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SPLITTING OF EXTENSIONS IN THE CATEGORY OF LOCALLY COMPACT ABELIAN GROUPS

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Abstract. Let $\pounds$ be the category of all locally compact abelian (LCA) groups. In this paper, the groups G in $\pounds$ are determined such that every extension $0 \to X \to Y \to G \to 0$ with divisible, $\sigma-$compact X in $\pounds$ splits. We also determine the discrete or compactly generated LCA groups H such that every pure extension $0 \to H \to Y \to X \to 0$ splits for each divisible group X in $\pounds$.

1. Introduction

Throughout, all groups are Hausdorff abelian topological groups and will be written additively. Let $\pounds$ denote the category of locally compact abelian (LCA) groups with continuous homomorphisms as morphisms. The Pontrjagin dual of a group $G \in \pounds$ is denoted by $\hat{G}$. A morphism is called proper if it is open onto its image and a short exact sequence $0 \to A \xrightarrow{\phi} B \xrightarrow{\psi} C \to 0$ in $\pounds$ is said to be proper exact if ϕ and ψ are proper morphisms. In this case the sequence is called an extension of A by C (in $\pounds$). Following [7], we let $Ext(C, A)$ denote the (discrete) group of extensions of A by C. A subgroup H of a group C is called pure if $nH = H \cap nC$ for all positive integers n. An extension $0 \to A \xrightarrow{\phi} B \xrightarrow{\psi} C \to 0$ is called a pure extension if $\phi(A)$ is pure in B. The elements represented by pure extensions of A by C form a subgroup of $Ext(C, A)$ which is denoted by $Pext(C, A)$. Assume that $\Im$ is any subcategory of $\pounds$. This paper is part of an investigation which answers the following question:
Under what conditions on $G \in \pounds$, $Pext(X, G) = 0$ or $Pext(G, X) = 0$ for all $X \in \Im$?. In [5], [6] and [13] the question is answered in some subcategories of $\pounds$ such as the category of torsion free locally

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compact abelian groups. In this paper, we answer the above question in the category of divisible, locally compact abelian groups. Let $\emptyset$ be the category of all divisible, locally compact abelian (LCA) groups. We show that a group G in $\pounds$ satisfies $Ext(G, X) = 0$ for all $\sigma-$compact groups X in $\emptyset$ if and only if $G = \mathbb{R}^n \oplus G'$ where n is a non-negative integer and G' contains a compact open subgroup K such that $K \cong \prod_{i\in I}\mathbb{Z}/p_i^{r_i}\mathbb{Z} \times \prod_p \triangle_p{}^{n_p}$ where only finitely many distinct primes p_i and positive integers r_i occur and n_p are cardinal numbers(see Theorem 2.7). We also determine the discrete or compactly generated groups H in $\pounds$ satisfying $Pext(X, H) = 0$ for all X in $\emptyset$ (see Proposition 3.1 and Theorem 3.3).

The additive topological group of real numbers is denoted by $\mathbb{R}$, $\mathbb{Q}$ is the group of rationals, $\mathbb{Z}$ is the group of integers, $\mathbb{Z}(n)$ is the cyclic group of order n. By G_d, we mean the group G with discrete topology, tG is the torsion part of G and G_0 is the identity component of G. The topological isomorphism will be denote by " $\cong$ ". For more on locally compact abelian groups, see [10].

2. Splitting extensions of divisible, locally compact abelian groups

In this section, we determine the structure of a group $G \in \pounds$ with the property that $Ext(G, X) = 0$ for all divisible, $\sigma-$compact groups X in $\pounds$.

Lemma 2.1. *Let G be a divisible, $\sigma-$compact group in $\pounds$ and C a compact torsion group. Then $Ext(C, G) = 0$.*

Proof. By Theorem 25.9 of [10], there exists a positive integer n such that $nC = 0$. Let $f : G \to G$, $f(g) = ng$ for all $g \in G$. Since G is divisible, f is surjective. By [6, Theorem 5.29], f is open. So f is a proper morphism. Consider the exact sequence $0 \to Kerf \to G \xrightarrow{f} G \to 0$. By Corollary 2.10 of [8], we have the exact sequence

$$\to Ext(C, Kerf) \to Ext(C, G) \xrightarrow{f_*} Ext(C, G) \to 0 \tag{2.1}$$

Since $f_*(Ext(C, G)) = nExt(C, G)$, it follows from sequence (2.1) that $nExt(C, G) = Ext(C, G)$. Hence $nExt(C, G) = 0$. So $Ext(C, G) = 0$. □

The statement of Theorem 2.2 can be found in a more general form as Theorem 1 in [6]. However, the suggested proof in [6] appears to be incomplete as it uses the incorrect Proposition 8 of [5]. We proved the Theorem by adding $\sigma-$compactness condition on G.

Theorem 2.2. *Let G be a divisible, $\sigma-$compact group in $\pounds$. Then, for each torsion group T in $\pounds$, $Ext(T, G) = 0$.*

Proof. Let G be a divisible, $\sigma-$compact group and T a torsion group. By Theorem 24.30 of [10], T contains an open compact subgroup C. Now we have the following exact sequence

$$... \to Ext(T/C, G) \to Ext(T, G) \to Ext(C, G) \to 0 \tag{2.2}$$

Since T/C is discrete and G is divisible, then $Ext(T/C, G) = 0$ (see [7], Proposition 2.17). By Lemma 2.1, $Ext(C, G) = 0$. It follows from (2.2) that , $Ext(T, G) = 0$. □

Recall that a discrete group A is said to be cotorsion if for any discrete torsion-free group B, $Ext(B, A) = 0$.

Theorem 2.3. *Let G be a compact group. Then $Ext(G, X) = 0$ for all divisible, $\sigma-$compact groups X in $\pounds$ if and only if $G \cong \prod_{i\in I}\mathbb{Z}/p_i^{r_i}\mathbb{Z}\times\prod_p \triangle_p{}^{n_p}$ ($\triangle_p$ denotes the group of p-adic integers and $\mathbb{Z}/p_i^{r_i}\mathbb{Z}$ is the cyclic group of order $p_i^{r_i}$) where only finitely many distinct primes p_i and positive integers r_i occur and n_p are cardinal numbers.*

Proof. Assume that G is a compact group such that $Ext(G, X) = 0$ for each divisible, $\sigma-$compact group X in $\pounds$. Then $Ext(\hat{\mathbb{Q}}, \hat{G}) \cong Ext(G, \mathbb{Q}) = 0$. So $Ext(\hat{\mathbb{Q}}, \hat{G}/t\hat{G}) = 0$. Consider the following exact sequence

$$0 = Hom(\widehat{\mathbb{Q}/\mathbb{Z}}, \hat{G}/t\hat{G}) \to Ext(\hat{\mathbb{Z}}, \hat{G}/t\hat{G}) \to Ext(\hat{\mathbb{Q}}, \hat{G}/t\hat{G}) = 0$$

Hence $\hat{G}/t\hat{G} \cong Ext(\hat{\mathbb{Z}}, \hat{G}/t\hat{G}) = 0$. Therefore $\hat{G}$ is a discrete, torsion group. Let X be any discrete torsion-free group. We have $Ext(X, \hat{G}) \cong Ext(G, \hat{X}) = 0$. So $\hat{G}$ is a cotorsion group. By [1, Corollary 54.4], a torsion group is cotorsion if and only if it is a direct sum of a bounded group and a divisible group. Hence G is a totally disconnected group which is a direct sum of a compact torsion group and a compact torsion-free group.

Conversely, let $G \cong \prod_{i\in I}\mathbb{Z}/p_i^{r_i}\mathbb{Z} \times \prod_p \triangle_p{}^{n_p}$ and X a divisible, $\sigma-$compact group. Then

$$Ext(G, X) \cong Ext(\prod_{i\in I}\mathbb{Z}/p_i^{r_i}\mathbb{Z}, X) \oplus Ext(\prod_p \triangle_p{}^{n_p}, X)$$

By Lemma 2.1, $Ext(\prod_{i\in I}\mathbb{Z}/p_i^{r_i}\mathbb{Z}, X) = 0$. By [6, Theorem 25.8], $\prod_p \triangle_p{}^{n_p}$ is a compact totally disconnected, torsion-free group. So $\widehat{\prod_p \triangle_p{}^{n_p}}$ is a discrete torsion divisible group. On the other hand, $Ext(\prod_p \triangle_p{}^{n_p}, X) \cong Ext(\hat{X}, \widehat{\prod_p \triangle_p{}^{n_p}})$. By [3, Corollary 10], $Ext(\hat{X}, \widehat{\prod_p \triangle_p{}^{n_p}}) = 0$. Hence, $Ext(\prod_p \triangle_p{}^{n_p}, X) = 0$. Therefore, $Ext(G, X) = 0$. □

Corollary 2.4. *Let C be a connected group in $\pounds$. Then $Ext(C, X) = 0$ for all divisible, $\sigma-$compact groups X in $\pounds$ if and only if C is a vector group (i.e. $C \cong \mathbb{R}^n$ for some nonnegative integer n).*

Proof. If C is a vector group, the result is clear. Suppose $Ext(C, X) = 0$ for all divisible, $\sigma-$compact groups X in $\pounds$. Since C is connected, $C = K \oplus \mathbb{R}^n$ where K is a compact connected group (see [10, Theorem 9.14]). Hence $Ext(K, X) = 0$ for all divisible, $\sigma-$compact groups X. Therefore, by Theorem 2.3, $K = 0$. So $C \cong \mathbb{R}^n$. □

Corollary 2.5. *Let C be a locally connected group. Then $Ext(C, X) = 0$ for all divisible, $\sigma-$compact groups X in $\pounds$ if and only if $C \cong \mathbb{R}^n \oplus E$ where E is a discrete group.*

Proof. Let C be a locally connected group in $\pounds$ and $Ext(C, X) = 0$ for all divisible, $\sigma-$compact groups X. By, p. 19 of [2] and p. 38 of [3], $C \cong \mathbb{R}^n \oplus E \oplus \hat{D}$ where $\mathbb{R}^n$ is a vector group with $n \geq 0$, E a discrete group, and D is a discrete torsion-free abelian group in which every subgroup of finite rank is free. So $Ext(E, X) = 0$, $Ext(\hat{D}, X) = 0$. Since $\hat{D}$ is a compact connected group, by Theorem 2.3, $\hat{D} = 0$. So $G \cong \mathbb{R}^n \oplus E$.

Conversely, if $C \cong \mathbb{R}^n \oplus E$, then the result is clear. □

Corollary 2.6. *Let G be a locally compact abelian group containing a non-trivial compact open subgroup. Then the following are equivalent:*

(1) *every compact open subgroup K of G has the form $K \cong \prod_{i\in I}\mathbb{Z}/p_i^{r_i}\mathbb{Z} \times \prod_p \triangle_p{}^{n_p}$ where only finitely many distinct primes p_i and positive integers r_i occur and n_p are cardinal numbers.*

(2) *some compact open subgroup K of G has the form $K \cong \prod_{i\in I}\mathbb{Z}/p_i^{r_i}\mathbb{Z} \times \prod_p \triangle_p{}^{n_p}$ where only finitely many distinct primes p_i and positive integers r_i occur and n_p are cardinal numbers.*

Proof. (1) $\Rightarrow$ (2): It is clear.

(2) $\Rightarrow$ (1): Let K be a compact open subgroup of G of the form $K \cong \prod_{i\in I}\mathbb{Z}/p_i^{r_i}\mathbb{Z} \times \prod_p \triangle_p{}^{n_p}$. By Theorem 2.3, $Ext(K, X) = 0$ where X is any divisible, $\sigma-$compact group in $\pounds$. Now the exact sequence $0 \to K \to G \to G/K \to 0$ induces the following exact sequence

$$... \to Ext(G/K, X) \to Ext(G, X) \to Ext(K, X) \to 0$$

Since G/K is discrete and X is divisible, then $Ext(G/K, X) = 0$. Hence $Ext(G, X) = 0$. Let L be any compact open subgroup of G. Again the sequence $0 \to L \to G \to G/L \to 0$ induces the exact sequence

$$... \to Ext(G/L, X) \to Ext(G, X) \to Ext(L, X) \to 0$$

Since $Ext(G, X) = 0$, then $Ext(L, X) = 0$. Now by Theorem 2.3, L has the required form. □

Theorem 2.7. *Let $G \in \pounds$. Then $Ext(G, X) = 0$ for all divisible, $\sigma-$compact groups X in $\pounds$ if and only if $G = \mathbb{R}^n \oplus G'$ where n is a non-negative integer and G' contains a compact open subgroup K such that $K \cong \prod_{i\in I}\mathbb{Z}/p_i^{r_i}\mathbb{Z} \times \prod_p \triangle_p{}^{n_p}$ where only finitely many distinct primes p_i and positive integers r_i occur and n_p are cardinal numbers.*

Proof. Let $G \in \pounds$ and $Ext(G, X) = 0$ for all divisible, $\sigma-$compact groups X in $\pounds$. By [10, Theorem 24.30], $G = \mathbb{R}^n \oplus G'$ where G' contains an open compact subgroup K. For a locally compact abelian group X consider the exact sequence

$$... \to Ext(G'/K, X) \to Ext(G', X) \to Ext(K, X) \to 0 \tag{2.3}$$

If X is divisible, then by [7, Proposition 2.17], $Ext(G'/K, X) = 0$. So $Ext(G', X) \cong Ext(K, X)$. By assumption, $Ext(G', X) = 0$. So $Ext(K, X) = 0$. Now by Theorem 2.3, $K \cong \prod_{i\in I}\mathbb{Z}/p_i^{r_i}\mathbb{Z} \times \prod_p \triangle_p{}^{n_p}$.

Conversely, let $G = \mathbb{R}^n \oplus G'$ and G' contains an open compact subgroup of K such that $K \cong \prod_{i\in I}\mathbb{Z}/p_i^{r_i}\mathbb{Z} \times \prod_p \triangle_p{}^{n_p}$. Again, $Ext(G'/K, X) = 0$ where X is divisible. By sequence (2.3), $Ext(G', X) = Ext(K, X)$. Now Theorem 2.3 implies that $Ext(K, X) = 0$. Hence $Ext(G, X) = Ext(G', X) = 0$. □

As an immediate result we have the following:

Corollary 2.8. *Let G be a torsion-free group in $\pounds$. Then $Ext(G, X) = 0$ for all divisible, $\sigma-$compact groups X in $\pounds$ if and only if $G = \mathbb{R}^n \oplus G'$ where G' contains an open compact subgroup K such that $K \cong \prod_p \triangle_p{}^{n_p}$.*

3. Splitting pure extensions by divisible, locally compact abelian groups

In this section, we determine the discrete or compactly generated group G in $\pounds$ such that $Pext(X, G) = 0$ for all divisible groups X in $\pounds$.

Proposition 3.1. *Let A be a discrete group. Then $Pext(X, A) = 0$, for all divisible groups X in $\pounds$ if and only if $A = 0$.*

Proof. It is clear that if $A = 0$ then $Pext(X, A) = 0$, for all divisible groups X in $\pounds$. Suppose that A is a discrete group and $Pext(X, A) = 0$ for each divisible group X in $\pounds$. Consider the exact sequence

$$Hom(\widehat{\mathbb{Q}/\mathbb{Z}}, A/tA) \rightarrow Ext(\hat{\mathbb{Z}}, A/tA) \rightarrow Ext(\hat{\mathbb{Q}}, A/tA)$$

By Corollary 2, p. 377 of [14], $Hom(\widehat{\mathbb{Q}/\mathbb{Z}}, A/tA) \cong Hom(\widehat{A/tA}, \mathbb{Q}/\mathbb{Z})$. Now by [10, Theorem 24.25], since A/tA is discrete torsion-free, then $\widehat{A/tA}$ is connected. On the other hand $\mathbb{Q}/\mathbb{Z}$ is discrete. So $Hom(\widehat{A/tA}, \mathbb{Q}/\mathbb{Z}) = 0$. Hence $Hom(\widehat{\mathbb{Q}/\mathbb{Z}}, A/tA) = 0$. By assumption $Pext(\widehat{\mathbb{Q}}, A) = 0$. Consequently $Pext(\hat{\mathbb{Q}}, A/tA) = 0$. Hence $Ext(\hat{\mathbb{Q}}, A/tA) = Pext(\hat{\mathbb{Q}}, A/tA) = 0$. By [7, Proposition 2.17], A/tA is isomorphic to $Ext(\hat{\mathbb{Z}}, A/tA) = 0$. So $A/tA = 0$ i.e. $A = tA$. Since $Ext(\mathbb{Q}, A) = 0$, so A is a cotorsion group. It follows from [4, Corollary 54.4], that $A = B \oplus D$ where B is a bounded group and D is a divisible group. Clearly, $Ext(C, D) = Pext(C, D) = 0$ for each connected group C. Thus $D = 0$ (see [8, Theorem 3.3]). So $A = B$. Now we show that $B = 0$. If $B \neq 0$ then by [6, Corollary 10], there exists a torsion-free group X such that $Pext(X, B) \neq 0$. Suppose X^* is the minimal divisible extension of X. Since X is torsion-free, so is X^* (see [10]). Hence X^* is a divisible torsion-free group. By assumption $Pext(X^*, B) = 0$. Now we have the following exact sequence:

$$... \rightarrow Ext(X^*/X, B) \rightarrow Ext(X^*, B) \rightarrow Ext(X, B) \rightarrow 0$$

Recall that since X^* and X are torsion-free groups, $Ext(X^*, B) = Pex(X^*, B)$,$Ext(X, B) = Pext(X, B)$. Since $Pext(X, B) \neq 0$, then $Pext(X^*, B) \neq 0$. It is a contradiction. Hence $B = 0$. □

Recall that a locally compact abelian group G will be called an $\pounds - cotorsion$ if and only if $Ext(X, G) = 0$ for each torsion-free group X in $\pounds$ (see [6]).

Theorem 3.2. *Let G be a compact group. Then $Pext(X, G) = 0$ for all divisible groups X in $\pounds$ if and only if $G \cong (\mathbb{R}/\mathbb{Z})^\sigma$ where σ is a cardinal number .*

Proof. Let X be a torsion-free group in $\pounds$ and X^* the minimal divisible extension of X. It follows from [10, A.13] that X^* is torsion-free. Consider the exact sequence

$$.. \rightarrow Ext(X^*/X, G) \rightarrow Ext(X^*, G) \rightarrow Ext(X, G) \rightarrow 0$$

Since $Ext(X^*, G) = Pext(X^*, G) = 0$, so $Ext(X, G) = 0$. It follows that G is an $\pounds - cotorsion$. Since G is compact, by [6, Corollary 9], G is connected. Note that every connected group in $\pounds$ is divisible (see [10, Theorem 24.25]). Suppose C is a connected group in $\pounds$. Then $Ext(C, G) = Pext(C, G) = 0$. Now by [8, Theorem 3.3], $G \cong (\mathbb{R}/\mathbb{Z})^\sigma$.

Conversely, if $G \cong (\mathbb{R}/\mathbb{Z})^\sigma$, then by [14, Theorem 3.2], $Ext(X, G) = 0$ for each $X \in \pounds$. Note that if G is divisible, then $Ext(X, G) = Pex(X, G)$. □

Theorem 3.3. *Let G be a compactly generated group in $\pounds$. Then $Pext(X, G) = 0$, for all divisible groups X in $\pounds$ if and only if $G \cong \mathbb{R}^n \oplus (\mathbb{R}/\mathbb{Z})^\sigma$ where σ is a cardinal number.*

Proof. Let G be a compactly generated group in $\pounds$. By [14, Theorem 2.5], $G \cong \mathbb{R}^n \oplus \mathbb{Z}^m \oplus F$ where F is a compact group and n, m are nonnegative integers. Suppose for all divisible groups X in $\pounds$, $Pext(X, G) = 0$. We have $Pext(X, G) = Pext(X, \mathbb{R}^n) \oplus Pext(X, \mathbb{Z}^m) \oplus Pext(X, F)$.Hence $Pext(X, \mathbb{R}^n) = 0$, $Pext(X, \mathbb{Z}^m) = 0$, $Pet(X, F) = 0$. Since $\mathbb{Z}^m$ is discrete and $Pext(X, \mathbb{Z}^m) = 0$, then $m = 0$ by Proposition 3.1. By Theorem 3.2, $Pext(X, F) = 0$ implies that $F \cong (\mathbb{R}/\mathbb{Z})^\sigma$ where σ is a cardinal number. So $G \cong \mathbb{R}^n \oplus (\mathbb{R}/\mathbb{Z})^\sigma$.

Conversely, Suppose $G \cong \mathbb{R}^n \oplus (\mathbb{R}/\mathbb{Z})^\sigma$. Then $Pext(X, G) = Pext(X, \mathbb{R}^n) \oplus Pext(X, (\mathbb{R}/\mathbb{Z})^\sigma)$, for all divisible groups X in $\pounds$. Now $Pext(X, \mathbb{R}^n) = 0$. By Theorem 3.1, $Pext(X, (\mathbb{R}/\mathbb{Z})^\sigma) = 0$. Hence $Pext(X, G) = 0$. □

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References

[1] D. L. Armacost, *The structure of locally compact abelian groups*, **68**, Marcel Dekker Inc., New York, 1981.
[2] J. Braconnier, Sur les groupes topologiques localement compacts, *J. Math. Pures Appl. (9)*, **27** (1948) 1-85.
[3] J. Dixmier, Quelques proprietes des groupes abeliens localement compacts. *Bull. Sci. Math.*, **81** (1957) 38-48.
[4] L. Fuchs, *Infinite Abelian Groups*, Vol. I, Pure and Applied Mathematics, **36**, Academic Press, New York-London, 1970.
[5] R. O. Fulp, Homological study of purity in locally compact groups, *Proc. London Math. Soc. (3)*, **21** (1970) 502–512.
[6] R. O. Fulp, Splitting locally compact abelian groups, *Michigan Math. J.*, **19**(1972) 47-55.
[7] R. O. Fulp and P. Griffith, Extensions of locally compact abelian groups. I, *Trans. Amer. Math. Soc.*, **154** (1971) 341-356.
[8] R. O. Fulp and P. Griffith, Extensions of locally compact abelian groups II, *Trans. Math. Soc.*, **154** (1971) 357-363.
[9] S. Hartman and A. Hulanicki, Les sous-groupes purs et leurs duals, *Fund. Math.*, **45** (1957) 71-77.
[10] E. Hewitt and K. Ross, *Abstract Harmonic Analysis*, Vol I, Second Edition, Springer Verlag, Berlin-New York, 1979.
[11] P. Loth, Topologically pure extensions. Abelian groups, rings and modules, Contemp. Math., *Amer. Math. Soc.*, Providence, RI, **273** (2001) 191–201.
[12] P. Loth, On pure subgroups of locally compact abelian groups, *Arch. Math. (Basel)*, **81** (2003) 255-257.
[13] P. Loth, Pure extensions of locally compact abelian groups, *Rend. Sem. Mat. Univ. Padova*, **116** (2006) 31-40.
[14] M. Moskowitz, Homological algebra in locally compact abelian groups, *Trans. Amer. Math. Soc.*, **127** (1967) 361-404.
[15] C. Schochet, The UCT, the Minor sequence, and canonical decomposition of Kasparon groups, *K-Theory*, **10** (1996) 49-72.

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