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ON n-KAPPE GROUPS

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Abstract. Let G be an infinite group and $n \in \{3, 6\} \cup \{2^k | k \in \mathbb{N}\}$. In this paper, we prove that G is an n-Kappe group if and only if for any two infinite subsets X and Y of G, there exist $x \in X$ and $y \in Y$ such that $[x^n, y, y] = 1$.

1. Introduction

Let w be a word in a free group of rank $n > 0$. Let $\mathcal{V} = \mathcal{V}(w)$ be the variety of groups defined by the law $w(x_1, \ldots, x_n) = 1$. Define $\mathcal{V}^* = \mathcal{V}(w^*)$ to be the class of all groups G such that given infinite subsets $X_1, X_2, \ldots, X_n$ of G, there exist $x_i \in X_i$, $1 \leq i \leq n$, satisfying $w(x_1, \ldots, x_n) = 1$. In [7], P. Longobardi *et al.* posed the question of when the equality $\mathcal{F} \cup \mathcal{V} = \mathcal{V}^*$ holds, where $\mathcal{F}$ is the class of all finite groups (see Problem 15.1 in [10]).

The origin of this question is a problem of Erdös (see [8]). There is no example, so far, of an infinite group in $\mathcal{V}^* \setminus \mathcal{V}$. In considering this question, many authors have obtained the equality for certain words (see [1, 2, 3, 4, 8]).

For a group G, we denote the set of right 2-Engel elements of G by $R_2(G) = \{a \in G | [a, x, x] = 1 \text{ for every } x \in G\}$. A well-known result of W. Kappe [5] states that $R_2(G)$ is always a characteristic subgroup of G. Let n be an integer, a group G is called an n-Kappe group if $[y^n, x, x] = 1$ for all x and y in G. Clearly, G is an n-Kappe group if and only if the factor group $G/R_2(G)$ has finite exponent dividing n. In this paper, we answer the above mentioned problem affirmatively for the word $w = [y^n, x, x]$, where $n \in \{3, 6\} \cup \{2^k | k \in \mathbb{N}\}$. More precisely, we shall prove that every infinite $\mathcal{K}_n^*$-group is a $\mathcal{K}_n$-group, where $\mathcal{K}_n = \mathcal{V}(w)$ and $\mathcal{K}_n^* = \mathcal{V}(w^*)$.

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2. **Results**

In this section we present our main result. First, we state the following lemma, which will be used frequently.

Lemma 2.1 ([1]). *Let G be an infinite $\mathcal{V}(w^*)$-group, where w is a word in the free group of rank 2. Let A be an infinite abelian subgroup of G and $y_1, \ldots, y_n \in G$. Then there exists an infinite subset T of the set $B = \{a \in A | w(a, y_i) = w(y_i, a) = 1,\ \forall\, i = 1, \ldots, n\}$ such that $t_1 t_2^{-1} \in B$ for all distinct elements t_1 and t_2 in T. Also, $A \setminus B$ is finite.*

Let G be a group and x an element of G. As usual, $C_G(x)$ denotes the centralizer of x in G.

Lemma 2.2. *Let G be an infinite $\mathcal{K}_n^*$-group and g an element of G. Then $C_G(g^n)$ is infinite.*

Proof. Assume that $C_G(x^n)$ is finite for some $x \in G$. First we construct by induction on m, a sequence $(x_m)_{m \in \mathbb{N}}$ of elements of G satisfying the following properties:

(i) $x_i x_j^{-1} \notin C_G(x^n)$, if $i \neq j$,

(ii) $x_i x_j^{-1} x_k x_h^{-1} \notin C_G(x^n)$, if i, j, h and k are pairwise different.

For the induction step, assume that $x_1, \ldots, x_m$ satisfy (i) and (ii). Let

$$D = \bigcup_{i=1}^{m} C_G(x^n) x_i \cup \bigcup_{i,j,h=1}^{m} C_G(x^n) x_i x_j^{-1} x_h \cup \bigcup_{i,j,h=1}^{m} x_i x_j^{-1} x_h C_G(x^n)^{x_h}.$$

Since D is finite, there exists $x_{m+1} \in G$ which is not in D. It is easy to verify that $x_1, \ldots, x_m, x_{m+1}$ satisfy (i) and (ii).

Now let $\mathbb{N} = A \cup B$ with A and B infinite and disjoint. Then also the sets $\{x^{x_i} : i \in A\}$ and $\{x^{x_j} : j \in B\}$ are infinite. Since $G \in \mathcal{K}_n^*$, there exist $s \in A$ and $t \in B$ such that $[(x^{x_s})^n, x^{x_t}, x^{x_t}] = 1$. Thus $[(x^n)^{x_s x_t^{-1}}, x, x] = 1$, it follows that $x^{(x^n)^{x_s x_t^{-1}}} \in C_G(x)$. Let $A_1 = A \setminus \{s\}$ and $B_1 = B \setminus \{t\}$. Then, arguing as before, we find $v \in A_1$ and $w \in B_1$ such that $x^{(x^n)^{x_v x_w^{-1}}} \in C_G(x)$. Hence, by induction, we have infinite subsets $I \subseteq A$ and $J \subseteq B$ such that $x^{(x^n)^{x_s x_t^{-1}}} \in C_G(x)$, for an infinite number of pairs (s, t) (with all s's different and all t's different). But $C_G(x)$ is finite, so there exist $i_0 \in I$ and $j_0 \in J$ such that, for infinitely many $i \in I$ and $j \in J$ we have $x^{(x^n)^{x_{i_0} x_{j_0}^{-1}}} = x^{(x^n)^{x_i x_j^{-1}}}$. From this we conclude that $(x^n)^{x_{i_0} x_{j_0}^{-1}} ((x^n)^{x_i x_j^{-1}})^{-1} \in C_G(x)$, is a finite set, and hence there exist $i_1 \neq i_2 \in I$ and $j_1 \neq j_2 \in J$, such that $(x^n)^{x_{i_0} x_{j_0}^{-1}} ((x^n)^{x_{i_1} x_{j_1}^{-1}})^{-1} = (x^n)^{x_{i_0} x_{j_0}^{-1}} ((x^n)^{x_{i_2} x_{j_2}^{-1}})^{-1}$, and so $(x^n)^{x_{i_1} x_{j_1}^{-1}} = (x^n)^{x_{i_2} x_{j_2}^{-1}}$. Thus we get $x_{i_1} x_{j_1}^{-1} x_{j_2} x_{i_2}^{-1} \in C_G(x^n)$, which contradicts (ii). Therefore, $C_G(x^n)$ is infinite, as required. □

In the next results, n is an element of $\{3, 6\} \cup \{2^k | k \in \mathbb{N}\}$. The following corollary will be used repeatedly later.

Corollary 2.3. *Let G be an infinite group in $\mathcal{K}_n^*$. Then G has an infinite abelian subgroup.*

Proof. We show that in any infinite group $G \in \mathcal{K}_n^*$, there exists an element $x \neq 1$ with $C_G(x)$ infinite. Then the result will follow, arguing as in Corollary 2.5 of [6]. If there exists an element $g \in G$ such that $g^n \neq 1$, then by Lemma 2.2, g^n has the required property. Suppose that G has exponent dividing

n. If $n \in \{3,6\}$ then G is locally finite (see page 425 of [9]) and if $n \in \{2^k | k \in \mathbb{N}\}$ then G is an infinite 2-group, therefore by Corollary 2.5 of [6], G has an infinite abelian subgroup, A say. Now every non-trivial element of A has the required property. □

Let G be a group, g an element of G and X a subset of G. We denote the subsets $\{gx : x \in X\}$, $\{x^g : x \in X\}$ and $\{g^x : x \in X\}$ by gX, X^g and g^X respectively.

Lemma 2.4. *Let G be an infinite group in $\mathcal{K}_n^*$ and x and y elements of G such that $C_G(x)$ is infinite and $C_G(y)$ is finite. Then $[y^n, x, x] = 1$.*

Proof. We may assume that $y^n \neq 1$. Then $C_G(y^n)$ is infinite by Lemma 2.2, then there exists an infinite abelian subgroup $A_{y^n} \leq C_G(y^n)$ by Corollary 2.3. By applying Lemma 2.1, there exists an infinite subset $T_{y^n} \subseteq B_{y^n} \subseteq A_{y^n}$, where $B_{y^n} = \{a \in A_{y^n} | [a^n, x, x] = [x^n, a, a] = 1\}$ such that $t_1 t_2^{-1} \in B_{y^n}$ for all distinct elements $t_1, t_2 \in T_{y^n}$. Now we claim that $y^{T_{y^n}}$ is an infinite subset of G. If $y^{T_{y^n}}$ is finite then there exists an element $t_0 \in T_{y^n}$ and an infinite subset T_0 of T_{y^n} such that $y^{t_0} = y^t$, for all $t \in T_0$. Thus $C_G(y)$ is infinite, that gives a contradiction. Therefore $y^{T_{y^n}}$ is infinite. Now we consider two cases for $x^{T_{y^n}}$. In the first case, we assume that $x^{T_{y^n}}$ is finite. Then $C_G(x) \cap C_G(y^n)$ is infinite. Hence there exists an infinite abelian subgroup A of $C_G(x) \cap C_G(y^n)$. It is clear that y^A and xA are infinite. Thus there are t_1 and t_2 in A such that $[(y^n)^{t_1}, x^{t_2}, x^{t_2}] = [y^n, x, x] = 1$.

In the second case, we suppose that $x^{T_{y^n}}$ is infinite. Then there are elements t_1 and t_2 of T_{y^n} such that $1 = [(y^n)^{t_1}, x^{t_2}, x^{t_2}] = [(y^n)^{t_1 t_2^{-1}}, x, x]^{t_2} = [y^n, x, x]$. This completes the proof. □

Lemma 2.5. *Let G be an infinite group in $\mathcal{K}_n^*$ and $x, y \in G$ such that $C_G(x) \cap C_G(y)$ is infinite. Then $[y^n, x, x] = 1$.*

Proof. Let A be an infinite abelian subgroup of $C_G(x) \cap C_G(y)$, by Corollary 2.3. It is clear that Ax and Ay are infinite subsets of G. Thus, there exist elements a and b of A such that $[(by)^n, ax, ax] = 1$. Then $[b^n y^n, ax, ax] = 1$ and therefore $[y^n, x, x] = 1$. □

Lemma 2.6. *Let G be an infinite group in $\mathcal{K}_n^*$ and y an element of G such that $C_G(y)$ is infinite. Then $[y^n, x, x] = 1$, for all elements x of G.*

Proof. Let x be an arbitrary element of G. Since $C_G(y)$ is infinite, there exists an infinite abelian subgroup $A \leq C_G(y)$ which contains y, by Corollary 2.3. We break up the proof into two cases:

Case (i): Let $C_G(x)$ be finite. Then we conclude from Lemma 2.1 that A has a cofinite subset $B = \{a \in A : [a^n, x, x] = [x^n, a, a] = 1\}$. Now if $A_0 = \{a \in A : a^n = 1\}$ is infinite, then yA_0 and x^{A_0} are infinite. By the property $\mathcal{K}_n^*$, there exist a and b of A_0 such that $[(yb)^n, x^a, x^a] = [y^n, x^a, x^a] = 1$ and so $[y^n, x, x] = 1$. Now suppose that A_0 is finite. Since $x^{b^n} \in C_G(x)$ for all elements b of B, there exists an infinite subset X of B and an element a_0 of B such that $x^{a_0^n} = x^{a^n}$ for all elements a of X. This implies that $a^n a_0^{-n} \in C_G(x)$ for all elements a of X. Consequently, there exists an infinite subset $X_0 \subseteq X$ and an element $a_1 \in X$ such that $a^n = a_1^n$ for all elements a of X_0. It follows that $a a_1^{-1} \in A_0$ for all elements a of X_0, which contradicts the assumption that A_0 is finite.

Case (ii): Let $C_G(x)$ be infinite. Then we may assume that $C_G(x)\cap C_G(y)$ is finite, by Lemma 2.5. Let $B_0=\{a\in A:[a^n,x,x]=1=[a^n,yx,yx]\}$, which is cofinite by Lemma 2.1. It is easy to check that $B_0=\{a\in A:[a^n,x,x]=1=[a^n,x,y]\}$. By applying the property $\mathcal{K}_n^*$ for infinite subsets yB_0 and x^{B_0}, we get $[(b_1y)^n,x^{b_2},x^{b_2}]=1$ for some elements b_1 and b_2 of B_0. Then $[b_1^ny^n,x,x]=1$ and so $[[b_1^n,x][y^n,x],x]=1$. Now we have $[y^n,x,x]=1$, which completes the proof. □

Main Theorem. *Every infinite $\mathcal{K}_n^*$-group is a $\mathcal{K}_n$-group.*

Proof. Let G be an infinite $\mathcal{K}_n^*$-group and x and y two elements of G such that $y^n\neq 1$. By Lemmas [2.4, 2.5 and 2.6], we may assume that $C_G(y)$ and $C_G(x)$ are finite. By Corollary 2.3, there exists an infinite abelian subgroup A of $C_G(y^n)$. Since $C_G(y)$ and $C_G(x)$ are finite, y^A and x^A are infinite subsets of G. Thus there exist elements a and b in A such that $[(y^a)^n,x^b,x^b]=1$. It follows that $[y^n,x,x]=1$, therefore G is a $\mathcal{K}_n$-group. □

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