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RESTRICTIONS ON COMMUTATIVITY RATIOS IN FINITE GROUPS

R. HEFFERNAN*, D. MACHALE AND Á. NÍ SHÉ

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Abstract. We consider two commutativity ratios $\mathrm{Pr}(G)$ and $f(G)$ in a finite group G and examine the properties of G when these ratios are 'large'. We show that if $\mathrm{Pr}(G) > \frac{7}{24}$, then G is metabelian and we give threshold results in the cases where G is insoluble and G' is nilpotent. We also show that if $f(G) > \frac{1}{2}$, then $f(G) = \frac{n+1}{2n}$, for some natural number n.

This paper is dedicated to the memory of our friend and colleague Dr. Fran Barry.

1. Introduction

For a finite group G the probability that two group elements chosen at random with replacement commute, which we denote by $\mathrm{Pr}(G)$, is given by

$$\mathrm{Pr}(G) = \frac{|\{(a,b) \in G \times G \mid ab = ba\}|}{|G|^2} = \frac{\sum_{g \in G} |C_G(g)|}{|G|^2}.$$

We call $\mathrm{Pr}(G)$ the commutativity ratio of G, but others sometimes refer to it as the 'commutativity degree' of the group G; this phrase has another well-established meaning in the theory of p-groups and so we will avoid it here. Alternative notations for $\mathrm{Pr}(G)$ in the literature include $\mathrm{d}(G)$, $\mathrm{mc}(G)$, $\mathrm{R}(G)$ and $\mathrm{cp}(G)$.

Gustafson notes in [11] that $\mathrm{Pr}(G) = k(G)/|G|$ where $k(G)$ is the number of conjugacy classes of G. Clearly the value of $\mathrm{Pr}(G)$ indicates, in some sense, how commutative the group G is. Moreover, $\mathrm{Pr}(G) = 1$ if and only if G is abelian.

Several other 'commutativity ratios' appear in the literature. Along with $\mathrm{Pr}(G)$ we will be interested in the following: suppose that the degrees of the irreducible complex representations of G are

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∗Corresponding author.

$d_1, d_2, \ldots, d_k$, where G has $k = k(G)$ conjugacy classes. Let $T(G) = \sum_{i=1}^{k} d_i$ and let $f(G) = T(G)/|G|$. Note that, like $\Pr(G)$, it is clear that $f(G) = 1$ if and only if G is abelian.

We will make use of Philip Hall's notion of isoclinism [12]. Two groups H and K are said to be *isoclinic*, written $H \sim K$, if there exist isomorphisms $\theta : H/Z(H) \to K/Z(K)$ and $\phi : H' \to K'$ such that the isomorphism ϕ is induced by the isomorphism θ. Isoclinism is an equivalence relation on finite groups. The equivalence classes are called *isoclinism families* and each family contains a *stem-group* G with the property that $Z(G) \subseteq G'$.

The function $\Pr(G)$ satisfies several properties that one might expect from an indicator of commutativity: $\Pr(G)$ is a multiplicative function, $\Pr(G)$ is an isoclinic invariant, if $H \leq G$ then $\Pr(H) \geq \Pr(G)$, and if $N \lhd G$ then $\Pr(G/N) \geq \Pr(G)$.

The function $f(G)$ satisfies the first three of these properties [2] but surprisingly not the last. A counterexample can be found using the Small Groups library [10]. Throughout the paper, the ith group of order n in the Small Groups library will be denoted by $[n, i]$. Small group $[128, 731]$ has small group $[64, 150]$ as a homomorphic image but $f([128, 731]) = 15/32 = 0.46875 > f([64, 150]) = 7/16 = 0.4375$. This counterexample has minimal order.

In this paper we will investigate the properties of a group G if either $\Pr(G)$ or $f(G)$ is 'large'.

2. Notation and terminology

Throughout, G will denote a finite group with centre $Z = Z(G)$ and commutator subgroup G'; p will denote a prime and $\mathcal{G}_p$ will denote the set of all finite groups with order divisible by the prime p but by no smaller prime. The cyclic group of order n will be denoted by C_n, the dihedral group of order $2n$ by D_n, $n > 2$, and the dicyclic group of order $4n$ by Q_n, $n > 1$. The symmetric and alternating groups on n points will be denoted by S_n and A_n, respectively. If $x \in G$, then x^G will denote the conjugacy class of x in G and $C_G(x)$ will denote the centraliser of x in G. If H is a subgroup of G, then $C_G(H)$ will denote the centraliser of H in G. Finally, the set of degrees of irreducible characters of G will be denoted by $c.d.(G)$, i.e. $c.d.(G) = \{\chi(1) \mid \chi \in \mathrm{Irr}(G)\}$. As mentioned above, the ith group of order n in the Small Groups library will be denoted by $[n, i]$. Other notation is standard.

3. Preliminary results

Lemma 3.1 ([14], [18])**.** *If $G \in \mathcal{G}_p$, then*

$$\Pr(G) \leq \frac{1}{p^2}\left[1 + \frac{p^2 - 1}{|G'|}\right].$$

In particular, if G is non-abelian, then

$$\Pr(G) \leq \frac{p^2 + p - 1}{p^3}.$$

Lemma 3.2 ([14], [18])**.** *If $G \in \mathcal{G}_p$, then*

$$f(G) \leq \frac{1}{p} + \frac{p - 1}{p|G'|}.$$

In particular, if G is non-abelian, then

$$f(G) \leq \frac{2p-1}{p^2}.$$

Lemma 3.3 ([1])**.** *If $G' \cong A_4$ then $\Pr(G) \leq \frac{5}{24}$.*

Lemma 3.4 ([13])**.** *If $G' \cong Q_2$ then $Pr(G) = \frac{1}{6} + \frac{1}{2^{2s+1}}$ for some $s \geq 1$.*

We say a group K is *incompetent* [15] if $K \ncong G'$ for any group G.

Lemma 3.5. *(i) If K is a finite group such that K has a characteristic non-central cyclic subgroup, then K is incompetent.*

(ii) The dihedral group D_n of order $2n$ is incompetent for all $n \geq 3$.

(iii) The dicyclic group Q_n of order $4n$ is incompetent for all $n \geq 3$.

(iv) Q_2 and A_4 are competent. All other non-abelian groups of order less than 16 are incompetent.

(v) If G is a non-abelian group of order pq, where p and q are distinct odd primes, then G is incompetent.

Proof. The proof of part (i) can be found in [16]. The other parts are easy consequences of part (i) and the fact that $\mathrm{SL}_2(3)' \cong Q_2$ and $S_4' \cong A_4$. □

Lemma 3.6. *Let G be a non-abelian group in $\mathcal{G}_p$ and let A be an abelian subgroup of index p in G. For $x \in G \setminus A$, let $q = |A : C_A(x)|$. Then*

$$f(G) = \frac{p+q-1}{pq}.$$

Proof. First, note that the hypotheses of the Lemma imply that $Z = C_A(x)$.

Now, $|G : C_G(a)| = p$ for any $a \in A$, since for any such a one has $C_G(a) = A$. Also, $|G : C_G(x)| = q$ for any $x \in G \setminus A$, since for any such x one has $C_G(x) = \langle Z, x\rangle$. Thus $G/C_G(x) \cong A/Z$.

So

$$\begin{aligned} k(G) &= |Z| + \frac{|A \backslash Z|}{p} + \frac{|G \backslash A|}{q} \\ &= |Z| + \frac{(q-1)|Z|}{p} + \frac{(p-1)|A|}{q} \\ &= |Z| + \frac{(q-1)|Z|}{p} + \frac{(p-1)q|Z|}{q} \\ &= |Z| \left(1 + \frac{q-1}{p} + p - 1\right) \\ &= \frac{|G|}{pq}\left[\frac{(q-1)}{p} + p\right]. \end{aligned}$$

We will now find $|G'|$ and $f(G)$. Since $|G : A| = p$ and $G \in \mathcal{G}_p$, we have $\mathrm{c.d.}(G) = \{1, p\}$. Thus $|G| = |G : G'| + p^2[k(G) - |G : G'|]$ so that $|G'| = q$. It follows that

$$T(G) = |G : G'| + p[k(G) - |G : G'|]$$

so that

$$f(G) = \frac{p+q-1}{pq}$$

as required. □

A prime divisor p of the order of a non-abelian group G is called *essential* if the Sylow p-subgroup of G is not contained in $Z(G)$.

Lemma 3.7 ([3]). *Let p be the smallest essential prime divisor of the order $|G|$ of a non-abelian group G. If $f(G) > \frac{1}{p}$, then one of the following holds:*

(1) G has an abelian subgroup of index p;

(2) $|G'| = p$;

(3) $C_p \times C_p \cong G' \subseteq Z(G)$, $\mathrm{c.d.}(G) \subseteq \{1, p, p^2\}$. If $p^2 \in \mathrm{c.d.}(G)$, then all irreducible characters of degree p^2 are such that their kernels intersect with G' and do so in the same subgroup of order p;

(4) $|G'| \geq p$, $\mathrm{c.d.}(G) = \{1, p\}$, $G/Z(G)$ has order p^3 and exponent p. If $G' \subseteq Z(G)$, then G has exponent p.

Conversely, if any of the above hold, then $f(G) > \frac{1}{p}$.

Lemma 3.8 ([14, 2]). $\mathrm{Pr}(G)$ *and* $f(G)$ *are isoclinic invariants.*

Lemma 3.9. *(1) If $H \leq G$, then $f(G) \geq f(H)$.*

(2) $f(G) \leq \frac{1}{d} + \frac{1}{|G'|}\left(1 - \frac{1}{d}\right)$, where d is the minimal degree of a non-linear representation of G.

(3) $f(G) \geq \mathrm{Pr}(G) \geq f(G)^2$.

(4) If $K \trianglelefteq G$, then $\mathrm{Pr}(G) \leq \mathrm{Pr}(K)\,\mathrm{Pr}(G/K)$.

Proof. The proof Parts 1 and 3 may be found in [2]. Part 2 follows from the fact that G has $k(G)$ irreducible complex characters. The proof of Part 4 may be found in [14]. □

4. **Main theorems**

Our first result confirms a conjecture of Berkovich [4, p. 327]:

Theorem 4.1. *If G is a non-abelian group such that $\mathrm{Pr}(G) > \frac{7}{24}$, then G' is abelian.*

Proof. Let G be a group such that $\mathrm{Pr}(G) > \frac{7}{24}$ and G' is non-abelian. We will obtain a contradiction. Note that, by Lemma 3.8 we can assume that G is a stem-group, so $Z(G) \subseteq G'$.

Claim 1: G' has order 16.

Proof: by Lemma 3.1, if $G \in \mathcal{G}_3$,

$$\mathrm{Pr}(G) \leq \frac{1}{9}\left(1 + \frac{8}{|G'|}\right) = \frac{|G'|+8}{9|G'|}$$

which implies $|G'| \leq 4$ and so $|G'|$ is abelian. Thus $G \in \mathcal{G}_2$. So,

$$\mathrm{Pr}(G) \leq \frac{1}{4}\left(1 + \frac{3}{|G'|}\right)$$

which implies $|G'| < 18$. Since G' is non-abelian we thus have $|G'| \leq 16$.

By Lemma 3.5 the non-abelian groups of order less than 16 are incompetent except for Q_2 and A_4. If $G' \cong A_4$ then Lemma 3.3 gives that $\Pr(G) \leq \frac{5}{24} < \frac{7}{24}$ and if $G' \cong Q_2$ then Lemma 3.4 gives that $\Pr(G) \leq \frac{1}{6} + \frac{1}{2^{2+1}} = \frac{7}{24}$.

By Lemma 3.5, the groups Q_4 and $\mathrm{QD}(4) \cong \langle r, s \mid r^8 = s^2 = 1, srs = r^3 \rangle$ are both incompetent. Finally, Ó' Murchú [16] shows that small groups $[16, 3]$ and $[16, 4]$ are incompetent.

Now let $H = C_G(G')$.

Claim 2: $|G/H| \geq 12$.

Proof: $H \trianglelefteq G$. If G/H is abelian, then H contains G' and G' is abelian, giving a contradiction. Furthermore, $G'/Z(G')$ is a subgroup of G/H. The former is a 2-group and must have order at least 4, if G' is non-abelian. Thus, G/H is a non-abelian group of order a multiple of 4, hence $|G/H| \geq 8$. To prove the claim, it suffices to show that G/H cannot have order 8. Suppose it does. Since G has an abelian 2-complement, so does H. Let A be an abelian 2-complement in H. Let $C = C_G(A)$. Then C contains G', hence is a normal subgroup of G. But A is just the $2'$-part of $Z(C)$, hence A is characteristic in C. Hence A is normal in G, thus in fact contained in $Z(G)$. Since $Z(G) < G_2$, it follows that $A = 1$. But if $|G/H| = 8$, then A is also a 2-complement in G, in other words, G is a 2-group, giving a contradiction.

Claim 3: Let $a \in G$. If $|G : C_G(a)| \leq 3$, then $C_G(a)$ contains G'.

Proof: This follows from the fact that G has a normal Sylow 2-subgroup.

Claim 4: $|Z(G)| \leq 4$.

Proof: $Z(G) < G'$, G' is non-abelian and $|G'| = 16$.

To finish the proof, let there be k elements of G whose centralisers do not contain G'. By Claim 1, $k \geq 11|G|/12$. By Claim 2, the centraliser of any of these k elements contains at most $|G|/4$ elements. We have at most 4 elements in $Z(G)$, by Claim 3, and any element outside $Z(G)$ has at most $|G|/2$ elements in its centraliser. Adding up, we get that

$$\Pr(G) \leq \frac{4}{|G|} + \frac{1}{2}\left(\frac{1}{12} - \frac{4}{|G|}\right) + \frac{11}{12} \cdot \frac{1}{4} = \frac{13}{48} + \frac{2}{|G|}. \tag{4.1}$$

Hence if $\Pr(G) > 7/24$, we require $13/48 + 2/|G| > 7/24$ so $|G| < 96$.

But $|G|$ is a multiple of 16 and not a 2-power. This only leaves two possibilities, $|G| = 48$ or 80. The second can be ruled out by refining Claim 1, for in this case, Claim 1 can be strengthened to $|G/H| \geq 20$. Then Equation 4.1 can be strengthened to

$$\Pr(G) <= \frac{4}{|G|} + \frac{1}{2}\left(\frac{1}{20} - \frac{4}{|G|}\right) + \frac{19}{20} \cdot \frac{1}{4}$$

which is strictly less than $7/24$ when $|G| = 80$.

This leaves $|G| = 48$. If $\Pr(G) > 7/24 = 14/48$, then in fact $\Pr(G) \geq 15/48$. If we substitute into Equation 4.1, then it's easy to check that the only possibility is $\Pr(G) = 15/48$, $|Z(G)| = 4$,

$Z(G) = Z(G') = H$, $k = 44 = 11|G|/12$, and every element of $G - Z(G)$ has a centraliser of index 4 in G. But this is impossible, for if $a \in G' - Z(G)$, then the index of $C_G(a)$ cannot be a multiple of 4.

This completes the proof. □

We note that this result is best possible since $\Pr(\mathrm{SL}_2(3)) = 7/24$ but $\mathrm{SL}_2(3)' \cong Q_2$.

Theorem 4.2. *Let G be a non-abelian group. If $f(G) > 1/2$, then $f(G) = \frac{n+1}{2n}$ for some $n \in \mathbb{N}$.*

Proof. Let $G \in \mathcal{G}_5$. Then

$$f(G) \leq \frac{2p-1}{p^2} \leq 1/2$$

for all $p \geq 5$. Hence $G \in \mathcal{G}_2$ or $\mathcal{G}_3$.

Suppose that $G \in \mathcal{G}_3$. Since $f(G) > 1/2 > 1/3$, Lemma 3.7 says that one of the following holds:

(a) G has an abelian subgroup of index 3;

(b) $|G'| = 3$;

(c) $G' \simeq C_3 \times C_3$ and $G' \leq Z(G)$ and $\mathrm{c.d.}(G) \subseteq \{1, 3, 9\}$. Moreover, if there is an irreducible character of degree 9, then all such have kernels that intersect G' in one and the same subgroup of order 3;

(d) $|G'| > 3, \mathrm{c.d.}(G) = \{1, 3\}, |G : Z(G)| = 27$ and $G/Z(G)$ has exponent 3. Moreover, if $G' \leq Z(G)$, then G' has exponent 3.

We now consider each of these sub-cases separately.

Case (a): G has an abelian subgroup A of index 3.

For $x \in G\backslash A$, let $q = |A : C_A(x)|$. Then Lemma 3.6 gives that $1/2 < f(G) = \frac{3+q-1}{3q} = \frac{q+2}{3q}$ which implies that $q < 4$. Since $G \in \mathcal{G}_3$ and $f(G) > 1/2$, we must have $q = 3$ and so $f(G) = 5/9$.

Case (b): $|G'| = 3$.

Since $G \in \mathcal{G}_3$, $|G'| = 3$ implies that $f(G) = \frac{3^n+2}{3^{n+1}}$. If $f(G) > 1/2$, then $3^n < 4$ so that $n \in \{0, 1\}$. We cannot have $n = 0$ since G is non-abelian. Thus $n = 1 \Rightarrow f(G) = 5/9$.

Case (c): $G' \simeq C_3 \times C_3$ and $G' \leq Z(G)$ and $\mathrm{c.d.}(G) \subseteq \{1, 3, 9\}$. *Moreover, if there is an irreducible representation of degree* 9, *then all such have kernels that intersect G' in one and the same subgroup of order* 3.

From Passman [17], if $\mathrm{c.d.}(G) \subseteq \{1, p, p^2\}$, then $\bigcap \ker \theta$, where θ runs over all irreducible characters of degree p^2, is trivial for $p = 3$. Thus $\mathrm{c.d.}(G) = \{1, 3\}$ so the degree equation for G gives

$$|G| = \frac{|G|}{9} + 9\left[k(G) - \frac{|G|}{9}\right]$$

whence $k(G) = \frac{17}{81}|G|$. Thus $f(G) = 11/27 < 1/2$.

Case (d): $|G'| > 3, \mathrm{c.d.}(G) = \{1, 3\}, |G : Z(G)| = 27$ and $G/Z(G)$ has exponent 3. *Moreover, if $G' \leq Z(G)$, then* $\exp G' = 3$.

From MacHale et. al. [6], we see that $|G : Z(G)| = 27$ implies that G/Z is isomorphic to the group $C_3 \times C_3 \times C_3$ or to the non-abelian group of order 27 and of exponent 3. Since $f(G)$ is an isoclinic invariant, we may assume without loss of generality that G is a stem-group. If G/Z is abelian, then $G' \simeq C_3 \times C_3$ or $G' \simeq C_3 \times C_3 \times C_3$. Also $G' = Z(G)$ so that $|G| = 729$ or $|G| = 243$. The corresponding values of $f(G)$ are $20/81, 11/27$ respectively. Each of these values is less than $1/2$.

If G/Z is non-abelian, then $(G/Z)' \simeq C_3$ and $|G'| = 9$ or 27. If $|G'| = 9$, then, assuming that G is a stem-group, we have $Z(G) \simeq C_3$ so that $|G| = 81$ and so $f(G) = 11/27$. If $|G'| = 27$, then $|G| = 243$ and $f(G) = 87/243 < 1/2$.

Now we suppose that $G \in \mathcal{G}_2$. Since $f(G) > 1/2$, then Lemma 3.7 again shows that one of the following holds:

(a) G has an abelian subgroup of index 2;
(b) $|G'| = 2$;
(c) $G' \simeq C_2 \times C_2$, G' is central and $\mathrm{c.d.}(G) \subseteq \{1, 2, 4\}$. If there is an irreducible representation of degree 4, then all such have kernels that intersect G' in one and the same subgroup of order 2;
(d) $|G'| > 2$, $\mathrm{c.d.}(G) = \{1, 2\}$, $|G : Z(G)| = 8$ and G/Z has exponent 2. If G' is central, then exp $G' = 2$.

We will again deal with each of these cases separately.

Case (a): G has an abelian subgroup of index 2.

Let A be an Abelian subgroup of index 2 in G. Let $q = |A : C_A(x)|$. Then, from Lemma 3.6, $f(G) = \frac{q+1}{2q} > 1/2$ for all $q \geq 1$. Thus $f(G)$ is of the required form.

Case (b): $|G'| = 2$.

In this case we have $f(G) = \frac{2^s+1}{2^{s+1}} > 1/2$ which is of the required form.

Case (c): $G' \simeq C_2 \times C_2$, G' is central and $\mathrm{c.d.}(G) \subseteq \{1, 2, 4\}$. *If there is an irreducible representation of degree* 4, *then all such have kernels that intersect G' in one and the same subgroup of order* 2.

If $\mathrm{c.d.}(G) = \{1, 2\}$, then from Passman [17], $|G : Z|$ divides 8. Since $G' \simeq C_2 \times C_2$, the only possibility is that $G/Z \simeq C_2 \times C_2 \times C_2$ so that, if G is a stem-group, $|G| = 32$ and $f(G) = 5/8 = \frac{4+1}{2 \cdot 4}$. If $\mathrm{c.d.}(G) = \{1, 4\}$, then $|G| = |G : G'| + 16m$ where m denotes the number of non-linear representations of G. Since $|G : G'| = |G|/4$, it follows that $m = \frac{3}{64}|G|$ so that $f(G) = 1/4 + 3/16 = 7/16 < 1/2$. If $\mathrm{c.d.}(G) = \{1, 2, 4\}$, then, since all irreducible representations of degree 4 have kernels that intersect G' in one and the same subgroup of order 2, it follows that the intersection of these kernels is non-trivial. From Passman, $|G : Z| = 16$. Also G' is central so that G is nilpotent class 2. If G is a stem-group, then $|G| = 64$. The only groups G of order 64 with $|G : Z| = 16, G' \simeq Z(G) \simeq C_2 \times C_2$ and $\mathrm{c.d.}(G) =$

$\{1,2,4\}$ are groups $[64,226]$ to $[64,240]$ inclusive. For each of these groups, $f(G) = 9/16 = \frac{8+1}{2\cdot 8}$.

Case (d): $|G'| > 2$, $\mathrm{c.d.}(G) = \{1,2\}$, $|G : Z(G)| = 8$ *and* G/Z *has exponent* 2. *If* G' *is central, then* $\exp G' = 2$.

Thus $G/Z \simeq C_2 \times C_2 \times C_2$ so that, if G is a stem-group, $G' \simeq Z(G) \simeq C_2 \times C_2$ or $C_2 \times C_2 \times C_2$. If $G' \simeq C_2 \times C_2$, then $|G| = 32$ so that $T(G) = 8 + 6 \cdot 2$. Thus $f(G) = 5/8 > 1/2$. If $G' \simeq C_2 \times C_2 \times C_2$, then $T(G) = 8 + 14 \cdot 2$ whence $f(G) = 9/16 > 1/2$. Each of these values is of the required form. □

Dixon [9] shows that if $\mathrm{Pr}(G) > \frac{1}{12}$, then G is not a simple group. We will now prove that if $f(G) > 4/15$, then G is not a simple group. Our proof relies on Dixon's result.

Theorem 4.3. *Let* G *be a non-abelian group. If* $f(G) > 4/15$*, then* G *is not simple.*

Proof. Assume that there exists a simple, non-abelian group G with the property that $f(G) > 4/15$. Then, by Lemma 3.9, $\mathrm{Pr}(G) > (f(G))^2 > 16/225$.

If $\mathrm{Pr}(G) > 1/12$, then G is not simple so we assume that $1/12 \geq \mathrm{Pr}(G) > 16/225$. Thus $k(G) > \frac{16}{225}|G|$ so that $|G| < \frac{225}{16}k(G) < 15k(G)$.

Now we suppose that $|x^G| \geq 19$ for each $x \neq 1, x \in G$. Then, since G is simple, $|x^G| > 19$ for each $x \neq 1, x \in G$ so that

$$\begin{aligned} |G| &> 1 + 19(k(G) - 1) = 1 + 19k(G) - 19 \\ &= 19k(G) - 18 = 15k(G) + 4k(G) - 18 \\ &> |G| + 4 \cdot 5 - 18 = |G| + 2 > |G|, \end{aligned}$$

a contradiction.

Thus there exists $x \in G, x \neq 1$ such that $|x^G| = n$ for some $n \in \{6,10,12,14,15,18\}$. From the Atlas of Finite Groups [8], we see that there is no simple subgroup H of S_n for $n \in \{6,10,12,14,15,18\}$ with $f(H) > \frac{4}{15}$ which gives us our contradiction and so the theorem is proved. □

If we are prepared to make use of deeper results, we can prove the following

Theorem 4.4. *If* G *is a group such that* $f(G) > \frac{4}{15}$*, then* G *is soluble.*

Proof. Suppose that G is an insoluble group of minimal order such that $f(G) > 4/15$. Since G has minimal order and $f(G') \geq f(G)$, G must be perfect and so G has only one linear representation. Since

$$f(G) \leq \frac{1}{d} + \frac{1}{|G'|}\left(1 - \frac{1}{d}\right)$$

where d is the minimum degree of a nonlinear representation of G and since $G = G'$, G must have an irreducible representation of degree 2 or 3, as otherwise we would have $f(G) \leq \frac{1}{4} + \frac{3}{4}|G|$ and, since $f(G) > 4/15$, this would imply $|G| < 45$, contradicting the insolubility of G.

So, let θ be a representation of degree 2 or 3 and $K = \ker(\theta)$. By minimality of G, G/K cannot be soluble. Now let M be the kernel of θ composed with the natural projection onto $\mathrm{PGL}_d(\mathbb{C})$. Then G/M is an insoluble subgroup of $\mathrm{PGL}_d(\mathbb{C})$ and $\Pr(G/M) > (4/15)^2$. By the classification of finite subgroups of $\mathrm{PGL}_2(\mathbb{C})$ and $\mathrm{PGL}_3(\mathbb{C})$ [5] the only possibility for G/M is A_5. Thus G/K must have A_5 as a central quotient. By the classification of finite subgroups of $\mathrm{SL}_2(\mathbb{C})$ [19] and $\mathrm{SL}_3(\mathbb{C})$ [5, 20] we must have $G/K = A_5$ or $2I$.

Finally, since $\Pr(A_5) = 1/12$, $\Pr(2I) = 3/40$ and $f(G)^2 \leq \Pr(G) \leq \Pr(K)\Pr(G/K)$, we must have $\Pr(K) = 1$, that is K must be abelian.

Let M be a maximal abelian normal subgroup of G. Thus $G/M = A_5$ or $2I$. It follows that any irreducible representation of G is either an irreducible representation of G/M or has degree greater than or equal to 4. Since $f(A_5) = 4/15$ and $f(2I) = 1/4 < 4/15$, it follows from the degree equation that also $f(G) \leq 4/15$, giving a contradiction. □

We note that this result is best possible since $f(A_5) = 4/15$.

Lemma 4.5. *If G is a non-abelian group of odd order such that G' is non-nilpotent then $|G'| \geq 75$.*

Proof. Suppose G is a finite group such that $|G|$ is odd and G' is non-nilpotent. Recall that if $|G'| = p, p^2$ or pq where p and q are primes such that $p < q$ and p divides $q - 1$, then G' is abelian, giving a contradiction.

Lemma 3.5 states that if H is non-abelian of order pq, where p and q are odd primes with $p > q$ and $p \equiv 1 \pmod{q}$ then H is incompetent. Note also that, while there are two non-abelian groups of order 27, both of these are nilpotent and that the groups of order 45 are abelian.

Now, suppose $|G'| = 63$. There are two non-nilpotent groups of order 63, namely $[63, 1]$ and $[63, 3]$. Suppose $G' \cong [63, 3] \cong C_3 \times G_{21}$. Let

$$[63, 3] \cong \langle x, a, b \mid x^3 = a^7 = b^3 = [a, x] = [b, x] = 1, a^b = a^2\rangle$$

where x generates C_3 and a and b generate G_{21}. Then $\langle a\rangle$ is a characteristic non-central cyclic subgroup of G' and so, by Lemma 3.5, G' is incompetent giving a contradiction.

Suppose $G' \cong [63, 1]$ which may be given by the presentation

$$\langle a, b \mid a^7 = b^9 = 1, a^b = a^2\rangle.$$

Then $Z(G') = \{1, b^3, b^6\}$ and $\langle a\rangle$ is a characteristic non-central cyclic subgroup of G' and so, by Lemma 3.5, G' is incompetent giving a contradiction.

So, $|G'| \geq 75$. The group $[75, 2]$ has no characteristic non-central cyclic subgroup and may be competent. Thus it is possible that $|G'| = 75$. □

Lemma 4.5 suggests the following results:

Theorem 4.6. *If G is a non-abelian group of odd order such that $\Pr(G) > \frac{83}{675}$, then G' is nilpotent.*

Proof. Suppose $|G|$ is odd and $\Pr(G) > \frac{83}{675}$. If $G \in \mathcal{G}_p$ with $p \geq 11$ then, by Lemma 3.1,

$$\Pr(G) \leq \frac{p^2+p-1}{p^3} = \frac{131}{1331}(\approx 0.09842) < \frac{83}{675}(\approx 0.12296)$$

so we can assume that $G \in \mathcal{G}_3$, $G \in \mathcal{G}_5$ or $G \in \mathcal{G}_7$.

If $G \in \mathcal{G}_3$ then

$$\frac{83}{675} < \Pr(G) \leq \frac{1}{9}\left(1 + \frac{8}{|G'|}\right) = \frac{1}{9} + \frac{8}{9|G'|}$$

which implies $|G'| < 75$. By Lemma 4.5, if G is an odd-order group and $|G'| < 75$ then G' is nilpotent.

If $G \in \mathcal{G}_5$ then

$$\frac{83}{675} < \Pr(G) \leq \frac{1}{25}\left(1 + \frac{24}{|G'|}\right) = \frac{1}{25} + \frac{24}{25|G'|}$$

which implies $|G'| \leq 11$ and again, by Lemma 4.5, G' is nilpotent. Finally, if $G \in \mathcal{G}_7$ then

$$\frac{83}{675} < \Pr(G) \leq \frac{1}{49}\left(1 + \frac{48}{|G'|}\right) = \frac{1}{49} + \frac{48}{49|G'|}$$

which implies $|G'| < 1$ giving a contradiction. □

Theorem 4.7. *If G is a non-abelian group of odd order such that $f(G) > \frac{77}{225}$, then G' is nilpotent.*

Proof. Suppose that $f(G) > \frac{77}{225}$. If $G \in \mathcal{G}_p$ with $p \geq 7$ then, by Lemma 3.2,

$$f(G) \leq \frac{13}{49}(\approx 0.2653) < \frac{77}{225}(= 0.34222\ldots)$$

so we may assume that $G \in \mathcal{G}_3$ or $G \in \mathcal{G}_5$.

If $G \in \mathcal{G}_3$ then, by Lemma 3.2,

$$\frac{77}{225} < \frac{1}{3} + \frac{2}{3|G'|}$$

which implies $|G'| < 75$ and so, by Lemma 4.5, G' is nilpotent.

If $G \in \mathcal{G}_5$ then, by Lemma 3.2,

$$\frac{77}{225} < \frac{1}{5} + \frac{4}{5|G'|}$$

which implies $|G'| \leq 5$ and so, by Lemma 4.5, G' is nilpotent. □

We note that the results of Theorem 4.6 and 4.7 may not be best possible. The only odd order-group G with $|G| \leq 2023$ and G' non-nilpotent is $[1053, 51]$ which has derived subgroup isomorphic to $[351, 12]$. We can use GAP to see that $\Pr([1053, 51]) = \frac{1}{81}(\approx 0.012346\ldots)$ which is significantly less than $\frac{83}{675}(\approx 0.12296\ldots)$. Similarly, $f(1053/51) = \frac{31}{351}(\approx 0.088319\ldots)$ which is significantly less than $\frac{77}{225}(\approx 0.34222\ldots)$.

If G is a finite supersoluble group, then G' is nilpotent. However, not every finite nilpotent group is the commutator subgroup of a supersoluble group. In fact, Q_2 is the smallest nilpotent group which is not the commutator subgroup of a supersoluble group. Burnside [7] showed that Q_2 is not the commutator subgroup of a 2-group and it is easy to deduce that Q_2 is not the commutator subgroup of a nilpotent group. However, $\mathrm{SL}_2(3)' \cong Q_2$ showing that Q_2 is the commutator subgroup of a soluble group.

Theorem 4.8. *There is no finite supersoluble group G such that $G' \cong Q_2$.*

Proof. Suppose a supersoluble G exists with $G' \cong Q_2$. Consider $G' \cap Z$. This is a characteristic abelian subgroup of Q_2 and contains the unique involution in Q_2. Thus $G' \cap Z \cong C_2$. So $(G/Z)' \cong G'/(G' \cap Z) \cong C_2 \times C_2$.

Now, by [15] if $K' \cong C_2 \times C_2$, then either K is nilpotent or $K/Z(K) \cong A_4$. Now if G/Z is nilpotent, then G is nilpotent, contradicting Burnside's result, so $(G/Z)/Z(G/Z) \cong A_4$, a contradiction since A_4 is not supersoluble. □

Theorem 4.9. *If $G' \cong C_{2^n}$, then G is nilpotent of class at most $n+1$.*

Proof. We use induction on n. For $n = 1$, $G' \cong C_2$. The unique involution in G' is invariant under all automorphisms of G and is therefore central in G. Thus $G' \subseteq Z$ and so G is nilpotent of class 2.

Assume that if $G' \cong C_{2^k}$ for $k < n$, then G is nilpotent of class at most $k+1$. Suppose that $G' \cong C_{2^n}$.

Now $|G' \cap Z| > 1$ since the unique involution in G' is central in G. So, $(G/Z)' \cong G'/(G' \cap Z)$ and so $(G/Z)'$ is a cyclic 2-group of order less than 2^n. By our inductive hypothesis G/Z is nilpotent of class at most n. So G is nilpotent of class at most $n+1$. □

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Robert Heffernan

Department of Mathematics, University of Connecticut, Storrs, USA

Email: robert.heffernan@uconn.edu

Des MacHale

Department of Mathematics, University College Cork Cork, Ireland

Email: d.machale@ucc.ie

Áine Ní Shé

Department of Mathematics, Cork Institute of Technology Cork, Ireland

Email: aine.nishe@cit.ie