that $Aut(Z_2) \cong \{e\}$ and $Aut(Z_3) \cong Z_2$. However, $Aut(\Gamma_{Z_2}) \cong Aut(\Gamma_{Z_3}) \cong S_2$. Thereby, by checking, we have G is isomorphic to Z_3. Now we shall claim the following conclusions and prove that G is isomorphic to the Klein group $Z_2 \times Z_2$ if $|G| \geq 4$.

Step 1. G is abelian.

Let ψ be a mapping from Γ_G to itself. We define $u^\psi = u^{-1}$ for all u in $V(\Gamma_G)$. Clearly, ψ is a bijection and $(|x|, |y|) = 1$ if and only if $(|x^{-1}|, |y^{-1}|) = 1$ for all $x, y \in G$. That is, xy is an edge of Γ_G if and only if $x^\psi y^\psi$ is an edge of Γ_G. Thus $\psi \in Aut(\Gamma_G)$. By the hypothesis, $\psi \in Aut(G)$. So $(xy)^\psi = x^\psi y^\psi = (xy)^{-1} = y^{-1}x^{-1} = x^{-1}y^{-1}$, namely, $xy = yx$ and hence G is a abelian group.

Step 2. G is an abelian p-group for some prime integer p.

Let $x(\neq e)$ be an element of G. If there exists another an element $y(\neq e)$ of G such that $(|x|, |y|) = 1$, then $xy \neq x$ and $xy \neq y$. By Step 1, we know that $|xy| = |x||y| > 2$. Let ϕ be an one-to-one mapping from Γ_G to itself, where $(xy)^\phi = (xy)^{-1}, ((xy)^{-1})^\phi = xy$, and $u^\phi = u$ for all u of $V(\Gamma_G)$ such that $u \neq xy$ and $u \neq (xy)^{-1}$. It is easy to see that $\phi \in Aut(\Gamma_G)$. Hence $\phi \in Aut(G)$, that is, $(xy)^\phi = x^\phi y^\phi = xy = (xy)^{-1}$. It means $|xy| = 2$, a contradiction. It follows that every two non-identity elements of Γ_G are non-adjacent. Consequently, Γ_G is a star. In the light of Proposition 2.6, we can see that G is an abelian $p-$group for some prime integer p.

Step 3. G is non-cyclic.

If G is a cyclic group, then we can know that $Aut(G)$ is abelian. By Step 2, G is a $p-$group. It indicates that $Aut(\Gamma_G)$ is isomorphic to $S_{|G|-1}$ by Proposition 4.1. But $S_{|G|-1}$ is non-abelian for $|G| \geq 4$, a contradiction.

Step 4. G is an abelian p-group, where p is even.

By Step 2, G is an abelian p-group for some prime integer p. If p is odd, then there exists an element x in G such that $|x| > 2$. Note that G has at least two subgroups of order p as G is non-cyclic(see Step 3). Then we can take $e \neq y \in G$ such that $y \notin \langle x \rangle$ and $|y|||x|$. Then there is a graph automorphism ξ of Γ_G fixing x and y, and ξ puts xy mapping into its inverse. Since ξ is also an automorphism of group G, the inverse of xy is equal to itself. Since $(xy)^{|x|} = y^{|x|} = e$, we have $|xy| \mid |x|$. Now that $|x|$ is odd, $|xy| > 2$, a contradiction.

Step 5. G is an elementary abelian 2-group.

If not, then there exists $x \in G$ such that $|x| > 2$. By the proof above, we can see that G is an abelian p-group, and p is even. Therefore, we can take $e \neq y \in G$ such that $|y| = 2$. Note that $(xy)^{|y|} = x^{|y|}$, it implies $|xy| = |x| > 2$. This is a contradiction since there is a graph automorphism fixing x and y and putting xy mapping into its inverse. It means $exp(G) = 2$, that is, G is an elementary abelian 2-group.

Step 6. Finishing the proof.

Let $|G| = 2^n$. By Step 5, Γ_G is a star $K_{1,2^n-1}$. Then $Aut(\Gamma_G)$ is the symmetric group S_{2^n-1}. While $Aut(G)$ is the general linear group $GL(n,2)$. It is obvious that they are equal if and only if $n = 2$. That is, G is isomorphic to the Klein group $Z_2 \times Z_2$.

By checking, the converse of the proposition is obvious. □

Theorem 4.3. *Let G be a cyclic group. Then $Aut(\Gamma_G)$ is an elementary abelian 2-group if and only if G is isomorphic to one of the groups Z_2, Z_3 or Z_6.*

Proof. Clearly, $Aut(\Gamma_{Z_2}) \cong S_2$, $Aut(\Gamma_{Z_3}) \cong S_2$ and $Aut(\Gamma_{Z_6}) \cong Z_2 \times Z_2$. Thus $Aut(\Gamma_G)$ is an elementary abelian 2−group for $G \cong Z_2, Z_3$ or Z_6. For the converse, we assume that $Aut(\Gamma_G)$ is an elementary abelian 2−group, where G is a cyclic group. Obviously, we can see that every element of $Aut(\Gamma_G)$ is self-inverse and Γ_G contains at least an end-vertex. Let the number of end-vertices of Γ_G be n. If $n \geq 3$, then it is easy to see that a subgroup of $Aut(\Gamma_G)$ is isomorphic to S_3, a contradiction. Hence $n \leq 2$. By Theorem 3.2 and Theorem 3.5, we have G is isomorphic to one of the groups Z_2, Z_3 or Z_6, as desired. □

Remark 4.4. *If $Aut(\Gamma_G)$ is an elementary abelian 2-group for a group G, then we claim that Γ_G is highly symmetric as the order of every element of $Aut(\Gamma_G)$ equals 2. Theorem 4.3 determines all cyclic groups whose coprime graphs is highly symmetric.*

For a group G, an interesting question is what $Aut(\Gamma_G)$ is isomorphic to G. For instance, if $G = S_3$, then we have $Aut(\Gamma_G) \cong G$. We close this section by the following question.

Question 4.5. *Is it possible to characterize all finite groups G having the property that $Aut(\Gamma_G) \cong G$?*

5. The coprime graphs of the dihedral groups

For $n \geq 3$, the dihedral group $D_{2n} = \langle r, s : s^2 = r^n = 1, s^{-1}rs = r^{-1} \rangle$. As a list,

$$D_{2n} = \{r^1, r^2, \ldots, r^n = e, sr^1, sr^2, \ldots, sr^n\}.$$

Theorem 5.1. *Let $\Gamma_{D_{2n}}$ be the coprime graph of D_{2n} and let n be odd. Then*
(1) $deg_{\Gamma_{D_{2n}}}(sr^i) = n$ *for any* $1 \leq i \leq n$;
(2) $deg_{\Gamma_{D_{2n}}}(r^i) \geq n$ *for any* $1 \leq i \leq n$;
(3) $\Gamma_{D_{2n}}$ *is not Eulerian;*
(4) $\Gamma_{D_{2n}}$ *is Hamiltonian;*
(5) $\Gamma_{D_{2n}}$ *is not planar.*

Proof. (1) Clearly, the order of sr^i equals 2 for any $1 \leq i \leq n$ by the definition of D_{2n}. Since n is odd, we can see that the order of r^j is odd for any $1 \leq j \leq n$. Hence we have that $(|sr^i|, |r^j|) = 1$, that is, sr^i and r^j are connected by an edge. Note that $\{sr^1, sr^2, \ldots, sr^n\}$ is an independence set of $\Gamma_{D_{2n}}$. Thus $deg_{\Gamma_{D_{2n}}}(sr^i) = n$, as required.

(2) It follows from (1).

(3) Since $deg_{\Gamma_{D_{2n}}}(s)$ is an odd integer by (1), we can see that $\Gamma_{D_{2n}}$ is not Eulerian(see [5], Theorem 6.1, p.137).

(4) In view of (1) and (2), we have that $deg_{\Gamma_{D_{2n}}}(x) \geq \frac{2n}{2}$ for every $x \in V(\Gamma_{D_{2n}})$. In the light of Corollary 6.7 of [5] on page 148, $\Gamma_{D_{2n}}$ is Hamiltonian.

(5) If $n = 3$, then $\Gamma_{D_6} \cong K_{1,2,3}$. It is easy to prove that Γ_{D_6} is not planar. Now we assume that $n > 3$. Since $\{s, sr, sr^2\}$ and $\{r, r^2, r^3\}$ are independent and every vertex in $\{s, sr, sr^2\}$ is adjacent to every vertex in $\{r, r^2, r^3\}$, $\Gamma_{D_{2n}}$ contains a subgraph which is isomorphic to $K_{3,3}$. It is well known that $K_{3,3}$ is non-planar. Thereby $\Gamma_{D_{2n}}$ is not planar. $\square$

Corollary 5.2. *Let n be an odd prime. Then $\Gamma_{D_{2n}} \cong K_{1,n-1,n}$.*

Theorem 5.3. *Let $n = 2^k p_1^{r_1} p_2^{r_2} \cdots p_m^{r_m}$, where p_i is prime integer for any $1 \leq i \leq m$, r_i is non-negative integer for any $1 \leq i \leq m$ and k is a positine integer. Then*

(1) The number of end-vertices of $\Gamma_{D_{2n}}$ is $\sum_{d|n} \varphi(d)$, where $2p_1 p_2 \cdots p_m$ is a divisor of d. In particular, $\Gamma_{D_{2n}}$ contains an end-vertices;

(2) $\Gamma_{D_{2n}}$ is not Eulerian;

(3) $\Gamma_{D_{2n}}$ is not Hamiltonian;

(4) $\Gamma_{D_{2n}}$ is not planar.

Proof. (1) Note that n is even and $\langle r \rangle$ is cyclic group of order n. Then e is only one vertex which is adjacent to every element of order d, where $2p_1 p_2 \cdots p_m \mid d$. Clearly, the number of the elements of d is $\sum_{d|n} \varphi(d)$, as required.

(2) It follows from (3) of Theorem 5.1.

(3) By (1), we can see that $\Gamma_{D_{2n}}$ contains an end-vertices. Hence $\Gamma_{D_{2n}}$ has a cut-vertex(In fact e is a cut-vertex of $\Gamma_{D_{2n}}$). That is, $\Gamma_{D_{2n}}$ cannot be Hamiltonian.

(4) It follows from Proposition 2.10. $\square$

Corollary 5.4. *Let $n = 2^k$ for some positive integer k. Then $\Gamma_{D_{2n}}$ is isomorphic to $K_{1,2^{k+1}-1}$.*

Corollary 5.5. *$\Gamma_{D_{2n}}$ is planar if and only if $n = 2^k$ for some positive integer k.*

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