ON THE TOTAL CHARACTER OF FINITE GROUPS

S. K. PRAJAPATI* AND B. SURY

Communicated by Vahid Dabbaghian

Abstract. For a finite group $G$, we study the total character $\tau_G$ afforded by the direct sum of all the non-isomorphic irreducible complex representations of $G$. We resolve for several classes of groups (the Camina $p$-groups, the generalized Camina $p$-groups, the groups which admit $(G, Z(G))$ as a generalized Camina pair), the problem of existence of a polynomial $f(x) \in \mathbb{Q}[x]$ such that $f(\chi) = \tau_G$ for some irreducible character $\chi$ of $G$. As a consequence, we completely determine the $p$-groups of order at most $p^5$ (with $p$ odd) which admit such a polynomial. We deduce the characterization that these are the groups $G$ for which $Z(G)$ is cyclic and $(G, Z(G))$ is a generalized Camina pair and, we conjecture that this holds good for $p$-groups of any order.

1. Introduction

In this paper, $G$ denotes a finite group. Let $\text{Irr}(G)$ and $\text{nl}(G)$ be the set of all irreducible characters of $G$ and the set of all nonlinear irreducible characters of $G$ respectively. Then $\text{lin}(G) = \text{Irr}(G) \setminus \text{nl}(G)$ is the set of linear characters of $G$. Suppose $\rho$ is the direct sum of all the non-isomorphic irreducible complex representations of $G$. The character $\tau_G$ afforded by $\rho$ is called the total character of $G$, that is, $\tau_G = \sum_{\chi \in \text{Irr}(G)} \chi$. Since $\tau_G$ is stable under the action of the Galois group of the splitting field of $G$, $\tau_G(g) \in \mathbb{Z}$ for all $g \in G$.

The dimension $\tau_G(1)$ of $\rho$ seems to have remarkable connections with the geometry of the group. For instance, in the case of the symmetric group $G = S_n$, $\tau_G(1)$ is the number of involutions of $S_n$ ([10]) and, in the case of $G = \text{GL}(n,q)$, $\tau_G(1)$ is the number of symmetric matrices in $\text{GL}(n,q)$ ([5]).


Keywords: Finite groups, Group Characters, Total Characters.

Received: 22 July 2013, Accepted: 29 January 2014.

*Corresponding author.
It is a consequence of a well known theorem due to Burnside and Brauer (\cite{7} Theorem 4.3) that, the total character of the group $G$ is a constituent of $1 + \chi + \cdots + \chi^{m-1}$ if $\chi$ is a faithful character which takes exactly $m$ distinct values on $G$. S. M. Gagola, Jr. & M. L. Lewis classified (in \cite{1}) all the solvable groups for which $\tau_G$ equals $\chi^2$, for some $\chi \in \text{Irr}(G)$. A. Mann also studied the decomposition of $\chi^2$ and proved:

“A nonabelian group $G$ has a faithful irreducible character $\chi$ such that $\text{Irr}(\chi^2) \subseteq \text{lin}(G)$ if and only if $|G'| = 2$ and $Z(G)$ is cyclic”.

Here, $\text{Irr}(\chi^2)$ is the set of all irreducible constituents of $\chi^2$ (\cite{1} Theorem 22.7).

Motivated by this, K. W. Johnson raised the following question:

Does there exist an irreducible character $\chi$ of $G$ and a monic polynomial $f(x) \in \mathbb{Z}[x]$ such that $f(\chi) = \tau_G$? (see \cite{18}).

The aim of the article is to answer a weaker version of this question for several classes of $p$-groups including all $p$-groups of order at the most $p^5$; we examine the existence of a polynomial $f(x) \in \mathbb{Q}[x]$ and $\chi \in \text{Irr}(G)$ such that $f(\chi) = \tau_G$. We call such a polynomial $f(x) \in \mathbb{Q}[x]$, if it exists, a Johnson polynomial of $G$. This problem has been studied for dihedral groups $D_{2n}$ in \cite{18} where it is proved that $D_{2n}$ has a Johnson polynomial if and only if $8 \nmid n$. To describe the classes of groups to which our results apply, we recall some definitions.

A pair $(G, N)$ is said to be a generalized Camina pair (abbreviated GCP) if $N$ is normal in $G$ and, all nonlinear irreducible characters of $G$ vanish outside $N$ (\cite{12}). There are a number of equivalent conditions for $(G, Z(G))$ to be a GCP. An equivalent condition we will refer to is:

A pair $(G, Z(G))$ is a GCP if and only if for all $g \in G \setminus Z(G)$, the conjugacy class of $g$ in $G$ is $gG'$. In this case, one can easily observe that $G' \subseteq Z(G)$ and $\chi(1) = |G/Z(G)|^{1/2}$ for all $\chi \in \text{nl}(G)$. For such types of groups, the first author and R. Sarma investigated (in \cite{19}) the existence of a Johnson polynomial. The following theorem was proved in \cite{19}.

**Theorem 1.1.** (\cite{19} Theorem 3.2) Let $(G, Z(G))$ be a GCP. Then $G$ has a Johnson polynomial if and only if $Z(G)$ is cyclic. In fact, if $Z(G)$ is cyclic then a Johnson polynomial of $G$ is given by

$$f(x) = d^2 \sum_{j=1}^{r} (x/d)^{lj} + d \sum_{j=1}^{m} (x/d)^{lj},$$

where $d = |G/Z(G)|^{1/2}$, $r = |Z(G)/G'|$, $m = |Z(G)|$ and $l = |G'|$. In particular, $f(x) = d^2(x/d)^m + d\sum_{j=1}^{m-1}(x/d)^j$ when $Z(G)$ is cyclic and $Z(G) = G'$.

Further, the above theorem was used by the authors in \cite{19} to classify all the nonabelian $p$-groups of order $p^4$ ($p$ an odd prime) which have a Johnson polynomial. The purpose of this article is to examine the existence of a Johnson polynomial for $p$-groups of order greater than $p^4$. In this direction, we examine the family of Camina $p$-groups and generalized Camina groups. As a consequence, we are able to obtain a complete classification of groups of order $p^5$ which admit a Johnson polynomial.

A pair $(G, N)$ is said to be a Camina pair if $1 < N < G$ is a normal subgroup of $G$ and for every element $g \in G \setminus N$, $gN \subseteq Cl_G(g)$, the conjugacy class of $g$. In the special case $N = G'$, the group $G$
is said to be a Camina group. More generally, a group $G$ is said to be a generalized Camina group if $Cl_G(g) = gG'$ for every element $g \in G \setminus G'Z(G)$. It is known (see [13]) that a nilpotent, generalized Camina group $G$ is isoclinic to Camina group which is a $p$-group; the prime $p$ is said to be associated to $G$.

Then, our main results can be stated as follows:

**Theorem A.** Let $G$ be a Camina $p$-group. Then $G$ has a Johnson polynomial if and only if the nilpotency class of $G$ is 2 and $Z(G)$ is cyclic.

**Theorem B.** Let $(G, Z(G))$ be a Camina pair and let $(G/Z(G), Z(G/Z(G)))$ be a generalized Camina pair. Then $G$ does not possess a Johnson polynomial.

**Theorem C.** Let $G$ be a nilpotent, generalized Camina group with associated prime $p$. Then $G$ has a Johnson polynomial if and only if the nilpotency class of $G$ is 2 and $Z(G)$ is cyclic.

In the last section, we apply the above theorems to obtain the complete list of all groups of order $p^5$ (with $p$ odd) which admit a Johnson polynomial. This is proved using case-by-case considerations (using a description of all groups of order $p^5$ by R. James ([8, Section 4.5])) but, in particular, we deduce the following:

**Theorem D.** Let $G$ be a nonabelian $p$-group of order $p^5$ with $p$ odd. Then $G$ has a Johnson polynomial if $Z(G)$ is cyclic and $(G, Z(G))$ is a GCP.

In view of Theorem 1.1 and the above theorems, it seems reasonable to pose the following conjecture for $p$-groups:

**Conjecture:** A nonabelian $p$-group (with $p$ odd) admits a Johnson polynomial if and only if $Z(G)$ is cyclic and $G' \leq Z(G)$.

2. Notations and Preliminaries

Throughout, $C_n$ denotes the cyclic group of order $n$. Suppose $G$ is a finite group. Then $Z(G)$, $G' = G_2$ and $cd(G)$ denote respectively the center, the commutator subgroup and the set of irreducible character degrees of $G$. If $a, b \in G$, then $b^a = b^{-1}ab$ and $[a, b] = a^{-1}b^{-1}ab$. For $g \in G$, $Cl_G(g)$ denotes its conjugacy class $\{x^{-1}gx : x \in G\}$. The nilpotency class of a nilpotent group $G$ is the number $n$ such that $G_n \neq 1$ and $G_{n+1} = 1$, where $G_2 = [G, G] = G'$ and $G_{i+1} = [G_i, G]$ for $i \geq 2$. Further, if $H$ is a subgroup of $G$ and $\chi$ a character of $G$, $\chi|_H$ denotes the restriction of $\chi$ to $H$. Suppose $N$ is a normal subgroup of $G$. Then we denote by $\text{Irr}(G/N) = \text{Irr}(G) \setminus \text{Irr}(G/N)$.

We start by recalling some basic results that we will need later.

**Lemma 2.1.** [7, Theorem 2.32]

1. If $G$ has a faithful irreducible character, then $Z(G)$ is cyclic.
2. If $G$ is a $p$-group and $Z(G)$ is cyclic, then $G$ has a faithful irreducible character.
Proposition 2.2. An abelian group has a Johnson polynomial if and only if it is cyclic. In fact, if $G$ is a cyclic group of order $n$ then $f(x) = 1 + x + \cdots + x^{n-1}$ is a Johnson polynomial of $G$ and $f(\chi) = \tau_G$ for every faithful irreducible character of $G$.

Proof. Let $f(x)$ be a Johnson polynomial of $G$. Suppose, to the contrary, $G$ is non-cyclic. Then by Lemma 2.1 $\ker(\chi) \neq \{1\}$ for all $\chi \in \text{Irr}(G)$. Since $G$ is an abelian group, $\tau_G$ is the regular character of $G$. Hence

$$\tau_G(g) = \begin{cases} |G| & \text{if } g = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Thus for $g \neq 1 \in \ker(\chi)$, we have $\tau_G(g) = f(\chi(g)) = f(\chi(1)) = \tau_G(1)$, which is a contradiction.

Conversely, let $G = \langle a \rangle$ be the cyclic group of order $n$. Set $\zeta_n = e^{2\pi i / n}$ and $f(x) = \sum_{i=0}^{n-1} x^i$. Consider the linear character $\lambda : G \rightarrow \mathbb{C}^*$ defined by $a \mapsto \zeta_n$. Then $\lambda$ is a faithful irreducible character and $f(\lambda) = \sum_{i=0}^{n-1} \lambda^i = \tau_G$. □

Lemma 2.3. Let $G$ be a non-abelian group. Then $\sum_{\chi \in \text{lin}(G)} \chi(g) = 0$ for each $g \in G \setminus G_2$.

In this article, whenever we prove a certain group $G$ does not possess a Johnson polynomial, we use the following simple observation.

Proposition 2.4. Let $\chi$ be an irreducible character of $G$. If $g_1, g_2 \in G$ are such that $\chi(g_1) = \chi(g_2)$ but $\tau_G(g_1) \neq \tau_G(g_2)$, then there does not exist $f(x) \in \mathbb{C}[x]$ such that $f(\chi) = \tau_G$.

Proposition 2.5. Let $G$ be a non-abelian group. Suppose $f(x) \in \mathbb{Q}[x]$ is a Johnson polynomial of $G$ and $\chi \in \text{Irr}(G)$ is such that $f(\chi) = \tau_G$. Then $\chi$ is a nonlinear faithful character.

Proof. Suppose $f(x) \in \mathbb{Q}[x]$ and $\chi \in \text{lin}(G)$. Since $G$ is non-abelian, $\text{nl}(G)$ is non-empty. Pick $\psi \in \text{nl}(G)$. Then the inner product of $\psi$ with $f(\chi)$ is zero but with $\tau_G$ is 1. Hence $f(\chi) \neq \tau_G$.

Suppose $f(x) \in \mathbb{Q}[x]$ is a Johnson polynomial of $G$ and $\chi \in \text{nl}(G)$ is such that $f(\chi) = \tau_G$ with $\ker(\chi) \neq \{1\}$. Since $\cap_{\chi \in \text{Irr}(G)} \ker(\chi) = \{1\}$, $\tau_G(1) = \tau_G(g)$ for all $g \neq 1 \in G$. Take $g \neq 1 \in \ker(\chi)$. Then $\tau_G(1) = f(\chi(1)) = f(\chi(g)) = \tau_G(g)$, which is a contradiction. □

3. Camina $p$-Groups

In this section, we investigate the existence question of a Johnson polynomial for Camina $p$-groups. A. R. Camina in [2] initiated the study of these groups. We start by recalling the definition.

Definition 3.1. ([2]) Suppose $N$ is a normal subgroup of $G$. A pair $(G, N)$ is a Camina pair if $1 < N < G$ is a normal subgroup of $G$ and for every element $g \in G \setminus N$, $gN \subseteq \text{Cl}_G(g)$.

It is clear that if $(G, N)$ is a Camina pair and if $H$ is normal in $G$ and $H \leq N$ then $(G/H, N/H)$ is also a Camina pair. The following lemma gives a number of equivalent condition for a pair $(G, N)$ to be a Camina pair.

Lemma 3.2. ([17] Lemma 3) Let $N$ be a normal subgroup of $G$ and let $g \in G \setminus N$. Then following are equivalent: 

...
(1) $\chi(g) = 0$ for all $\chi \in \text{Irr}(G|N)$,
(2) $|C_G(g)| = |C_{G/N}(gN)|$,
(3) $gN \subseteq Cl_G(g)$.

It is easy to see that if $(G, N)$ is a Camina pair, then $Z(G) \leq N \leq G'$.

Camina groups have been studied by many authors [3, 15, 16]. By Lemma 3.2, it is clear that if $G$ is Camina group, then $\chi(g) = 0$ for all $\chi \in \text{nl}(G)$ and $g \in G \setminus G'$. In [3], Dark and Scoppola proved:

**Theorem 3.3.** ([3]) If $G$ is a finite Camina $p$-group, then the nilpotency class of $G$ is at most 3, i.e., $G_4 = \{1\}$.

**Lemma 3.4.** [15, Corollary 2.3] Let $G$ be a $p$-group of nilpotency class $r$. If $(G, G_k)$ is a Camina pair, then $G_i/G_{i+1}$ has exponent $p$ for $k - 1 \leq i \leq r$.

**Theorem 3.5.** [15, Theorem 5.2] Let $G$ be a Camina $p$-group of nilpotency class 3 and let $|G/G_2| = p^m$, $|G_2/G_3| = p^n$. Then

(1) $(G, G_3)$ is a Camina pair,
(2) $m = 2n$ and $n$ is even.

**Corollary 3.6.** [15, Corollary 5.3] If $G$ is a Camina $p$-group of nilpotency class 3, then $Z_2(G) = G_2$ and $Z(G) = G_3$.

**Remarks on Camina $p$-groups of class 3.**

Suppose $G$ is a Camina $p$-group of nilpotency class 3. Then by Lemma 3.4, $G/G_2$, $G_2/G_3$, and $G_3$ are elementary abelian $p$-groups and by Corollary 3.6 we have $G_3 = Z(G)$. Now by Theorem 3.5, we have $(G, G_3)$ is a Camina pair, $|G/G_2| = p^{2n}$, $|G_2/G_3| = p^n$ and $|G/G_3| = p^{3n}$ where $n$ is even. We will show that $\text{nl}(G) = \text{Irr}(G|G_3) \sqcup \text{nl}(G/G_3)$ and $cd(G) = \{1, p^n, p^{3n/2}\}$.

Take $\chi \in \text{Irr}(G|G_3)$. Now $\chi|_{G_3} = \chi(1)\lambda$ for some $\lambda \in \text{Irr}(G_3)$. Thus

\[
|G| = \sum_{g \in G} |\chi(g)|^2 = \sum_{g \in G_3} |\chi(g)|^2 \quad \text{(since $(G, G_3)$ is a Camina pair)}
\]
\[
= \sum_{g \in G_3} |\chi(1)\lambda(g)|^2
\]
\[
= |\chi(1)|^2 |G_3|.
\]

Hence $\chi(1)^2 = |G/G_3| = p^{3n}$ for all $\chi \in \text{Irr}(G|G_3)$. Thus we have a bijection

$$
\Phi : \text{Irr}(G_3) \setminus \{1_{G_3}\} \longrightarrow \text{Irr}(G|G_3) \text{ defined by}
$$

\[
\Phi(\lambda)(g) := \begin{cases} 
p^{3n/2}\lambda(g) & \text{if } g \in G_3, \\
0 & \text{otherwise},
\end{cases}
\]

(3.1)
where $1_{G_3}$ is the trivial character of $G_3$. Therefore $|\text{Irr}(G|G_3)| = |G_3| - 1$.

Since $(G, G_2)$ is a Camina pair, $(G/G_3, G_2/G_3)$ is also a Camina pair. By Corollary 3.6, we have $Z(G/G_3) = Z_2(G)/G_3 = G_2/G_3 = [G/G_3, G/G_3]$. Thus $G/G_3$ is a Camina $p$-group of nilpotency class 2. Now take $\chi \in \text{nl}(G/G_3)$. Then $\chi|_{G_2/G_3} = \chi(1)\lambda$ for some $\lambda \in \text{Irr}(G/G_3)$. Now

$$|G/G_3| = \sum_{gG_3 \in G/G_3} |\chi(gG_3)|^2 = \sum_{gG_3 \in G/G_3} |\chi(gG_3)|^2 \quad (\text{since } G/G_3 \text{ is a Camina group})$$

$$= \sum_{g \in G} |\chi(1)\lambda(gG_3)|^2$$

$$= \chi(1)^2|G_2/G_3|,$$

Hence $\chi(1)^2 = |G/G_2| = p^{2n}$ for all $\chi \in \text{nl}(G/G_3)$. Thus we have a bijection

$$\Psi : \text{Irr}(G_2/G_3) \setminus \{1_{G_2/G_3}\} \rightarrow \text{nl}(G/G_3) \text{ such that}$$

$$\Psi(\lambda)(g) := \begin{cases} p^{3n/2}(\lambda \circ \eta)(g) & \text{if } g \in G_2, \\ 0 & \text{otherwise,} \end{cases}$$

(3.2)

where $\eta : G \rightarrow G_3$ is the natural homomorphism and $1_{G_2/G_3}$ is the trivial character of $G_2/G_3$. Therefore we have $|\text{nl}(G/G_3)| = |G_2/G_3| - 1 = p^n - 1$. Now

$$|G| = \sum_{\chi \in \text{lin}(G)} \chi(1)^2 = |G/G_2| + (|G_3| - 1)|G/G_3| + (|G_2/G_3| - 1)|G/G_2|.$$  

This shows that $\text{nl}(G) = \text{Irr}(G|G_3) \sqcup \text{nl}(G/G_3)$ as a disjoint union and $\text{cd}(G) = \{1, p^n, p^{3n/2}\}$.

Now, we can compute the total character of a Camina $p$-group of nilpotency class 3.

**Proposition 3.7.** Let $G$ be a Camina $p$-group of nilpotency class 3. Then the total character $\tau_G$ is given by,

$$\tau_G(g) = \begin{cases} p^{2n} + (|G_3| - 1)p^{3n/2} + (p^n - 1)p^n & \text{if } g = 1, \\ p^{2n} - p^{3n/2} + (p^n - 1)p^n & \text{if } g \in G_3 \setminus \{1\}, \\ p^{2n} - p^n & \text{if } g \in G_2 \setminus G_3, \\ 0 & \text{otherwise}. \end{cases}$$

(3.3)

**Proof.** By Theorem 3.5, we have $|G/G_2| = p^{2n}$, $|G_2/G_3| = p^n$ and $|G/G_3| = p^{3n}$ where $n$ is even. In view of (3.1) and (3.2), we have all the nonlinear irreducible character of $G$. Hence, if $g = 1$, then

$$\tau_G(1) = \sum_{\chi \in \text{lin}(G)} \chi(1) + \sum_{\chi \in \text{nl}(G)} \chi(1)$$

$$= p^{2n} + (|G_3| - 1)p^{3n/2} + (p^n - 1)p^n.$$

If $g \in G \setminus G_2$, then by Lemma 2.3 and (3.1), (3.2), we get

$$\tau_G(g) = \sum_{\chi \in \text{Irr}(G)} \chi(g) = \sum_{\chi \in \text{lin}(G)} \chi(g) = 0.$$
If \( g \neq 1 \in G_3 \), then
\[
\tau_G(g) = \sum_{\chi \in \text{lin}(G)} \chi(g) + \sum_{\chi \in \text{nl}(G)} \chi(g) = |G/G_2| + \sum_{\chi \in \text{Irr}(G|G_3)} \chi(g) + \sum_{\chi \in \text{nl}(G/G_3)} \chi(g) = p^{2n} - p^{3n/2} + (p^n - 1)p^n \quad \text{(by (3.1) and (3.2))}.
\]

Finally, if \( g \in G_2 \setminus G_3 \), then
\[
\tau_G(g) = \sum_{\chi \in \text{lin}(G)} \chi(g) + \sum_{\chi \in \text{nl}(G)} \chi(g) = |G/G_2| + \sum_{\chi \in \text{Irr}(G|G_3)} \chi(g) + \sum_{\chi \in \text{nl}(G/G_3)} \chi(g) = p^{2n} - p^n \quad \text{(by (3.1) and (3.2))}.
\]

This completes the proof. \( \square \)

Now, we are ready to characterize Camina \( p \)-groups which admit a Johnson polynomial (Theorem A).

**Proof of Theorem A.** By Theorem 3.3, the nilpotency class of \( G \) is at most 3, i.e., \( G_4 = 1 \). Suppose \( G \) is of nilpotency class equal to 3. If \( Z(G) \) is not cyclic then by Lemma 2.1, \( G \) has no faithful irreducible character. Therefore, from Proposition 2.5, \( G \) has no Johnson polynomial. Now suppose \( Z(G) \) is cyclic and \( \chi \) is a faithful irreducible character of \( G \). Let \( f(x) \in \mathbb{C}[x] \) with \( f(\chi) = \tau_G \). From (3.1) and (3.2), it is clear that \( \chi \in \text{Irr}(G|G_3) \) and \( \chi(g) = 0 \) for all \( g \in G \setminus G_3 \). Now take \( h \in G_2 \setminus G_3 \). Then from (3.3), we have \( f(\chi(h)) = f(0) = \tau_G(h) = p^{2n} - p^n \). If \( g \in G \setminus G_2 \), then from (3.3), we get \( f(\chi(g)) = f(0) = \tau_G(g) = 0 \). Therefore, we have a contradiction to the existence of a Johnson polynomial.

Next suppose that nilpotency class of \( G \) is 2 i.e., \( 1 < G_2 \leq Z(G) \). Since \( G \) is a Camina group, each nonlinear irreducible character of \( G \) vanishes outside \( G_2 \). Therefore, \( G_2 = Z(G) \). Thus \( (G, Z(G)) \) is a generalized Camina pair and hence from Theorem 1.1, the proof is complete. \( \square \)

### 4. Groups for which \( (G, Z(G)) \) is a Camina pair

In [14], M. L. Lewis began the study of those groups \( G \) for which \( (G, Z(G)) \) is a Camina pair and, proved that such a group \( G \) must be a \( p \)-group for some prime \( p \). The next lemma ([15] Lemma 2.1) was proved by Macdonald in a more general setting where \( G \) is a \( p \)-group with \( (G, N) \) as a Camina pair. In the case \( N = Z(G) \), this reduces to the following.

**Lemma 4.1.** ([15]) Let \( G \) be a \( p \)-group of nilpotency class \( r \) and let \( (G, Z(G)) \) be a Camina pair. Then \( Z(G) = G_r \).

**Remarks on \( \text{Irr}(G|Z(G)) \) when \( (G, Z(G)) \) is a Camina pair.**
Suppose \((G, Z(G))\) is a Camina pair. Then by Lemma 3.2, \(\chi(g) = 0\) for all \(\chi \in \text{Irr}(G/Z(G))\) and for all \(g \in G \setminus Z(G)\). Let \(1_{Z(G)}\) be the trivial character of \(Z(G)\). Now take any \(\chi \in \text{Irr}(G/Z(G))\). Then,

\[
|G| = \sum_{g \in G} |\chi(g)|^2 = \sum_{g \in Z(G)} |\chi(1)\lambda(g)|^2,
\]

where \(\lambda \in \text{Irr}(Z(G)) \setminus \{1_{Z(G)}\}\). Therefore, \(\chi(1)^2 = |G/Z(G)|\). Hence we have a bijection

\[
\Phi : \text{Irr}(Z(G)) \setminus \{1_{Z(G)}\} \rightarrow \text{Irr}(G/Z(G)) \text{ such that}
\]

\[
(4.1) \quad \Phi(\lambda)(g) := \begin{cases} |G/Z(G)|^{1/2} \lambda(g) & \text{if } g \in Z(G) \\ 0 & \text{otherwise.} \end{cases}
\]

**Proposition 4.2.** Let \((G, Z(G))\) be a Camina pair and let \((G/Z(G), Z(G/Z(G)))\) be a generalized Camina pair. Then the total character \(\tau_G\) is given by the following expressions:

\[
\tau_G(1) = |G/G_2| + (|Z(G)| - 1)|G/Z(G)|^{1/2} + m|G/Z_2(G)|^{1/2}, \quad \text{where } m = |Z(G/Z(G))| - |Z_2(G)/G_2|;
\]

\[
\tau_G(g) = |G/G_2| - |G/Z(G)|^{1/2} + (|Z(G/Z(G))| - |Z_2(G)/G_2|)|G/Z_2(G)|^{1/2} \quad \text{when } 1 \neq g \in Z(G);
\]

\[
\tau_G(g) = |G/G_2| - |Z_2(G)/G_2||G/Z_2(G)|^{1/2} \quad \text{if } g \in G_2 \setminus Z(G);
\]

\[
\tau_G(g) = 0 \quad \text{if } g \in G \setminus G_2.
\]

**Proof.** Since \((G, Z(G))\) is Camina pair, \(\text{Irr}(G/Z(G))\) is given by (4.1). Therefore, there are \(|Z(G)| - 1\) nonlinear irreducible characters of degree \(|G/Z(G)|^{1/2}\). It is given that \((G/Z(G), Z(G/Z(G)))\) is a generalized Camina pair. So,

\[
|G/Z(G), G/Z(G)| = G_2Z(G)/Z(G) \subseteq Z(G/Z(G)) = Z_2(G)/Z(G).
\]

Since \((G, Z(G))\) is a Camina pair, \(Z(G) \subseteq G_2\). Hence \(G_2Z(G)/Z(G) = G_2/Z(G)\). There is a bijection

\[
(4.2) \quad \Psi(\lambda)(g) := \begin{cases} |G/Z_2(G)|^{1/2} \chi(g) & \text{if } g \in Z(G) \\ 0 & \text{otherwise} \end{cases}
\]

(see [19 Theorem 3.1]). Thus \(G\) has \(|Z(G/Z(G))| - |Z_2(G)/G_2|\) nonlinear irreducible characters with \(Z(G)\) in their kernels and, degree of each such character is \(|G/Z_2(G)|^{1/2}\). Now

\[
\sum_{\chi \in \text{Irr}(G/Z(G))} \chi(1)^2 + \sum_{\chi \in \text{Irr}(G/Z(G))} \chi(1)^2 + \sum_{\chi \in \text{Irr}(G/Z(G))} \chi(1)^2
\]

\[
= |G/G_2| + (|Z(G)| - 1)|G/Z(G)| + (|Z(G/Z(G))| - |Z_2(G)/G_2||G/Z_2(G)| = |G|.
\]

This shows that \(\text{nl}(G) = \text{Irr}(G/Z(G)) \cup \text{nl}(G/Z(G))\).

Since \((G/Z(G), Z(G/Z(G)))\) is a generalized Camina pair, use [19 Proposition 3.1] to get,

\[
(4.3) \quad \tau_{G/Z(G)}(g) = \begin{cases} |G/G_2| + m|G/Z_2(G)|^{1/2} & \text{if } g \in Z(G) \\ |G/G_2| - |Z_2(G)/G_2||G/Z_2(G)|^{1/2} & \text{if } g \in G_2 \setminus Z(G) \\ 0 & \text{otherwise,} \end{cases}
\]
where \( m = |Z(G/Z(G))| - |Z_2(G)/G_2| \). We use \( \tau_{G/Z(G)} \) to calculate \( \tau_G \).

Next, if \( g = 1 \), then
\[
\tau_G(1) = \sum_{\chi \in \text{lin}(G)} \chi(1) + \sum_{\chi \in \text{Irr}(G/Z(G))} \chi(g) + \sum_{\chi \in \text{nl}(G/Z(G))} \chi(g)
\]
(4.4)
\[
= |G/G_2| + (|Z(G)| - 1)|G/Z(G)|^{1/2} + m|G/Z_2(G)|^{1/2},
\]
where \( m = |Z(G/Z(G))| - |Z_2(G)/G_2| \).

If \( g \neq 1 \in Z(G) \), then by (4.1) and (4.2) we have
\[
\tau_G(g) = \sum_{\chi \in \text{lin}(G)} \chi(g) + \sum_{\chi \in \text{Irr}(G/Z(G))} \chi(g) + \sum_{\chi \in \text{nl}(G/Z(G))} \chi(g)
\]
(4.5)
\[
= |G/G_2| - |G/Z(G)|^{1/2} + (|Z(G)/Z(G)| - |Z_2(G)/G_2|)|G/Z_2(G)|^{1/2}.
\]

If \( g \in G_2 \setminus Z(G) \), then then by (4.1), (4.2) and (4.3), we have
\[
\tau_G(g) = \sum_{\chi \in \text{lin}(G)} \chi(g) + \sum_{\chi \in \text{Irr}(G/Z(G))} \chi(g) + \sum_{\chi \in \text{nl}(G/Z(G))} \chi(g)
\]
(4.6)
\[
= |G/G_2| + 0 - |Z_2(G)/G_2||G/Z_2(G)|^{1/2}.
\]

If \( g \in G \setminus G_2 \), then then by (4.1), (4.2) and (4.3), one can easily get that
\[
\tau_G(g) = \sum_{\chi \in \text{lin}(G)} \chi(g) + \sum_{\chi \in \text{Irr}(G/Z(G))} \chi(g) + \sum_{\chi \in \text{nl}(G/Z(G))} \chi(g)
\]
(4.7)
\[
= 0.
\]

This completes the proof. □

**Proof of Theorem B.** In view of Proposition 2.4 and 4.2, \( G \) has no Johnson polynomial. □

5. **Generalized Camina groups**

In this section, we study the total character for a generalized Camina group and characterize those groups which admit a Johnson polynomial. We begin by recalling the important notion of isoclinism introduced by Philip Hall.

**Definition 5.1.** Let \( G, H \) be finite groups. \( G \) and \( H \) are said to be isoclinic if there exist isomorphisms \( \theta : G/Z(G) \rightarrow H/Z(H) \) and \( \phi : G_2 \rightarrow H_2 \) such that
\[
[\theta(g_1Z(G)), \theta(g_2Z(G))] = \phi([g_1Z(G), g_2Z(G)]) \text{ for all } g_1, g_2 \in G.
\]

The notion of isoclinism was first introduced by P. Hall [6] who proved that two isoclinic nilpotent groups have the same nilpotency class. It is also known that isoclinic groups of the same order have the same character degrees. Recall:

**Definition 5.2.** [13] A group \( G \) is said to be a Generalized Camina group if \( Cl_G(g) = gG_2 \) for every \( g \in G \setminus G_2Z(G) \).
This generalization of a Camina group was introduced by M. L. Lewis in [13]. It is clear from the
definition that if $G$ is a generalized Camina group, then either $G$ has nilpotence class 2 or $G/Z(G)$ is
a Camina group. The author proved that $G$ is a nilpotent generalized Camina group if and only if $G$
is isoclinic to a nilpotent Camina group $H$ and $H$ must be $p$-group ([13]). Lewis also pointed out that
a Camina group which is isoclinic to $G$ will be a $p$-group for the same prime $p$; one calls $p$, the prime
associated to $G$.

**Definition 5.3.** Let $N$ be a normal subgroup of $G$ and let $\chi \in \text{Irr}(G)$. We say that $\chi$
is fully ramified with respect to $G/N$ if $\chi \downarrow_N = e\theta$ and $\theta \uparrow^G = e\chi$ for some $\theta \in \text{Irr}(N)$ and some integer $e$.

In [13], Lewis proved the following theorem:

**Theorem 5.4.** [13, Theorem 3] Let $G$ be a nilpotent, generalized Camina group of nilpotency class 3.
Then following are true:

1. $G/G_2Z(G)$, $G_2Z(G)/Z(G)$, and $G_3 = G_2 \cap Z(G)$ are elementary abelian $p$-groups for some
prime $p$.
2. $|G/G_2Z(G)| = p^{2n}$ and $|G_2Z(G)/Z(G)| = |G_2/G_3| = p^n$ for some even integer $n$.
3. $cd(G) = \{1, p^n, p^{3n/2}\}$.
4. $Z(G/G_3) = G_2Z(G)/G_3$ and $G_2Z(G)/G_3 = G_2/G_3 \times Z(G)/G_3$.
5. Every character in $\text{nl}(G/G_3)$ is fully ramified with respect to $G/G_2Z(G)$ and every character
in $\text{Irr}(G|G_3)$ is fully ramified with respect to $G/Z(G)$.

**Remarks on Generalized Camina groups of nilpotency class 3.**

Suppose $G$ is a nilpotent, generalized Camina group of nilpotency class 3. Then from the above
theorem, we have $|G/Z(G)| = p^{3n}$ and one can observe that there are two bijections namely,

$$
\Phi_1 : \text{Irr}(Z(G)|G_3) \longrightarrow \text{Irr}(G|G_3) \quad \text{such that}
$$

$$
\Phi_1(\lambda)(g) := \begin{cases} 
p^{3n/2}\lambda(g) & \text{if } g \in Z(G), \\
0 & \text{otherwise},
\end{cases}
$$

(5.1)

and

$$
\Psi_1 : \text{Irr}(G_2Z(G)/G_3|G_2/G_3) \longrightarrow \text{nl}(G/G_3) \quad \text{such that}
$$

$$
\Psi_1(\lambda)(g) := \begin{cases} 
p^n(\lambda \circ \eta)(g) & \text{if } g \in G_2Z(G), \\
0 & \text{otherwise},
\end{cases}
$$

(5.2)

where $\eta : G \longrightarrow G/G_3$ is the natural homomorphism. Therefore $G$ has $|Z(G)| - |Z(G)/G_3|$ nonlinear
irreducible characters of degree $p^{3n/2}$ and $(|G_2/G_3| - 1)|Z(G)/G_3|$ nonlinear irreducible characters of
degree $p^n$, and $\text{nl}(G) = \text{Irr}(G|G_3) \sqcup \text{nl}(G/G_3)$. 
Lemma 5.5. Let $G$ be a generalized Camina group of nilpotency class 3. Then

\[ (5.3) \quad \sum_{\lambda \in \text{Irr}(G)} \lambda(g) = \begin{cases} -|Z(G)/G_3| & \text{if } g \in G_3, \\ 0 & \text{if } g \in Z(G) \setminus G_3 \end{cases} \]

and

\[ (5.4) \quad \sum_{\lambda \in \text{Irr}(G_2Z(G)/G_3|G_2/G_3)} \lambda(g) = \begin{cases} (p^n - 1)|Z(G)/G_3| & \text{if } g \in G_3, \\ -|Z(G)/G_3| & \text{if } g \in G_2 \setminus Z(G), \\ 0 & \text{otherwise}, \end{cases} \]

where $|G_2/G_3| = p^n$.

Proposition 5.6. Let $G$ be a generalized Camina group of nilpotency class 3. Then the total character $\tau_G$ is given by,

\[ (5.5) \quad \tau_G(g) = \begin{cases} |G/G_2| + rp^{3n/2} + (p^n - 1)|Z(G)/G_3|p^n & \text{if } g = 1, \\ |G/G_2| - |Z(G)/G_3|p^{3n/2} + (p^n - 1)|Z(G)/G_3|p^n & \text{if } g \neq 1 \in G_3, \\ |G/G_2| - |Z(G)/G_3|p^n & \text{if } g \in G_2 \setminus Z(G), \\ 0 & \text{otherwise}, \end{cases} \]

where $r = |Z(G)| - |Z(G)/G_3|$.

Proof. If $g = 1$, then

\[
\tau_G(1) = \sum_{\chi \in \text{lin}(G)} \chi(1) + \sum_{\chi \in \text{nil}(G)} \chi(1) = |G/G_2| + (|Z(G)| - |Z(G)/G_3|)p^{3n/2} + (|G_2/G_3| - 1)|Z(G)/G_3|p^n = |G/G_2| + (|Z(G)| - |Z(G)/G_3|)p^{3n/2} + (p^n - 1)|Z(G)/G_3|p^n.
\]

If $g \neq 1 \in G_3$, then

\[
\tau_G(g) = \sum_{\chi \in \text{lin}(G)} \chi(g) + \sum_{\chi \in \text{nil}(G)} \chi(g) = |G/G_2| + \sum_{\chi \in \text{Irr}(G_3)} \chi(g) + \sum_{\chi \in \text{nil}(G/G_3)} \chi(g).
\]

Now use (5.1), (5.2) and Lemma 5.5 to get

\[ \tau_G(g) = |G/G_2| - |Z(G)/G_3|p^{3n/2} + (|G_2/G_3| - 1)|Z(G)/G_3|p^n. \]

If $g \in Z(G) \setminus G_3$, then

\[
\tau_G(g) = \sum_{\chi \in \text{lin}(G)} \chi(g) + \sum_{\chi \in \text{nil}(G)} \chi(g) = 0 + \sum_{\chi \in \text{Irr}(G_3)} \chi(g) + \sum_{\chi \in \text{nil}(G/G_3)} \chi(g) \quad \text{(by Lemma 2.3)}
\]

\[ = 0 \quad \text{(use (5.1), (5.2) and Lemma 5.5).} \]
If $g \in G_2 \setminus Z(G)$, then
\[
\tau_G(g) = \sum_{\chi \in \text{lin}(G)} \chi(g) + \sum_{\chi \in \text{nl}(G)} \chi(g)
\]
\[
= |G/G_2| + \sum_{\chi \in \text{Irr}(G/G_3)} \chi(g) + \sum_{\chi \in \text{nl}(G/G_3)} \chi(g)
\]
\[
= |G/G_2| - |Z(G)/G_3| p^n \quad \text{(use (5.1), (5.2) and Lemma 5.5)}.
\]

If $g \in G_2 Z(G)$ but neither in $G_2$ nor in $Z(G)$, then
\[
\tau_G(g) = \sum_{\chi \in \text{lin}(G)} \chi(g) + \sum_{\chi \in \text{nl}(G)} \chi(g)
\]
\[
= 0 + \sum_{\chi \in \text{Irr}(G/G_3)} \chi(g) + \sum_{\chi \in \text{nl}(G/G_3)} \chi(g) \quad \text{(by Lemma 2.3)}
\]
\[
= 0 \quad \text{(use (5.1), (5.2) and Lemma 5.5)}.
\]

Finally, if $g \in G \setminus G_2 Z(G)$, then by Lemma 2.3, (5.1) and (5.2), we get
\[
\tau_G(g) = \sum_{\chi \in \text{Irr}(G)} \chi(g) = \sum_{\chi \in \text{lin}(G)} \chi(g) = 0.
\]

This completes the proof. \qed

We can now characterize nilpotent, generalized Camina groups (Theorem C).

**Proof of Theorem C.** Let $G$ be a nilpotent, generalized Camina group with associated prime $p$. If $p = 2$, then $G$ has nilpotency class 2 and for $p$ odd, $G$ has nilpotency class at most 3 (see [13, Theorem 2]). Now if nilpotency class is 2, then $(G, Z(G))$ is a generalized Camina pair and hence the result follows from Theorem 1.1.

Next suppose $G$ has nilpotency class 3. If $Z(G)$ is not cyclic then by Lemma 2.1, $G$ has no faithful irreducible character. Therefore from Proposition 2.5, $G$ has no Johnson polynomial. Now suppose $Z(G)$ is cyclic. Therefore $G$ has a faithful irreducible character $\chi$ (say). Let $f(x)$ be a Johnson polynomial and let $f(\chi) = \tau_G$. From (5.1) and (5.2), it is clear that $\chi \in \text{Irr}(G/G_3)$. Then, in view of Proposition 2.4 and 5.6, $G$ has no Johnson polynomial.

This completes the proof. \qed

6. $p$-groups of order $p^5$

In this final section, we completely classify the groups of order $p^5$ (for $p$ odd) which admit a Johnson polynomial. Throughout this section $p$ always denotes an odd prime. We will use not only the results of the previous sections but, more crucially, also use the classification of groups of order $p^5$ by R. James ([8, Section 4.5]).

We begin by recalling some well known results which we will use.
Theorem 6.1. [7, Theorem 6.15] If $G$ is a nonabelian $p$-group with $cd(G) = \{1, p\}$, then exactly one of the following holds:

1. $G$ has an abelian subgroup of index $p$,
2. $G/Z(G)$ is of order $p^3$ and exponent $p$.

Lemma 6.2. [7, Lemma 2.9] Let $H$ be a subgroup of $G$. Suppose $\chi$ is a character of $G$. Then

$$\langle \chi|_H, \chi|_H \rangle \leq |G/H|\langle \chi, \chi \rangle$$

with equality if and only if $\chi(g) = 0$ for all $g \in G \setminus H$.

Lemma 6.3. [1, Theorem 20] If $G$ is a $p$-group, then for each $\chi \in \text{Irr}(G)$, $\chi(1)^2$ divides $|G/Z(G)|$.

Here is an easy consequence of the above lemma.

Lemma 6.4. Let $G$ be a non-abelian group of order $p^4$. Then $cd(G) = \{1, p\}$.

Proof. Since $Z(G) \neq 1$, $|Z(G)| = p$ or $p^2$. Therefore $|G/Z(G)| = p^3$ or $p^2$. So by Lemma 6.3, the result follows. \qed

Theorem 6.5. [7, Theorem 6.15] Let $H$ be an abelian normal subgroup of $G$. Then $\chi(1)$ divides $|G/H|$ for all $\chi \in \text{Irr}(G)$.

As mentioned at the outset of this section, we will use the classification of groups of order $p^5$ by R. James ([8, Section 4.5]). More particularly, we will use the list of polycyclic presentations of these groups that the author compiled, and divided the non-abelian ones into families denoted by $\Phi_1, \cdots, \Phi_{10}$, according to isoclinism.

Lemma 6.6. If $G \in X = \{\Phi_2(41), \Phi_2(311)b, \Phi_5(2111), \Phi_5(15)\}$ (see [8, Section 4.5]), then $G$ has a Johnson polynomial.

Proof. First we consider the isoclinism family $\Phi_2$. There are two type of groups in this family with $Z(G)$ cyclic namely,

$$G = \Phi_2(41) = \langle \alpha, \alpha_1, \alpha_2 \mid [\alpha_1, \alpha] = \alpha^{p^3} = \alpha_2, \alpha_1^p = \alpha_2^p = 1 \rangle$$

and

$$H = \Phi_2(311)b = \langle \alpha, \alpha_1, \alpha_2, \gamma \mid [\alpha_1, \alpha] = \gamma^{p^2} = \alpha_2, \alpha_1^p = \alpha_2^p = 1 \rangle.$$

Here $|Z(G)| = |\langle \alpha^p \rangle| = p^3$, $|G_2| = |\langle \alpha^{p^3} \rangle| = p|Z(H)| = |\langle \gamma^p \rangle| = p^3$, $|H_2| = |\langle \gamma^{p^2} \rangle| = p$. By Lemma 6.3, we have $cd(G) = \{1, p\}$ and $cd(H) = \{1, p\}$. Now by Lemma 6.2, it is clear that $(G, Z(G))$ and $(H, Z(H))$ are generalized Camina pair. Hence by Theorem 1.1, $G$ and $H$ have a Johnson polynomial.

Now we discuss the isoclinism family $\Phi_5$. There are only two type of groups in this family and both have cyclic center. Here are the groups:

1. $\Phi_5(2111) = \langle \alpha_1, \alpha_2, \alpha_3, \alpha_4, \beta \mid [\alpha_1, \alpha_2] = [\alpha_3, \alpha_4] = \alpha_1^p = \beta, \alpha_2^p = \alpha_3^p = \alpha_4^p = \beta^p = 1 \rangle$;
2. $\Phi_5(15) = \langle \alpha_1, \alpha_2, \alpha_3, \alpha_4, \beta \mid [\alpha_1, \alpha_2] = [\alpha_3, \alpha_4] = \beta, \alpha_1^p = \alpha_2^p = \alpha_3^p = \alpha_4^p = \beta^p = 1 \rangle$. 

Note that both $\Phi_5(2111)$ and $\Phi_5(1^5)$ are extra-special $p$-groups. Therefore for these two groups $(G, Z(G))$ is a GCP (see [9 Theorem 2.18]). Since $G$ is an extra-special $p$-group, $Z(G) = G_2$ and $|Z(G)| = p$. Therefore by Theorem 1.1 the polynomial

$$f(x) = p^{n-1} \sum_{j=1}^{p-1} (x/p^n)^j + p^{2n}(x/p^n)^p$$

is a Johnson polynomial of $G$ and $f(\chi) = \tau_G$ for every $\chi \in \operatorname{nl}(G)$, where $G \in \Phi_5$. $\Box$

**Lemma 6.7.** If $G$ in the isoclinism family $\Phi_3$, then $G$ has no Johnson polynomial.

**Proof.** Let $G \in \Phi_3$. There are two type of groups in this family with $Z(G)$ cyclic namely, $\Phi_3'(2111)c$ and $\Phi_3'(311)b_r$ (see [3 Section 4.5]). For $p = 3$ and $p \geq 5$, we define these groups separately.

1. $G = \Phi_3'(2111)c = \langle \alpha, \alpha_1, \alpha_2, \alpha_3, \gamma \mid [\alpha_1, \alpha] = \alpha_2, [\alpha_2, \alpha] = \gamma, \alpha^p = \alpha_3, \alpha_p = \alpha_2^3 = \alpha_3^3 = 1 \rangle$ for $p = 3$;
2. $H = \Phi_3'(2111)c = \langle \alpha, \alpha_1, \alpha_2, \alpha_3, \gamma \mid [\alpha_1, \alpha] = \alpha_2, [\alpha_2, \alpha] = \gamma, \alpha^p = \alpha_3, \alpha_p = \alpha_2^3 = \alpha_3^3 = 1 \rangle (i = 1, 2, 3)$ for $p \geq 5$;
3. $K = \Phi_3'(311)b_r = \langle \alpha, \alpha_1, \alpha_2, \alpha_3 \mid [\alpha_1, \alpha] = \alpha_2, [\alpha_2, \alpha] = \gamma, [\alpha_3, \alpha] = \gamma^2, \alpha^p = \alpha_2^3 = \alpha_3^3 = 1 \rangle$ for $r = 1$ or $\nu$, where $\nu$ is a fixed quadratic non-residue mod $p$, and $p \geq 3$.

Observe that $|Z(G)| = |\gamma| = p^2$, $|Z(H)| = |\gamma| = p^2$ and $|Z(K)| = |\langle \alpha_1 \rangle| = p^2$.

First we will deal with $H$. Consider a normal abelian subgroup

$$N = \langle \alpha_1, \alpha_2, \gamma \mid [\alpha_1, \alpha] = \alpha_2, [\alpha_2, \alpha] = \gamma, \alpha_i^p = \gamma^2 = 1 (i = 1, 2) \rangle$$

of $H$ of index $p$. By Theorem 6.5, $cd(H) = \{1, p\}$. Since $N$ is a normal abelian subgroup of index $p$, every nonlinear irreducible characters of $H$ must be induced from $N$ and hence $\chi(H \setminus N) = 0$ for all $\chi \in \operatorname{nl}(H)$. Now

$$\overline{H} := H/Z(H) = \langle \alpha, \alpha_1, \alpha_2 \mid [\alpha_1, \alpha] = \alpha_2, \alpha^p = \alpha_2^3 = 1 \rangle (i = 1, 2)$$

is an extra-special $p$-group of order $p^3$. Therefore, $\overline{H}$ has $p - 1$ nonlinear irreducible characters of each of degree $p$ which vanish out side $Z(\overline{H})$ in $\overline{H}$ and on $Z(\overline{H})$ it is $p\lambda$, where $\lambda \in \operatorname{Irr}(Z(\overline{H})) \setminus \{1_{Z(\overline{H})}\}$. In particular, $H$ has $p - 1$ nonlinear irreducible characters which contains $Z(H)$ in their kernel.

Take $Q = \langle \gamma \rangle$. Then $\operatorname{Irr}(H|Z(H)) = \operatorname{Irr}(H/Q|Z(H)/Q) \cup \operatorname{Irr}(H/Q)$. Now, suppose $\chi \in \operatorname{Irr}(H|Q)$. Then $\chi$ is faithful. Let $\phi$ be an irreducible constituent of $\chi_{\phi_1}^H$, where $M = \langle \alpha_2, \gamma \rangle$. Since $\chi$ is faithful, $\phi$ is not $H$-invariant. Therefore, by Clifford’s theorem $\chi_{\phi_1}^H = \sum_{i=1}^p \phi_i$, where $\phi_1 = \phi$ and $p$ is the index of the inertia group $N$ of $\phi$ in $H$. Now $\phi_1^M = \lambda$, where $\lambda \in \operatorname{Irr}(Z(H)) \setminus \{1_{Z(H)}\}$ for each $1 \leq i \leq p$.

Therefore, by [7 Corollary 6.17], we have

$$\chi_{\phi_1}^H = \sum_{\sigma \in \operatorname{Irr}(M/Z(H))} \sigma \phi_1 = \rho_{M/Z(H)} \phi_1,$$

where $\rho_{M/Z(H)}$ is the regular character of $M/Z(H)$. Hence for each $\chi \in \operatorname{Irr}(H|Q)$, we have $\chi(M \setminus Z(H)) = 0$. 

Next, we consider $\chi \in \text{Irr}(H/Q|Z(H)/Q)$, where $H/Q = \langle \alpha, \alpha_1, \alpha_2, \gamma \mid [\alpha_1, \alpha] = \alpha_2, \alpha^p = \gamma^p = \alpha_i^p = 1 \ (i = 1, 2) \rangle$ and $Z(H/Q) = M/Q$. Since $(H/Q, Z(H/Q))$ is a generalized Camina pair, $\chi(\alpha_2) = p\lambda(\alpha_2)$, where $\lambda \in \text{Irr}(Z(H/Q)) \setminus \text{Irr}(Z(H/Q)/(H/Q)_2)$ (see [19] Theorem 3.1).

But then
\[
\tau_H(\alpha_2) = \sum_{\chi \in \text{lin}(H)} \chi(\alpha_2) + \sum_{\chi \in \text{Irr}(H/Z(H))} \chi(\alpha_2) + \sum_{\chi \in \text{Irr}(H/Q)} \chi(\alpha_2) \\
= |H/H_2| + \sum_{\lambda \in Z(H) \setminus \{1_{Z(H)}\}} p\lambda(\alpha_2) + \sum_{\chi \in \text{Irr}(H/Q)} \chi(\alpha_2) \\
= p^3 - p + 0 - p^2 + p \\
= p^3 - p^2.
\]

(6.2)

Now suppose $H$ has a Johnson polynomial $f(x)$ such that $f(\chi) = \tau_H$, where $\chi \in \text{nl}(H)$. Therefore $\chi$ is faithful and $\chi \in \text{Irr}(H/Q)$. Now $f(0) = f(\chi(\alpha_2)) = \tau_H(\alpha_2) = p^3 - p^2$ and $f(0) = f(\chi(\alpha)) = \tau_H(\alpha) = 0$. The resultant contradiction proves that $H$ can have no Johnson polynomial.

One can use a very similar argument to show that neither $G$ nor $K$ can have a Johnson polynomial. □

**Lemma 6.8.** If $G$ in the isoclinism family $\Phi_7$ or $\Phi_8$, then $G$ has no Johnson polynomial.

**Proof.** Suppose $G$ is in the isoclinism family $\Phi_7$. For $p = 3$ and $p \geq 5$, we will define these groups separately.

**For $p = 3$:**

1. $G = \Phi_7(2111)a = \langle \alpha, \alpha_1, \alpha_2, \alpha_3, \beta \mid [\alpha_i, \alpha] = \alpha_{i+1}, [\alpha_1, \beta] = \alpha_3, \alpha_3, \alpha_3 = \alpha_{i+1}^3 = \beta^3 = 1 \ (i = 1, 2) \rangle$;
2. $G = \Phi_7(2111)b_1 = \langle \alpha, \alpha_1, \alpha_2, \alpha_3, \beta \mid [\alpha_i, \alpha] = \alpha_{i+1}, [\alpha_1, \beta] = \alpha_3, \alpha_1^3 = \alpha_{i+1}^3 = \beta^3 = 1 \ (i = 1, 2) \rangle$;
3. $G = \Phi_7(2111)b_2 = \langle \alpha, \alpha_1, \alpha_2, \alpha_3, \beta \mid [\alpha_i, \alpha] = \alpha_{i+1}, [\alpha_1, \beta] = \alpha_3^2, \alpha_1^3 = \alpha_3, \alpha_1^3 = \alpha_{i+1}^3 = \beta^3 = 1 \ (i = 1, 2) \rangle$;
4. $G = \Phi_7(2111)c = \langle \alpha, \alpha_1, \alpha_2, \alpha_3, \beta \mid [\alpha_i, \alpha] = \alpha_{i+1}, [\alpha_1, \beta] = \alpha_3, \alpha_3^3 = \alpha_1^3 \alpha_3 = \alpha_{i+1} = \beta^3 = 1 \ (i = 1, 2) \rangle$;
5. $G = \Phi_7(1^5) = \langle \alpha, \alpha_1, \alpha_2, \alpha_3, \beta \mid [\alpha_i, \alpha] = \alpha_{i+1}, [\alpha_1, \beta] = \alpha_3, \alpha_3^3 = \alpha_1^3 \alpha_3 = \alpha_{i+1}^3 \beta^3 = 1 \ (i = 1, 2) \rangle$.

**For $p \geq 5$:**

1. $G = \Phi_7(2111)a = \langle \alpha, \alpha_1, \alpha_2, \alpha_3, \beta \mid [\alpha_i, \alpha] = \alpha_{i+1}, [\alpha_1, \beta] = \alpha_3, \alpha_1^p = \alpha_{i+1}^p = \beta^p = 1 \ (i = 1, 2) \rangle$;
2. $G = \Phi_7(2111)b_1 = \langle \alpha, \alpha_1, \alpha_2, \alpha_3, \beta \mid [\alpha_i, \alpha] = \alpha_{i+1}, [\alpha_1, \beta] = \alpha_3, \alpha_1^p = \alpha_{i+1}^p = \beta^p = 1 \ (i = 1, 2) \rangle$;
3. $G = \Phi_7(2111)b_2 = \langle \alpha, \alpha_1, \alpha_2, \alpha_3, \beta \mid [\alpha_i, \alpha] = \alpha_{i+1}, [\alpha_1, \beta] = \alpha_3, \alpha_1^p = \alpha_{i+1}^p = \beta^p = 1 \ (i = 1, 2) \rangle$ where $\nu$ is a fixed quadratic non-residue \( \pmod{p} \) and $2 \leq \nu \leq p - 1$;
(4) $G = \Phi_7(2111)c = \langle \alpha, \alpha_1, \alpha_2, \alpha_3, \beta \mid [\alpha_i, \alpha] = \alpha_{i+1}, [\alpha_1, \beta] = \alpha_3, \alpha_3 = \beta^p, \alpha^p = \alpha_1^p = \alpha_{i+1}^p = 1 \ (i = 1, 2) \rangle$;

(5) $G = \Phi_7(1^5) = \langle \alpha, \alpha_1, \alpha_2, \alpha_3, \beta \mid [\alpha_i, \alpha] = \alpha_{i+1}, [\alpha_1, \beta] = \alpha_3, \alpha^p = \alpha_1^p = \alpha_{i+1}^p = \beta^p = 1 \ (i = 1, 2) \rangle$.

It is clear that $|Z(G)| = |\langle \alpha_3 \rangle| = p$ and

$$G/Z(G) = H \times K = \langle \alpha, \alpha_1, \alpha_2 \mid [\alpha_1, \alpha] = \alpha_2, \alpha^p = \alpha_1^p = \alpha_3^p = 1 \rangle \times \langle \beta \rangle$$

is of order $p^4$ for all $G \in \Phi_7$, where $H$ is extra-special $p$-group of order $p^3$ and $K$ is a cyclic group of order $p$. Hence, by Lemma 6.4 we have $cd(G/Z(G)) = \{1, p \} \subseteq cd(G)$. Since $G$ has no abelian subgroup of index $p$ for all $G \in \Phi_7$, from Theorem 6.1 and Lemma 6.3 we get $cd(G) = \{1, p, p^2 \}$. From Lemma 6.2 it is easy to observe that if $\chi(1) = p^2$, then $\chi(g) = 0$ for all $g \in G \setminus Z(G)$. Hence $(G, Z(G))$ is a Camina pair. Since $H$ is an extra-special $p$-group, every nonlinear irreducible character $\phi$ of $H$ vanishes outside $Z(H) = \langle \alpha_2 \rangle$ in $H$ and $\phi|_{1Z(H)} = p\lambda$ for some $\lambda \in \text{Irr}(Z(H)) \setminus \{1_{Z(H)}\}$, where $1_{Z(H)}$ is the trivial character of $Z(H)$. Hence

$$nl(G/Z(G)) = \{ \phi \times \psi \mid \phi \in nl(H), \psi \in \text{Irr}(K) \}.$$ 

Now if $g = \alpha_2$, then

$$\tau_G(\alpha_2) = \sum_{\chi \in \text{Irr}(G)} \chi(\alpha_2) + \sum_{\chi \in \text{Irr}(G/Z(G))} \chi(\alpha_2) + \sum_{\chi \in \text{nl}(G/Z(G))} \chi(\alpha_2)$$

$$= |G/G_2| + 0 + \sum_{\phi \in \text{nl}(H)} (\phi \times \psi)(\alpha_2) \quad ((G/Z(G)) \text{ is a Camina pair})$$

$$= p^3 + \sum_{\lambda \in \text{Irr}(Z(H))} \sum_{\psi \in \text{Irr}(K)} (p\lambda \times \psi)(\alpha_2) \quad (H \text{ is an extra-special group})$$

$$= p^3 + p \sum_{\lambda \in \text{Irr}(Z(H))} p\lambda(\alpha_2)$$

(6.3)

$$= p^3 - p^2.$$ 

Since $(G, Z(G))$ is a Camina pair and $H$ is an extra-special group,

$$\tau_G(g) = \sum_{\chi \in \text{Irr}(G)} \chi(\alpha_2) + \sum_{\chi \in \text{Irr}(G/Z(G))} \chi(\alpha_2) + \sum_{\chi \in \text{nl}(G/Z(G))} \chi(\alpha_2) = 0$$

for all $g \in H \setminus Z(H)$. Now suppose $G$ has a Johnson polynomial $f(x)$ such that $f(\chi) = \tau_G$, where $\chi \in \text{nl}(G)$. Therefore $\chi$ is faithful and $\chi \in \text{Irr}(G/Z(G))$. Since $\chi \in \text{Irr}(G/Z(G))$, $\chi(\alpha_2) = 0$ for all $g \in G \setminus Z(G)$. In particular, $\chi(\alpha_1) = \chi(\alpha_2) = 0$. Now $f(0) = f(\chi(\alpha_2)) = \tau_G(\alpha_2) = p^3 - p^2$ whereas $f(0) = f(\chi(\alpha_1)) = \tau_G(\alpha_1) = 0$, which is a contradiction. Thus, $G$ cannot have a Johnson polynomial.

Next suppose $G$ is in the isoclinism family $\Phi_8$; $G := \Phi_8(32) = \langle \alpha_1, \alpha_2, \beta \mid [\alpha_1, \alpha_2] = \beta = \alpha_1^p, \beta^p = \alpha_2^p = 1 \rangle$. Here $|Z(G)| = |\langle \alpha_1^p \rangle| = p$ and

$$G/Z(G) = \langle \alpha_1, \alpha_2 \mid [\alpha_1, \alpha_2] = \alpha_1^p, \alpha_1^p = \alpha_2^p = 1 \rangle.$$
is of order \(p^4\). To show that \(cd(G) = \{1, p, p^2\}\), we may use the same argument as we do for the groups in the family \(\Phi_7\); hence we skip the details. Now one can observe that \((G, Z(G))\) is a Camina pair and \((G/Z(G), Z(G/Z(G)))\) is a generalized Camina pair. Therefore, by Theorem B, \(G\) has no Johnson polynomial. \(\square\)

**Lemma 6.9.** Let

\[ H = \langle \alpha, \alpha_1, \alpha_2, \alpha_3 \mid [\alpha_1, \alpha] = \alpha_2, [\alpha_2, \alpha] = \alpha_3, \alpha^p = \alpha_1^p = \alpha_2^p = 1 \rangle. \]

Then, \(H\) is a group of order \(p^4\) for an odd prime \(p \geq 5\) and

\[ \sum_{\chi \in \text{Irr}(H)} \chi(\alpha_2) = \sum_{\chi \in \text{Irr}(H)} \chi(\alpha_3) = -p. \]

**Proof.** Observe that \(Z(H) = \langle \alpha_3 \rangle\) and \(H_2 = \langle \alpha_2, \alpha_3 \rangle\). Since \(H\) has a normal abelian subgroup \(N = \langle \alpha_1, \alpha_2, \alpha_3 \rangle = \langle \alpha_1 \rangle \times \langle \alpha_2 \rangle \times \langle \alpha_3 \rangle\) of order \(p^3\), by Theorem 6.5, \(cd(H) = \{1, p\}\). Now, if we consider the group

\[ \overline{H} := H/Z(H) = \langle \alpha, \alpha_1, \alpha_2 \mid [\alpha_1, \alpha] = \alpha_2, \alpha^p = \alpha_1^p = \alpha_2^p = 1 \rangle, \]

we see that it is an extra-special \(p\)-group of order \(p^3\). Therefore, \(\overline{H}\) has \(p - 1\) nonlinear irreducible characters of degree \(p\) which vanish outside \(Z(\overline{H}) = \langle \alpha_2 \rangle\) and, for \(\chi \in \text{Irr}(\overline{H})\), we have

\[ (6.5) \quad \chi|_{Z(\overline{H})} = p\lambda \]

for some \(\lambda \in \text{Irr}(Z(\overline{H})) \setminus \{1_{Z(\overline{H})}\}\). In particular, we have all the nonlinear irreducible characters of \(H\) having \(Z(H)\) in their kernel. Now, let \(\psi \in \text{Irr}(H/Z(H))\). Since \(|Z(H)| = p\), \(\psi\) is faithful and hence \(\phi\) is not \(H\)-invariant, where \(\phi\) is an irreducible constituent of \(\psi|_{H_2}\). Therefore, by Clifford’s theorem

\[ \psi|_{H_2}^{H} = \sum_{i=1}^{p} \phi_i, \]

where \(\phi_1 = \phi\) and \(p\) is the index of the inertia group \(N\) of \(\phi\) in \(H\). Now \(\phi_i|_{Z(H)}^{H_2} = \lambda\), where \(\lambda \in \text{Irr}(Z(H)) \setminus \{1_{Z(H)}\}\) for each \(1 \leq i \leq p\). Therefore, by [7] Corollary 6.17, we have

\[ (6.6) \quad \psi|_{H_2}^{H} = \sum_{\beta \in \text{Irr}(H_2/Z(H))} \beta \phi_1 = \rho_{H_2/Z(H)} \phi_1, \]

where \(\rho_{H_2/Z(H)}\) is the regular character of \(H_2/Z(H)\). Hence for each \(\psi \in \text{Irr}(H/Z(H))\), we have \(\psi(H_2 \setminus Z(H)) = 0\).

Now

\[
\sum_{\chi \in \text{Irr}(H)} \chi(\alpha_2) = \sum_{\chi \in \text{Irr}(H/Z(H))} \chi(\alpha_2) + \sum_{\chi \in \text{Irr}(H/Z(H))} \chi(\alpha_2)
= \sum_{\lambda \in \text{Irr}(Z(H)) \setminus \{1_{Z(H)}\}} \lambda \alpha(\alpha_2) + \sum_{\chi \in \text{Irr}(H/Z(H))} \chi(\alpha_2) \quad \text{(Use (6.5))}
= -p + 0 = -p
\]
and
\[
\sum_{\chi \in \text{nl}(H)} \chi(\alpha_3) = \sum_{\chi \in \text{nl}(H/Z(H))} \chi(\alpha_3) + \sum_{\chi \in \text{Irr}(H/Z(H))} \chi(\alpha_3)
\]
\[= p(p - 1) + \sum_{\chi \in \text{Irr}(H/Z(H))} \chi(\alpha_3) \quad \text{(Since } \chi(\alpha_3) = p \text{ for all } \chi \in \text{nl}(H))
\]
\[= p(p - 1) - p^2 = -p
\]

(6.8)

This completes the proof of the lemma. \(\square\)

**Lemma 6.10.** If \(G \in \Phi_9\), then \(G\) has no Johnson polynomial.

**Proof.** Suppose \(G\) is in the isoclinism family \(\Phi_9\); these are defined as follows. For \(p = 3:\)

1. \(G = \Phi_9(2111)a = \langle \alpha, \alpha_1, \alpha_2, \alpha_3, \alpha_4 \ | \ [\alpha_i, \alpha] = \alpha_{i+1}, \alpha^3 = \alpha_4, \alpha_3^{\alpha_3} = \alpha_3^2, \alpha_4^{\alpha_3} = \alpha_4 \rangle = 1 \ (i = 1, 2, 3);\)
2. \(G = \Phi_9(2111)b_0 = \langle \alpha, \alpha_1, \alpha_2, \alpha_3, \alpha_4 \ | \ [\alpha_i, \alpha] = \alpha_{i+1}, \alpha_i^{\alpha_3} = \alpha_i, \alpha_i^{\alpha_4} = \alpha_i^{\alpha_4} = \alpha_3^2, \alpha_4^{\alpha_4} = \alpha_4 = 1 \ (i = 1, 2, 3);\)
3. \(G = \Phi_9(1^5) = \langle \alpha, \alpha_1, \alpha_2, \alpha_3, \alpha_4 \ | \ [\alpha_i, \alpha] = \alpha_{i+1}, \alpha^3 = \alpha_3, \alpha_3^{\alpha_3} = \alpha_3^2, \alpha_4^{\alpha_4} = \alpha_4 = 1 \ (i = 1, 2, 3).\)

For \(p \geq 5:\)

1. \(G = \Phi_9(2111)a = \langle \alpha, \alpha_1, \alpha_2, \alpha_3, \alpha_4 \ | \ [\alpha_i, \alpha] = \alpha_{i+1}, \alpha^p = \alpha_4, \alpha_1^{\alpha_4} = \alpha_1^{\alpha_4} = 1 \ (i = 1, 2, 3);\)
2. \(G = \Phi_9(2111)b_r = \langle \alpha, \alpha_1, \alpha_2, \alpha_3, \alpha_4 \ | \ [\alpha_i, \alpha] = \alpha_{i+1}, \alpha_i = \alpha_i^k, \alpha_i^{\alpha_i} = \alpha_i^{\alpha_i} = 1 \ (i = 1, 2, 3))\)
   \(\text{where } k = g^r \text{ for } r + 1 = 1, 2, \cdots, (p - 1, 3);\)
3. \(G = \Phi_9(1^5) = \langle \alpha, \alpha_1, \alpha_2, \alpha_3, \alpha_4 \ | \ [\alpha_i, \alpha] = \alpha_{i+1}, \alpha^p = \alpha_i^p = \alpha_i^{p+1} = 1 \ (i = 1, 2, 3).\)

Here \(|Z(G)| = |\langle \alpha_4 \rangle| = p\) and
\[G/Z(G) = \langle \alpha, \alpha_1, \alpha_2, \alpha_3 \ | \ [\alpha_i, \alpha] = \alpha_{i+1}, \alpha^p = \alpha_i^p = \alpha_i^3 = 1 \ (i = 1, 2)\]

is of order \(p^4\) for all \(G \in \Phi_9\). Note that \(G\) has an abelian normal subgroup \(N = \langle \alpha_1, \alpha_2, \alpha_3, \alpha_4 \rangle\) of index \(p\) for all \(G \in \Phi_9\). Therefore, by Theorem 6.5 we have \(cd(G) = \{1, p\}\) for all \(G \in \Phi_9\).

Now consider \(p \geq 5\). In this case
\[G/Z(G) = \langle \alpha, \alpha_1, \alpha_2, \alpha_3 \ | \ [\alpha_i, \alpha] = \alpha_{i+1}, \alpha^p = \alpha_i^p = \alpha_i^3 = 1 \ (i = 1, 2)\].

Since \(N = \langle \alpha_1, \alpha_2, \alpha_3, \alpha_4 \rangle\) is a normal abelian subgroup of index \(p\), every nonlinear irreducible character of \(G\) must be induced from \(N\) and hence \(\chi(G \setminus N) = 0\) for all \(\chi \in \text{nl}(G)\). Let \(K = \langle \alpha_3, \alpha_4 \rangle\). Now, let \(\chi \in \text{Irr}(G/Z(G))\). Since \(|Z(G)| = p\), \(\chi\) is faithful. Let \(\phi\) be an irreducible constituent of \(\chi^G_K\). Since \(\chi\) is faithful, \(\phi\) is not \(G\)-invariant. And hence by Clifford’s theorem, we have \(\chi^G_K = \sum_i \phi_i\), where \(\phi_1 = \phi\) and \(p\) is the index of the inertia group \(N\) of \(\phi\) in \(G\). Now \(\phi_i\}_{Z(G)} = \lambda\), where \(\lambda \in \text{Irr}(Z(G)) \setminus \{1_{Z(G)}\}\) for each \(1 \leq i \leq p\). Therefore, by [7] Corollary 6.17, we have
\[
\chi^G_K = \sum_{\gamma \in \text{Irr}(K/Z(G))} \gamma \phi_1 = \rho_{K/Z(G)} \phi_1,
\]
where $\rho_{K/Z(G)}$ is the regular character of $K/Z(G)$. Hence for each $\chi \in \text{Irr}(G|Z(G))$, we have $\chi(K \setminus Z(G)) = 0$. Therefore,

$$\tau_G(\alpha_3) = \sum_{\chi \in \text{lin}(G)} \chi(\alpha_3) + \sum_{\chi \in \text{nl}(G)} \chi(\alpha_3)$$

$$= |G/G_2| + \sum_{\chi \in \text{Irr}(G|Z(G))} \chi(\alpha_3) + \sum_{\chi \in \text{nl}(G/Z(G))} \chi(\alpha_3) \quad \text{(Since } \alpha_3 \in G_2)$$

$$= p^2 + 0 - p \quad \text{(Use Lemma 6.9)}$$

(6.10)

$$= p^2 - p$$

Now suppose $G$ has a Johnson polynomial $f(x)$ such that $f(\chi) = \tau_G$, where $\chi \in \text{nl}(G)$. Therefore $\chi$ is faithful and $\chi \in \text{Irr}(G|Z(G))$. By (6.9), we have $\chi(\alpha_3) = 0$ for all $\chi \in \text{Irr}(G|Z(G))$. Therefore, for $p \geq 5$, $G$ has no Johnson polynomial.

Very similarly, for $p = 3$, one can show that $G$ has no Johnson polynomial. This completes the proof of this lemma.

\[\square\]

**Lemma 6.11.** If $G \in \Phi_{10}$, then $G$ has no Johnson polynomial.

**Proof.** Suppose that $G \in \Phi_{10}$; these are defined as follows.

For $p = 3$:

1. $\Phi_{10}(2111)a_0 = \langle \alpha, \alpha_1, \alpha_2, \alpha_3, \alpha_4 \mid [\alpha_i, \alpha] = \alpha_{i+1}, [\alpha_1, \alpha_2] = \alpha_4 = \alpha^3, \alpha_2^3 = \alpha_3^3 = \alpha_4^3 = 1 \ (i = 1, 2, 3)\rangle$;
2. $\Phi_{10}(2111)a_1 = \langle \alpha, \alpha_1, \alpha_2, \alpha_3, \alpha_4 \mid [\alpha_i, \alpha] = \alpha_{i+1}, [\alpha_1, \alpha_2]^2 = \alpha_4^2 = \alpha^3, \alpha_2^3 = \alpha_3^3 = \alpha_4^3 = 1 \ (i = 1, 2, 3)\rangle$;
3. $\Phi_{10}(15^3) = \langle \alpha, \alpha_1, \alpha_2, \alpha_3, \alpha_4 \mid [\alpha_i, \alpha] = \alpha_{i+1}, [\alpha_1, \alpha_2] = \alpha_4 = \alpha^3 = \alpha_2^3 = \alpha_3^3 = \alpha_4^3 = 1 \ (i = 1, 2, 3)\rangle$.

For $p \geq 5$:

1. $\Phi_{10}(2111)a_r = \langle \alpha, \alpha_1, \alpha_2, \alpha_3, \alpha_4 \mid [\alpha_i, \alpha] = \alpha_{i+1}, [\alpha_1, \alpha_2]^k = \alpha_4^k = \alpha^p, \alpha_1^p = \alpha_{i+1}^p = 1 \ (i = 1, 2, 3)\rangle$ where $k = g^r$ for $r + 1 = 1, 2, \cdots, (p-1, 4)$ and $g$ is the smallest positive integer which is primitive root $\text{mod } p$;
2. $\Phi_{10}(2111)b_r = \langle \alpha, \alpha_1, \alpha_2, \alpha_3, \alpha_4 \mid [\alpha_i, \alpha] = \alpha_{i+1}, [\alpha_1, \alpha_2]^k = \alpha_4^k = \alpha^p, \alpha_1^p = \alpha_{i+1}^p = 1 \ (i = 1, 2, 3)\rangle$ where $k = g^r$ for $r + 1 = 1, 2, \cdots, (p-1, 3)$ and $g$ is the smallest positive integer which is primitive root $\text{mod } p$;
3. $\Phi_{10}(15^3) = \langle \alpha, \alpha_1, \alpha_2, \alpha_3, \alpha_4 \mid [\alpha_i, \alpha] = \alpha_{i+1}, [\alpha_1, \alpha_2] = \alpha_4 = \alpha^p = \alpha_1^p = \alpha_{i+1}^p = 1 \ (i = 1, 2, 3)\rangle$.

Here $|Z(G)| = |\langle \alpha_4 \rangle| = p$ and

$$G/Z(G) = \langle \alpha, \alpha_1, \alpha_2, \alpha_3 \mid [\alpha_i, \alpha] = \alpha_{i+1}, \alpha^p = \alpha_1^p = \alpha_{i}^{(p)} = \alpha_3^p = 1 \ (i = 1, 2)\rangle$$

is of order $p^4$ for all $G \in \Phi_{10}$. Note that $G$ has no abelian subgroup of index $p$ for all $G \in \Phi_{10}$. By Lemma 6.4 we have $cd(G/Z(G)) = \{1, p\} \subseteq cd(G)$. Therefore from Theorem 6.1 and Lemma 6.3 we
get $cd(G) = \{1, p, p^2\}$. If $\chi(1) = p^2$, then by Lemma 6.2, $\chi$ vanish outside $Z(G)$ in $G$ for all $G$ in $\Phi_{10}$. This shows that $(G, Z(G))$ is a Camina pair for all $G$ in $\Phi_{10}$. Now consider the group $G/Z(G)$ for $p \geq 5$.

$G/Z(G) = \langle \alpha, \alpha_1, \alpha_2, \alpha_3 \mid [\alpha_1, \alpha] = \alpha_2, [\alpha_2, \alpha] = \alpha_3, \alpha^p = \alpha_4^p = \alpha_5^p = \alpha_3 = 1 \rangle$.

Then

$$
\tau_G(\alpha_2) = \sum_{\chi \in \text{Irr}(G)} \chi(\alpha_2) + \sum_{\chi \in \text{nl}(G)} \chi(\alpha_2)
$$

$$
= |G/Z(G)| + \sum_{\chi \in \text{Irr}(G/Z(G))} \chi(\alpha_2) + \sum_{\chi \in \text{nl}(G/Z(G))} \chi(\alpha_2) \quad (\text{Since } \alpha_2 \in G_2)
$$

$$
= p^2 + 0 + \sum_{\chi \in \text{nl}(G/Z(G))} \chi(\alpha_2) \quad (\text{Since } (G, Z(G)) \text{ is a Camina pair})
$$

(6.11)

$$
= p^2 - p \quad (\text{Use Lemma 6.9})
$$

Next suppose that $G$ has a Johnson polynomial $f(x)$ such that $f(\chi) = \tau_G$, where $\chi \in \text{nl}(G)$. Therefore $\chi$ is faithful and $\chi \in \text{Irr}(G/Z(G))$. Since $(G, Z(G))$ is a Camina pair, $\chi(g) = 0$ for all $g \in G \setminus Z(G)$ and $\chi \in \text{Irr}(G/Z(G))$. In particular, $\chi(\alpha) = \chi(\alpha_2) = 0$. Now $f(0) = f(\chi(\alpha_2)) = \tau_G(\alpha_2) = p^2 - p$ and $f(0) = f(\chi(\alpha)) = \tau_G(\alpha) = 0$, which is a contradiction. Hence in this case, $G$ has no Johnson polynomial.

Next, for $p = 3$, the group $G/Z(G)$ is

$G/Z(G) = \langle \alpha, \alpha_1, \alpha_2, \alpha_3 \mid [\alpha_1, \alpha] = \alpha_2, [\alpha_2, \alpha] = \alpha_3, \alpha^3 = \alpha_3^3 = \alpha_2^3 = \alpha_3^3 = 1 \rangle$,

and one can use a very similar argument to show that $G$ has no Johnson polynomial in this case also. This completes the proof of the lemma.

Finally, we may summarize the results of this section in the form of Theorem D.

**Proof of Theorem D.** As above, we use the list of nonabelian $p$-group of order $p^5$ given by R. James [S Section 4.5]. From Lemma 2.1 and Proposition 2.5, it is clear that if $G$ has a Johnson polynomial then $Z(G)$ must be cyclic. The nonabelian $p$-groups of order $p^5$ with $Z(G)$ cyclic occur in the isoclinism family $\Phi_2, \Phi_3, \Phi_5, \Phi_7, \Phi_9, \Phi_{10}$ (see [S pages 620-621]). Therefore, the result follows from Lemmata 6.6, 6.7, 6.8, 6.10 and 6.11.

In view of the above results, it seems reasonable to pose the following conjecture for $p$-groups:

**Conjecture:** A $p$-group (with $p$ odd) admits a Johnson polynomial if and only if $Z(G)$ is cyclic and $G' \leq Z(G)$.

**Acknowledgments**

The first author was supported by National Board for Higher Mathematics (NBHM), India and the Indian Statistical Institute. The authors had conjectured a characterization of $p$-groups which admit a Johnson polynomial. The above conjecture is a modified version for which we would like to thank Professor Mark L. Lewis. We would also wish to thank the referees for the useful comments.
and suggestions.

REFERENCES


S. K. Prajapati
Stat Math Unit, Indian Statistical Institute, 8th Mile Mysore Road, Bangalore-560059, India
Email: skprajapati.iitd@gmail.com

B. Sury
Stat Math Unit, Indian Statistical Institute, 8th Mile Mysore Road, Bangalore-560059, India
Email: sury@isibang.ac.in