UNITS IN $\mathbb{F}_{2^k} D_{2n}$

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Abstract. Let $\mathbb{F}_q D_{2n}$ be the group algebra of $D_{2n}$, the dihedral group of order $2n$, over $\mathbb{F}_q = GF(q)$. In this paper, we establish the structure of $U(\mathbb{F}_{2^k} D_{2n})$, the unit group of $\mathbb{F}_{2^k} D_{2n}$ and that of its normalized unitary subgroup $V_* (\mathbb{F}_{2^k} D_{2n})$ with respect to canonical involution $*$ when $n$ is odd.

1. Introduction

Let $FG$ be the group algebra of a finite group $G$ over a field $F$ and $U(FG)$ be its unit group. The homomorphism $\varepsilon : FG \to F$ given by $\varepsilon \left( \sum_{g \in G} a_g g \right) = \sum_{g \in G} a_g$ is called the augmentation map of $FG$. The group of normalized units of $FG$, denoted by $V(FG)$ consists of all the units in $FG$ of augmentation $1$. The anti-automorphism $g \mapsto g^{-1}$ of $G$ extends linearly to an anti-automorphism $a \mapsto a^*$ of $FG$; this extension leaves $V(FG)$ setwise invariant and its restriction to $V(FG)$ followed by $v \mapsto v^{-1}$ gives an automorphism of $V(FG)$. The elements of $V(FG)$ fixed by this automorphism are the normalized unitary units of $FG$; they form a subgroup denoted by $V_* (FG)$. If $F$ is a field of characteristic $2$, then

$$V_* (FG) = \left\{ v \in U(FG) \mid v^* = v^{-1} \right\}.$$ 

In [1], A. Bovdi and L. Erdei described the unitary subgroup $V_* (\mathbb{F}_2 G)$ for all groups $G$ of order 8 and 16. The structure of $\mathbb{F}_{2^k} Q_8$ was established by L. Creedon and J. Gildea in [4], where $Q_8$ is the quaternion group of order 8. In [2], V. Bovdi and L. G. Kovács gave conditions on $F$ and $G$ for which $V_* (FG)$ is normal in $V(FG)$. Additionally in [3], V. Bovdi and A. L. Rosa determined the order of $V_* (\mathbb{F}_{2^k} G)$ for special cases of $G$. In [5], K. Kaur and M. Khan described the structure of $U(\mathbb{F}_2 D_{2p})$.
and \( V_*(\mathbb{F}_2 D_{2p}) \) for an odd prime \( p \). In continuation to this investigation, we study the structure of \( \mathcal{U}(\mathbb{F}_2 D_{2n}) \) and \( V_*(\mathbb{F}_2 D_{2n}) \) for odd \( n \).

The following presentation for \( D_{2n} \) shall be used

\[
\langle a, b \mid a^n, b^2, b^{-1}ab = a^{-1} \rangle
\]

2. Notations

We establish the basic notation where \( l \) and \( m \) are coprime integers, \( R \) is a ring, \( K \) is a field extension of \( F \), \( \alpha \in K \) is algebraic over \( F \), \( g \in G \) and \( X \) is any subset of \( G \).

- \( \text{ord}_m(l) \): multiplicative order of \( l \) modulo \( m \)
- \( \text{irr}_F(\alpha) \): minimal polynomial of \( \alpha \) over \( F \)
- \( C_n \): cyclic group of order \( n \)
- \( F^* \): \( F \setminus \{0\} \)
- \( \hat{X} \): \( \sum_{x \in X} x \)
- \( \hat{g} \): \( \langle g \rangle \)
- \( G^m \): external direct product of \( m \) copies of \( G \)
- \( R^m \): external direct sum of \( m \) copies of \( R \)
- \( M(n,F) \): algebra of all \( n \times n \) matrices over the field \( F \)
- \( GL(n,F) \): general linear group of all \( n \times n \) invertible matrices over the field \( F \)

3. Unit Group of \( \mathbb{F}_2 D_{2n} \)

The following lemma is a consequence of some known facts.

**Lemma 3.1.** Let \( (q,m) = 1 \), \( \xi \) be a primitive \( m \)th root of unity over \( \mathbb{F}_q \) and \( d_m = \text{ord}_m(q) \). Then \( \xi \) and \( \xi^{-1} \) are conjugates over \( \mathbb{F}_q \) if and only if \( d_m \) is even and \( q^{d_m/2} \equiv -1 \) mod \( m \). Moreover

\[
\mathbb{F}_q[\xi + \xi^{-1}] \cong \begin{cases} 
\mathbb{F}_{q^{d_m/2}} & \text{if } d_m \text{ is even and } q^{d_m/2} \equiv -1 \text{ mod } m \\
\mathbb{F}_{q^{d_m}} & \text{otherwise}
\end{cases}
\]

**Proof.** From ([6, Theorem 2.21, pp. 53]), it is known that automorphisms of \( \mathbb{F}_q(\xi) \) over \( \mathbb{F}_q \) are determined by their action on \( \xi \) and given by the assignments

\[
\begin{align*}
\xi & \mapsto \xi \\
\xi & \mapsto \xi^q \\
& \vdots \\
\xi & \mapsto \xi^{q^{d_m-1}}
\end{align*}
\]
Thus $\xi$ and $\xi^{-1}$ are conjugates over $\mathbb{F}_q$ if and only if $q^l \equiv -1 \mod m$ for some $l \in \mathbb{N}$. The rest follows.

In what follows, $q = 2^k$.

**Theorem 3.2.** Let $G$ be the dihedral group of order $2n$, $n$ odd. Then

$$\mathcal{U}(\mathbb{F}_qD_{2n}) \cong C_2^k \times C_{q-1} \times \prod_{m|n, m > 1} GL(2, \mathbb{F}_{q^{d_m/2}})^{\varphi(m)}$$

where

$$\varphi_m = \begin{cases} 
    d_m/2 & \text{if } d_m \text{ is even and } q^{d_m/2} \equiv -1 \mod m \\
    d_m & \text{otherwise}
\end{cases}$$

and $d_m = \text{ord}_m(q)$.

**Proof.** Let $\Phi_l(X)$ denote the $l$-th cyclotomic polynomial

$$\Phi_l(X) = \prod_{0 < j \leq l, (j,l) = 1} (X - \xi^j)$$

where $\xi$ is a primitive $l$-th root of unity over $\mathbb{F}_q$.

It is known that

$$\Phi_l(X) = f_{l,1}(X) \cdots f_{l,s_l}(X)$$

where the polynomials $f_{l,i}(X) \in \mathbb{F}_q[X]$ are irreducible over $\mathbb{F}_q \forall i, 1 \leq i \leq s_l = \frac{\varphi(l)}{d_l}$.

Let $m$ be a divisor of $n$, $m > 1$. Also, let

$$B_m = \begin{cases} 
    s_m & \text{if } d_m \text{ is even and } q^{d_m/2} \equiv -1 \mod m \\
    s_m/2 & \text{otherwise}
\end{cases}$$

and $\xi_{m,i}$ be a root of $f_{m,i}$ for each $i, 1 \leq i \leq s_m$.

If $B_m = s_m/2$, then in view of Lemma 3.1, we suppose that

$$\xi_{m,s_m/2+i} = \xi_{m,i}^{-1} \forall i, 1 \leq i \leq B_m$$

Let $E_1 = \mathbb{F}_q$ and $E_m = \bigoplus_{j=1}^{B_m} M \left( 2, \mathbb{F}_q \left[ \xi_{m,j} + \xi_{m,j}^{-1} \right] \right)$.

We now define the following $\mathbb{F}_q$-algebra homomorphisms

(a) $\theta_1 : \mathbb{F}_qD_{2n} \to E_1$ by the assignment $a \mapsto 1, b \mapsto 1$.

(b) $\theta_m : \mathbb{F}_qD_{2n} \to E_m$ given by

$$a \mapsto \left( \begin{array}{cc} 0 & 1 \\ \xi_{m,j} + \xi_{m,j}^{-1} & 1 \end{array} \right)_{j=1}^{B_m} B_m, b \mapsto \left( \begin{array}{cc} 1 & \xi_{m,j} + \xi_{m,j}^{-1} \\ 0 & 1 \end{array} \right)_{j=1}^{B_m} B_m.$$
Let $\theta : \mathbb{F}_q D_{2n} \to \bigoplus_{m|n} E_m$ be defined as

$$\theta = \bigoplus_{m|n} \theta_m$$

We claim that $\ker \theta = \mathbb{F}_q \hat{D}_{2n}$.

Let $A = \sum_{i=0}^{n-1} \alpha_i a^i + \sum_{i=0}^{n-1} \beta_i ba^i \in \ker \theta$ and $F(X) = \sum_{i=0}^{n-1} \alpha_i X^i$, $G(X) = \sum_{i=0}^{n-1} \beta_i X^i \in \mathbb{F}_q [X]$.

Since

$$\begin{bmatrix} 0 & 1 \\ 1 & \xi_{m,j} + \xi_{m,j}^{-1} \end{bmatrix} = Z_{m,j}^{-1} \begin{bmatrix} \xi_{m,j} & 0 \\ 0 & \xi_{m,j}^{-1} \end{bmatrix} Z_{m,j}$$

and

$$\begin{bmatrix} 1 & \xi_{m,j} + \xi_{m,j}^{-1} \\ 0 & 1 \end{bmatrix} = Z_{m,j}^{-1} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} Z_{m,j}$$

where

$$Z_{m,j} = \begin{bmatrix} 1 & \xi_{m,j} \\ 1 & \xi_{m,j}^{-1} \end{bmatrix} \in M(2, \mathbb{F}_q [\xi_{m,j}])$$

therefore

$$\begin{bmatrix} F(\xi_{m,j}) & G(\xi_{m,j}^{-1}) \\ G(\xi_{m,j}) & F(\xi_{m,j}^{-1}) \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

for all $(m, j) \in A$, where $A = \{ (m, j) \mid m \mid n, m > 1 \text{ and } 1 \leq j \leq B_m \}$.

Thus $F(X) = \alpha (1 + X + \cdots + X^{n-1})$ and $G(X) = \beta (1 + X + \cdots + X^{n-1})$ for some $\alpha, \beta \in \mathbb{F}_q$. Since $F(1) + G(1) = 0$, $\alpha = \beta$. As a result $A = \alpha \hat{D}_{2n}$ as claimed.

Also

$$\dim_{\mathbb{F}_q} \bigoplus_{m|n} E_m = 1 + 2 \sum_{m > 1, \ m|n} \varphi(m) = 1 + 2(n - 1) = 2n - 1$$

shows that $\theta$ is onto and hence $J(\mathbb{F}_q D_{2n}) \subseteq \ker \theta$. But $\dim_{\mathbb{F}_q} J(\mathbb{F}_q D_{2n}) \geq 1$. Thus $J(\mathbb{F}_q D_{2n}) = \ker \theta$.

Using Wedderburn Malcev theorem, we have

$$U(\mathbb{F}_q D_{2n}) \cong (1 + J(\mathbb{F}_q D_{2n})) \rtimes \left( \frac{\mathbb{F}_q D_{2n}}{J(\mathbb{F}_q D_{2n})} \right) \cong C_2^k \times C_{q-1} \times \prod_{m|n, m > 1} GL(2, \mathbb{F}_q^m)^{\varphi(m)}$$

$\square$
4. Structure of $V_*(\mathbb{F}_q D_{2n})$

For any $g, h \in G$, 

$$\mu_{g, h} = 1 + (g - 1)h\hat{g}$$

is called a bicyclic unit of $FG$ and 

(a) $\mu^{-1}_{g, h} = 1 - (g - 1)h\hat{g}$

(b) $\mu_{g, h} = 1 \iff h^{-1}gh \in \langle g \rangle$

We denote the group generated by the bicyclic units of $FG$ by $B(FG)$.

It is important to note that if $G = D_{2n}$, then $\mu_{a^i, h} = 1 \forall h \in G$ and $1 \leq i \leq n - 1$. Thus $B(\mathbb{F}_q G)$ is generated by the set 

$$\{ \mu_{a^i, h} | 0 \leq i \leq n - 1, h \in G \}$$

$$= \left\{ 1 + (a^j + a^{-j})(1 + a^ib) | 0 \leq i \leq n - 1, 1 \leq j \leq \frac{n - 1}{2} \right\}$$

In [5], it is shown that if $d = ord_p(2)$, then the structure of $B(\mathbb{F}_2 D_{2p})$ is given by

$$B(\mathbb{F}_2 D_{2p}) \cong \begin{cases} SL(2, \mathbb{F}_{2d/2})^{\frac{\mu-1}{d}} & \text{if } d \text{ is even} \\ SL(2, \mathbb{F}_{2d})^{\frac{\mu-1}{2d}} & \text{if } d \text{ is odd} \end{cases}$$

We shall now alter the proof for the same in [5] to see how the result gets generalized.

Let $B_1$ be the subgroup of $U(\mathbb{F}_q D_{2n})$ generated by the set 

$$\left\{ 1 + \alpha (a^j + a^{-j})(1 + a^ib) | 0 \leq i \leq n - 1, 1 \leq j \leq \frac{n - 1}{2}, \alpha \in \mathbb{F}_q \right\}$$

We shall need the following lemma.

**Lemma 4.1.** Let $L$ be an algebraic extension of $F$ and $\alpha, \beta_1, \cdots, \beta_n \in L$ such that $\alpha$ is not an $F$-conjugate of $\beta_i \forall i, 1 \leq i \leq n$. Then for each $\gamma \in F[\alpha]$, there exists $f_\gamma(x) \in F[x]$ such that 

$$f_\gamma(\alpha) = \gamma,$$

$$f_\gamma(\beta_i) = 0 \forall i, 1 \leq i \leq n$$

**Proof.** Let $g(x) = \prod_{i=1}^{n} P_i(x)$, where $P_i(x) = \text{irr}_F(\beta_i)$.

If $\delta = g(\alpha)$ and $[F[\alpha] : F] = m$, then $\gamma = \delta \sum_{j=0}^{m-1} a_i x^j$ for some $a_i \in F$. The lemma follows if we take 

$$f_\gamma(x) = g(x) \sum_{j=0}^{m-1} a_i x^j \in F[x]$$

**Theorem 4.2.** If $n$ is odd, then 

$$B_1 \cong \prod_{m|n, m > 1} SL(2, \mathbb{F}_{q^m})^{\frac{\nu(m)}{2e_m}}$$
where
\[ e_m = \begin{cases} 
    d_m/2 & \text{if } d_m \text{ is even and } q^{d_m/2} \equiv -1 \mod m \\
    d_m & \text{otherwise}
\end{cases} \]

and \( d_m = \text{ord}_m(q) \).

**Proof.** Let \( \theta' = \left( \sum_{m|n, m>1} \theta_m \right) \big| U(F_q D_{2n}) \) and \( \theta'' = \theta'|_{B_1} \). For any \( X \in B_1 \),
\[
\theta''(X)^2 = \theta''(X^2) = \theta''(1) = (I_{2\times2}, \ldots, I_{2\times2})
\]
Thus \( \theta''(B_1) \subseteq \prod_{m|n, m>1} SL(2, F_{q^{e_m}})_{\frac{e(m)}{2e_m}} \).

Now \( \ker \theta'' \subseteq \ker \theta' = \{ 1 + \alpha \hat{D}_{2n} \mid \alpha \in F_q \} \). Let \( \alpha \in F_q \) such that
\[
1 + \alpha \hat{D}_{2n} \in \ker \theta'' \subseteq B_1
\]
If \( \tau : F_q D_{2n} \to F_q (D_{2n}/\langle a \rangle) \) be the \( F_q \)-algebra homomorphism determined by the map
\[
a \mapsto \bar{1}, \ b \mapsto \bar{b},
\]
then
\[
\tau(X) = \bar{1} \ \forall \ X \in B_1 \\
\Rightarrow \tau(1 + \alpha \hat{D}_{2n}) = \bar{1} \\
\Rightarrow \bar{1} + \alpha(\bar{1} + \bar{b}) = \bar{1} \\
\Rightarrow (1 + \alpha)\bar{1} + \alpha \bar{b} = \bar{1}
\]
which is possible only if \( \alpha = 0 \). Thus \( \ker \theta'' = (1) \).

It remains to show that \( \theta'' : B_1 \to \prod_{(m,j) \in \mathcal{A}} SL(2, F_q[\xi_{m,j} + \xi_{m,j}^{-1}]) \) is onto.

Choose \( \alpha \in F_q^*, \ t \geq 0 \).

Also let \( A^{\alpha,t}_{m,j} = (A^{\alpha,t}_{m,j}) \in \prod_{(m,j) \in \mathcal{A}} SL(2, F_q[\xi_{m,j} + \xi_{m,j}^{-1}]) \) such that
\[
A^{\alpha,t}_{m,j} = \begin{pmatrix} 1 & \alpha(\xi_{m,j} + \xi_{m,j}^{-1})^t \\ 0 & 1 \end{pmatrix}
\]
Then
\[
Z_{m,j}A_{m,j}^{\alpha,t}Z_{m,j}^{-1} = (\xi_{m,j} + \xi_{m,j}^{-1})^{-1} \left( \begin{array}{cc} 1 & \xi_{m,j} \\ 1 & \xi_{m,j}^{-1} \end{array} \right) \left( \begin{array}{cc} 1 & \alpha(\xi_{m,j} + \xi_{m,j}^{-1})^t \\ 0 & 1 \end{array} \right) \left( \begin{array}{cc} \xi_{m,j}^{-1} & \xi_{m,j} \\ 1 & 1 \end{array} \right)
\]
(4.1)

Notice that for any \( u \in \mathbb{N} \),
\[
(\xi_{m,j} + \xi_{m,j}^{-1})^u = \sum_{i=0}^{\lfloor \frac{u-1}{2} \rfloor} c_{u,i} \left( \xi_{m,j}^u - 2^i + \xi_{m,j}^{-u-2i} \right), \text{ where } c_{u,i} = uC_i \mod 2
\]

Thus for any \( t \geq 0 \),
\[
(\xi_{m,j} + \xi_{m,j}^{-1})^{t-1} = c_0 + \sum_i c_i \left( \xi_{m,j}^i + \xi_{m,j}^{-i} \right), \ c_i \in \mathbb{F}_2
\]
Equation (4.1) reduces to
\[
\left( \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right) + \left( d_0 + \sum_i d_i \left( \xi_{m,j}^i + \xi_{m,j}^{-i} \right) \right) \left( \begin{array}{cc} 1 & 1 \\ 1 & 1 \end{array} \right), \ d_i = \alpha c_i
\]

Let
\[
X_{\alpha,t} = 1 + d_0(1+b) + \sum_i d_i \left( a^i + a^{-i} \right) (1+b)
\]
\[
= (1 + d_0(1+b))(1 + \sum_i d_i(a^i + a^{-i})(1+b))
\]

Then \( \theta'(X_{\alpha,t}) = A^{\alpha,t} \).

If
\[
Z_{\alpha,t} = \prod_{i=1}^{n-1} (1 + d_0(a^i + a^{-i})(1+b)) = 1 + d_0(1+b) + d_0D_{2n} \text{ and }
\]
\[
Y_{\alpha,t} = Z_{\alpha,t}(1 + \sum_i d_i(a^i + a^{-i})(1+b)),
\]
then \( \theta'(Y_{\alpha,t}) = \theta'(X_{\alpha,t}) = A^{\alpha,t} \) and \( Y_{\alpha,t} \in B_1 \). Thus \( \theta''(Y_{\alpha,t}) = A^{\alpha,t} \).

By (7, Theorem 3.2.10)
\[
SL(2, F) = \left\langle \left( \begin{array}{cc} 1 & x \\ 0 & 1 \end{array} \right), \left( \begin{array}{cc} 1 & 0 \\ y & 1 \end{array} \right) \mid x, y \in F \right\rangle
\]

In order to show that \( \theta'' \) is onto, we find an inverse image for the generators of
\[
\prod_{(m,j) \in A} SL(2, \mathbb{F}_q[\xi_{m,j} + \xi_{m,j}^{-1}]). \text{ We begin with a generator of the type}
\]
\[
C = \left( I_{2 \times 2}, \cdots , I_{2 \times 2}, \left( \begin{array}{cc} 1 & x_{u,v} \\ 0 & 1 \end{array} \right), I_{2 \times 2}, \cdots , I_{2 \times 2} \right)
\]
where \( x_{u,v} \in \mathbb{F}_q[\xi_{u,v} + \xi_{u,v}^{-1}] \) and the rest will be similar.

By Lemma 4.1, there exists \( P_{u,v}(x) \in \mathbb{F}_q[x] \) such that

1. \( P_{u,v}(\xi_{u,v} + \xi_{u,v}^{-1}) = x_{u,v} \)

2. \( P_{u,v}(\xi_{m,j} + \xi_{m,j}^{-1}) = 0 \) \( \forall \) \((m, j) \neq (u, v)\)

Let \( P_{u,v}(x) = \sum_{i=0}^w \alpha_i^{u,v} x^i \). This gives \( x_{u,v} = \sum_{i=0}^w \alpha_i^{u,v} (\xi_{u,v} + \xi_{u,v}^{-1})^i \). Now consider

\[
\prod_{i=0}^w A^{\alpha_i^{u,v}, i} = \left( \prod_{i=0}^w A^{\alpha_i^{m,j}, i} \right) = (B_{m,j})
\]

where

\[
B_{m,j} = \begin{pmatrix} 1 & P_{u,v}(\xi_{m,j} + \xi_{m,j}^{-1}) \\ 0 & 1 \end{pmatrix}
\]

Thus

\[
\theta'' \left( \prod_{i=1}^w Y_{\alpha_i^{u,v}, i} \right) = C
\]

and we are done. \( \square \)

Having obtained the structure of the subgroup \( B_1 \), it becomes obvious to figure out how the subgroups \( V_*(\mathbb{F}_qD_{2n}) \) and \( B_1 \) are related. The next theorem determines the same.

**Theorem 4.3.** \( V_*(\mathbb{F}_qD_{2n}) \cong B_1 \times (1 + \mathbb{F}_q\overline{D}_{2n}) \)

**Proof.** Since \( q = 2^k \),

\[
GL \left( 2, \mathbb{F}_q[\xi_{m,j} + \xi_{m,j}^{-1}] \right) = SL \left( 2, \mathbb{F}_q[\xi_{m,j} + \xi_{m,j}^{-1}] \right) \times \mathcal{Z} \left( GL \left( 2, \mathbb{F}_q[\xi_{m,j} + \xi_{m,j}^{-1}] \right) \right)
\]

where it is known that

\[
\mathcal{Z} \left( GL \left( 2, \mathbb{F}_q[\xi_{m,j} + \xi_{m,j}^{-1}] \right) \right) = \left\{ \begin{pmatrix} \alpha_{m,j} & 0 \\ 0 & \alpha_{m,j} \end{pmatrix} \middle| \alpha_{m,j} \in \mathbb{F}_q[\xi_{m,j} + \xi_{m,j}^{-1}] \right\}
\]

Hence

\[
\mathcal{U}(\mathbb{F}_qD_{2n}) \cong (1 + \mathbb{F}_q\overline{D}_{2n}) \times \mathbb{F}_q^* \times \prod_{(m,j) \in \mathcal{A}} SL \left( 2, \mathbb{F}_q[\xi_{m,j} + \xi_{m,j}^{-1}] \right) \times \mathcal{Z} \left( GL \left( 2, \mathbb{F}_q[\xi_{m,j} + \xi_{m,j}^{-1}] \right) \right)
\]

Let \( \mathbb{F}_q^* = \langle \zeta \rangle \) and \( \mathbb{F}_q[\xi_{m,j} + \xi_{m,j}^{-1}]^* = \langle \zeta_{m,j} \rangle \).

Before we proceed further, we shall need the following polynomials in \( \mathbb{F}_q[x] \):

1. For each \((m, j) \in \mathcal{A}, Q_{m,j} \in \mathbb{F}_q[x]\) such that

   \[
   Q_{m,j}(\xi_{m,j} + \xi_{m,j}^{-1}) = \zeta_{m,j}
   \]

   \[
   Q_{m,j}(\xi_{u,v} + \xi_{u,v}^{-1}) = 0 \) \( \forall \) \((u, v) \in \mathcal{A}\) such that \((u, v) \neq (m, j)\)

2. \( g(x) = \prod_{(m,j) \in \mathcal{A}} p_{m,j}(x), \) where \( p_{m,j}(x) = \text{irr}_{\mathbb{F}_q}(\xi_{m,j} + \xi_{m,j}^{-1}) \)
3. \( g'(x) = g(0)^{-1}g(x) \)

4. \( h_{(m,j)}(x) = \sum_{(u,v)\in A} Q_{u,v}(x)^q u - 1 + Q_{m,j}(x) \)

5. \( H_{(m,j)}(x) = h_{(m,j)}(x) + (1 + h_{(m,j)}(0))g'(x) \)

6. \( h'(x) = \sum_{(u,v)\in A} Q_{u,v}(x)^q u - 1 \)

7. \( H'(x) = h'(x) + (\zeta + h'(0))g'(x) \)

Now for each \((m, j) \in A\), let

\[
R_{m,j} = H_{m,j}(a + a^{-1}) \\
S = H'(a + a^{-1})
\]

Then

\[
\theta(R_{m,j}) = \theta \left( H_{m,j}(a + a^{-1}) \right) = H_{m,j} \left( \theta(a + a^{-1}) \right) = (1, U^{m,j}), \quad \text{where}
\]

\[
U^{m,j}_{u,v} = \begin{cases} \left( \frac{\zeta_{m,j}}{0} \right) & \text{if } (u, v) = (m, j) \\ I_{2 \times 2} & \text{otherwise} \end{cases}
\]

\[
\theta(S) = H' \left( \theta(a + a^{-1}) \right) = (\zeta, I_{2 \times 2}, \cdots, I_{2 \times 2})
\]

Therefore \( R_{m,j}, S \in \mathcal{U}(\mathbb{F}_q D_{2n}) \) and hence \( \theta'(R_{m,j}) = \theta(R_{m,j}) \) and \( \theta'(S) = \theta(S) \).

Observe that \( Z(\mathcal{U}(\mathbb{F}_q D_{2n})) \cong (1 + \mathbb{F}_q \hat{D}_{2n}) \times \mathbb{F}_q^* \times \prod_{m \mid n, m > 1} (\mathbb{F}_q^{e_m})^{\frac{\varphi(m)}{2e_m}} \).

As a result, the following hold true.

1. The only elements of order 2 in \( Z(\mathcal{U}(\mathbb{F}_q D_{2n})) \) are of the type \( 1 + \alpha \hat{D}_{2n}, \alpha \in \mathbb{F}_q \).

2. If \( G_1 = \langle S, R_{m,j} | (m, j) \in A \rangle \), then \( G_1 \leq Z(\mathcal{U}(\mathbb{F}_q D_{2n})) \) and \( |G_1| \) is odd

3. \( G_1 \cap V_*(\mathbb{F}_q D_{2n}) = (1) \) and hence \( G_1 \cap \left( B_1 \times (1 + \mathbb{F}_q \hat{D}_{2n}) \right) = (1) \)

4. \( |G_1| \geq |\theta'(G_1)| = (q - 1) \times \prod_{m \mid n, m > 1} \left( q^{e_m} - 1 \right)^{\frac{\varphi(m)}{2e_m}} \)

5. \( |\mathcal{U}(\mathbb{F}_q D_{2n})| \geq |G_1 \times B_1 \times (1 + \mathbb{F}_q \hat{D}_{2n})| \geq q(q - 1) \times \prod_{m \mid n, m > 1} \left( q^{e_m} - 1 \right) \left| SL(2, \mathbb{F}_q^{e_m}) \right| \frac{\varphi(m)}{2e_m} = q(q - 1) \times \prod_{m \mid n, m > 1} \left| GL(2, \mathbb{F}_q^{e_m}) \right| \frac{\varphi(m)}{2e_m} \)
\[ |\mathcal{U}(\mathbb{F}_q D_{2n})| \]

6. \[ V_*(\mathbb{F}_q D_{2n}) = \mathcal{B}_1 \times (1 + \mathbb{F}_q D_{2n}) \]

7. \[ \mathcal{U}(\mathbb{F}_q D_{2n}) \cong V_*(\mathbb{F}_q D_{2n}) \times \mathbb{F}_q^* \times \prod_{m|n, m>1} (\mathbb{F}_q^*)^{\frac{\varphi(m)}{2e_m}} \]

and hence the proof. \[ \square \]

5. Concluding Remarks

Remark 5.1. Note that for any odd prime \( p \) and \( i \in \mathbb{N} \), \( \text{ord}_{p^{i+1}}(q) = \text{ord}_{p^i}(q) \) or \( p \times \text{ord}_{p^i}(q) \). Hence

\[ \mathcal{U}(\mathbb{F}_q [D_{2p^n}]) \cong C_2^k \times C_{q^{-1}} \times \prod_{1 \leq m \leq n} \text{GL}(2, \mathbb{F}_{q^m})^{\frac{\varphi(p^m)}{2e_m}} \]

where

\[ e_m = \begin{cases} 
\frac{d_m}{2} & \text{if } d \text{ is even} \\
d_m & \text{if } d \text{ is odd}
\end{cases} \]

\( d_m \) being the multiplicative order of \( q \) modulo \( p^m \) \( \forall 1 \leq m \leq n \) and \( d = d_1 \).

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