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THE EXACT SPREAD OF M_{23} IS 8064

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Abstract. Let G be a finite group. We say that G has *spread* r if for any set of distinct non-trivial elements of G $X := \{x_1, \ldots, x_r\} \subset G^{\#}$ there exists an element $y \in G$ with the property that $\langle x_i, y\rangle = G$ for every $1 \leq i \leq r$. We say G has *exact spread* r if G has spread r but not $r+1$. The spreads of finite simple groups and their decorations have been much-studied since the concept was first introduced by Brenner and Wiegold in the mid 1970s. Despite this, the exact spread of very few finite groups, and in particular of the finite simple groups and their decorations, is known. Here we calculate the exact spread of the sporadic simple Mathieu group M_{23}, proving that it is equal to 8064. The precise value of the exact spread of a sporadic simple group is known in only one other case - the Mathieu group M_{11}.

1. Introduction

We recall the following generalization of the concept of 3/2-generation that was first introduced by Brenner and Wiegold in [3].

Definition 1.1. *Let G be a group. We say that G has* spread *r if for any set of distinct non-trivial elements $X := \{x_1, \ldots, x_r\} \subset G^{\#}$ there exists an element $y \in G$ with the property that $\langle x_i, y\rangle = G$ for every $1 \leq i \leq r$. We say that y is a* mate *to X. We say G has* exact spread *$r := s(G)$ if G has spread r but not $r+1$.*

There has been considerable interest in determining, or at least bounding, the exact spreads of the finite simple groups and in particular the exact spreads of the sporadic simple groups - see any of [1, 2, 5, 6, 7, 8] and the references therein. The precise value of the exact spread of a finite simple group is known in only a few cases. Here we prove the following.

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Theorem 1.2. $s(M_{23}) = 8064$

We remark that the exact spread of a sporadic simple group is known in only one other case: $s(\mathrm{M}_{11}) = 3$ [1, 8].

2. Proof of Theorem 1.2

Proof. First note that from [5, Table 1] (itself a heavily corrected version of [1, Table 1]) we have that $s(\mathrm{M}_{23}) \leq 8064$ and so it is sufficient to show that any set of 8064 elements from $\mathrm{M}_{23}^{\#}$ has a mate. Let $X \subset \mathrm{M}_{23}$ be a set of 8064 distinct elements.

Now, from the maximal subgroups of M_{23} listed in [4, p.71] we see that an element of order 23 is contained in only one maximal subgroup - a copy of the Frobenius group 23:11. Since the normalizer in M_{23} of a cyclic subgroup of order 11 is a Frobenius group 11:5 and each cyclic subgroup of 23:11 of order 11 is self normalizing it follows that each element of order 11 is contained in 55/11=5 distinct copies of 23:11. Therefore, the only way X can avoid having a mate of order 23 is if X consists of a well chosen configuration of elements of order 11.

Finally, note that of the maximal subgroups of M_{23} containing elements of order 11, which are each isomorphic to one of M_{22}, M_{11} or 23:11, none contain elements of order 14. It follows that even if X only contains elements of order 11, X must have a mate and so $s(\mathrm{M}_{23}) \geq 8064$. $\square$

We remark that what enabled us to improve the lower bound on the exact spread of M_{23} was the existence of a maximal frobenious subgroup. Several other sporadic subgroups have similar such maximal subgroups: J_1 [4, p.36], Ly [4, p.174], Th [4, p.177], J_4 [4, p.190], Fi_{24}' [4, p.207], $\mathbb{B}$ [4, p.217] and $\mathbb{M}$ [4, p.234]. In these cases, better upper bounds on the spreads would be required for similar arguments to be applied.

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