www.theoryofgroups.ir

International Journal of Group Theory
ISSN (print): 2251-7650 , ISSN (on-line): 2251-7669
Vol. 01 No.1 (2012), pp. 39-45.

www.ui.ac.ir

LOCALLY FINITE p-GROUPS WITH ALL SUBGROUPS EITHER SUBNORMAL OR NILPOTENT-BY-CHERNIKOV

G. CUTOLO* AND H. SMITH

Communicated by Patrizia Longobardi

Abstract. We pursue further our investigation, begun in [H. Smith, Groups with all subgroups subnormal or nilpotent-by-Chernikov, *Rend. Sem. Mat. Univ. Padova* 126 (2011), 245–253] and continued in [G. Cutolo and H. Smith, Locally finite groups with all subgroups subnormal or nilpotent-by-Chernikov. *Centr. Eur. J. Math.* (to appear)] of groups G in which all subgroups are either subnormal or nilpotent-by-Chernikov. Denoting by $\mathfrak{X}$ the class of all such groups, our concern here is with locally finite p-groups in the class $\mathfrak{X}$, where p is a prime, while an earlier article provided a reasonable classification of locally finite $\mathfrak{X}$-groups in which all of the p-sections are nilpotent-by-Chernikov. Our main result is that if G is a Baer p-group in $\mathfrak{X}$ then G is nilpotent-by-Chernikov .

1. Introduction

The main result of the article [13] is that a locally soluble-by-finite group G that lies in the class $\mathfrak{X}$ of groups that have all subgroups either subnormal or nilpotent-by-Chernikov (that is, subnormal or in $\mathfrak{NC}$) is soluble-by-finite, and such a group G is even soluble if it is not nilpotent-by-Chernikov. The original motivation for discussing the class $\mathfrak{X}$ is that if G is a group and *either* every subgroup of G is subnormal *or* G is locally graded and every subgroup of G lies in $\mathfrak{NC}$ then G is soluble — we refer the reader to the papers [8], [9] and [1] for these results, and to the introductions in [13] and [3] for more detail and for further discussion.

Further consideration of the class $\mathfrak{X}$ leads immediately to the problem of determining reasonable conditions on an $\mathfrak{X}$-group G that suffice to ensure that G is in $\mathfrak{NC}$. In view of the fact that there are groups G that have all subgroups subnormal but are not nilpotent-by-Chernikov (see [10] and [12]),

MSC(2010): Primary: 20E15; Secondary: 20F50, 20F19.

Keywords: locally finite groups, subnormal subgroups, nilpotent-by-Chernikov groups.

Received: 21 July 2011, Accepted: 08 December 2011.

* Corresponding author.

we restricted our discussion in [3] to locally finite groups in $\mathfrak{X}$, and we showed that there are locally finite groups in $\mathfrak{X}$ but not in $\mathfrak{NC}$. Indeed, we were able to provide a reasonable necessary and sufficient condition for a locally finite $\mathfrak{X}$-group G to fail to belong to $\mathfrak{NC}$, provided that it is assumed at the outset that all p-sections of G are nilpotent-by-Chernikov. This condition involves the presence of a certain section of G that is described in terms of the structure of locally finite fields, and here we must refer the reader to [3] for the details.

As far as locally finite groups in $\mathfrak{X}$ are concerned, what remains to be considered is primarily the case where G is a p-group for some prime p — an easy argument presented in Lemma 2 of [13] shows that a locally finite-nilpotent group in $\mathfrak{X} \setminus \mathfrak{NC}$ has exactly one primary component P that is not in $\mathfrak{NC}$, and G/P is in $\mathfrak{NC}$. Let us recall that a group H is a *Baer group* if every cyclic subgroup (equivalently, every finitely generated subgroup) of H is subnormal, and that every Baer group is locally nilpotent. Our main result is as follows.

Theorem 1. *Let G be a locally finite Baer group in the class $\mathfrak{X}$. Then $G \in \mathfrak{NC}$.*

Of course, what we would really like to know is whether every locally finite-nilpotent group in $\mathfrak{X}$ is nilpotent-by-Chernikov, and although we have made considerable progress towards answering this more general question, nevertheless we have to leave it unanswered here. However, it is perhaps worth pointing out the following.

Firstly, the property of being a Baer group is one that is immediately to hand if one wishes to show that a p-group with all subgroups subnormal is nilpotent-by-Chernikov — that a p-group (and hence any periodic group) with all subgroups subnormal is indeed nilpotent-by-Chernikov was established by Casolo in [2], and it constitutes a major result in this area.

Secondly, the property of being a Baer group, though not immediate, is quite easy to establish if one is assuming that G is a locally finite p-group in which all proper subgroups are in $\mathfrak{NC}$ and attempting to show that G is soluble (and hence in $\mathfrak{NC}$) — that a locally finite p-group (and hence an arbitrary locally graded group, because of results from [9]) with all proper subgroups in $\mathfrak{NC}$ is itself in $\mathfrak{NC}$ is due to Asar [1], and is also a major result, of course. Our point here is that if we assume (as we may) for a contradiction that G is perfect then G is the product of all its proper normal subgroups and so there are G-invariant subgroups $L \leq K$ with L a product of normal nilpotent subgroups and hence a Fitting group (hence a Baer group), K/L a product of normal divisible abelian Chernikov groups and hence abelian, and G/K a product of finite G-invariant subgroups (and hence an FC-group). Because G/K is perfect it is trivial in this case, and therefore so is K/L, which means that $G = L$ and G is a Baer group.

2. **Preliminary results**

Our first lemma collects, for convenience, a few results that are either well-known or very similar to results that are well-known.

Lemma 2.1.

(a) *Let G be a group and let H, K be subgroups of G such that $H \lhd K \lhd G$, H is nilpotent and K/H is Chernikov. Then H^G is nilpotent.*

(b) *Let G be a locally finite Baer group in the class $\mathfrak{X}$ and suppose that G has a normal nilpotent subgroup N such that G/N has finite exponent. Then G is nilpotent.*

(c) *Let G be a nilpotent p-group such that G' has exponent p. Then $G/Z(G)$ has finite exponent.*

(d) *Let G be a p-group with an elementary abelian normal subgroup A containing G', and let $C = C_G(A)$. Then $C/Z(G)$ is elementary abelian.*

(e) *Let G be a Baer p-group and suppose that G has a normal nilpotent subgroup N such that G/N is divisible. If G is residually nilpotent then G is nilpotent.*

Proof. Part (a) follows immediately from Lemma 1 of [5]. Let G be as stated in (b) and let H be an arbitrary non-subnormal subgroup of G. Then H is nilpotent, and by Theorem 3 of [11] we have that every subgroup of G is subnormal. Theorem 12 of [7] now gives G nilpotent. For (c), let m be the nilpotency class of G, choose n such that $p^n \geq m$ and let $x, y \in G$. Since the binomial coefficient $\binom{p^n}{j}$ is divisible by p if $1 \leq j < p^n$ we have from a standard commutator formula that $[x, y^{p^n}] = [x,_{p^n} y] = 1$, and the result follows. Now let G be as in (d) and let $g \in G$, $x \in C$. Then $[x^p, g] = [x, g]^p = 1$ and so $C^p \leq Z(G)$; also $[C', G] \leq [C, G, C][G, C, C] = 1$, as required.

Finally, let G be a group that satisfies the hypotheses of (e) and suppose first that $G/Z(N)$ is nilpotent. We claim that $X := Z(N) \leq Z(G)$, and assuming this false we have by residual nilpotency that $[X, G, G] < [X, G]$, so that we may factor and hence assume that $[X, G, G] = 1$. Choose an element a of X that has order p modulo $Z(G)$, and consider the homomorphism $\theta : G \to [X, G]$ given by $\theta(g) = [a, g]$ for all $g \in G$. The image of θ has exponent exactly p, but $G/\ker(\theta)$ is divisible and we have a contradiction that establishes the claim. It remains only to show that G/X is residually nilpotent, for then by induction on the class of N we may assume G/X nilpotent and hence G nilpotent, by the above. Since G/N is a divisible Baer p-group and hence abelian, there is a set $\{K_\lambda : \lambda \in \Lambda\}$ of G-invariant subgroups of N with trivial intersection such that each G/K_λ is nilpotent. Then $L := \bigcap_{\lambda \in \Lambda} XK_\lambda \leq N$, and $[L, N] \leq \bigcap_{\lambda \in \Lambda} K_\lambda = 1$. Thus $L = X$ and the result follows. □

Next we need some notation from [2]. An abelian p-group A is *small* if it is the direct product of a divisible Chernikov group and a group of finite exponent, otherwise A is *large*. The following result will be useful in helping us to reduce the proof of the theorem to that of the metabelian case. The proof of this lemma follows closely (and in most places exactly) that of Theorem 1 of [2] (and that of Lemma 2 of [14]).

Lemma 2.2. *Let G be a Baer p-group in the class $\mathfrak{X}$ and let H, K be subgroups of G such that $K \lhd H \lhd G$, K is nilpotent, G/H is abelian and H/K is divisible Chernikov. Suppose further that every nilpotent-by-abelian subgroup of G is in $\mathfrak{NC}$. Then $G \in \mathfrak{NC}$.*

Proof. Suppose the result false and let G be a counterexample with the rank r of H/K minimal for some H and K. Then $K < H$ and $r > 0$. Since K^G is nilpotent, by Lemma 2.1(a), we may assume $K \lhd G$. Since G is a Baer group H/K is central in G/K, and Lemma 10 of [2] applies: if G/H is small then there is a normal subgroup N/K of G/K that has finite exponent and is such that G/N is divisible Chernikov. Then N is nilpotent, by Lemma 2.1(b), and since G/N is abelian we have the contradiction $G \in \mathfrak{NC}$. Thus G/H is large.

Let W/K be a normal subgroup of G/K that is maximal such that $W \cap H = K$. Then $W' \leq K$ and so W is nilpotent-by-abelian and there is a normal subgroup Y of W such that $K \leq Y$ and W/Y is Chernikov; we may assume that W/Y is divisible and, by Lemma 2.1(a), that Y is G-invariant. If G/HW is small then G/Y is (divisible-Chernikov)-by-(finite exponent), and by Lemma 10 of [2] we see that G/Y is (finite-exponent)-by-divisible Chernikov. Again by Lemma 2.1(b) we obtain the contradiction $G \in \mathfrak{NC}$. Thus G/HW is large, and by Lemma 10 of [2] there is a large abelian subgroup X/W of G/W such that $X \cap HW = W$. Thus $X \cap H = K$ and X/K is abelian, and again by hypothesis there is a normal nilpotent subgroup U of X such that $K \leq U$ and X/U is divisible Chernikov. Since $U \lhd HU \lhd G$ and HU/U is Chernikov (since $U \geq K$) we see from Lemma 2.1(a) that U^G is nilpotent and contained in HU; also $H/H \cap U^G$ is divisible and hence of rank r, by minimality. Thus $(H \cap U^G)/K$ is finite. But $U^G/U = U(H \cap U^G)/U \cong (H \cap U^G)/(H \cap U) = (H \cap U^G)/K$, so U^G/U is finite. Since XU^G/U^G is divisible it is central in G/U^G; hence $X^G = XU^G$.

Next we observe that $X^G/X \cong U^G/(X \cap U^G) = U^G/U$, which is finite. Thus there is a positive integer n such that $M := (X^G)^{p^n}$ is contained in X, and so X contains the normal subgroup WM of G. Then $WM \cap H = K$, and by choice of W we have $M \leq W$ and hence $X^{p^n} \leq W$, contradicting the fact that X/W is large. This completes the proof of the lemma. □

3. **Proof of the theorem**

Throughout this final section G is a group that satisfies the hypotheses of the theorem. Firstly we present two results dealing with special cases of our main result.

Lemma 3.1. *If G^* is a section of G that is a residually nilpotent p-group then G^* is nilpotent.*

Proof. Let H be a non-subnormal subgroup of G^*. Then H is nilpotent-by-Chernikov and hence nilpotent-by-divisible, and it follows from Lemma 2.1(e) that H is nilpotent. Hence every subgroup of G^* is either subnormal or nilpotent, and by Theorem 3 of [11] we have that every subgroup of G^* is subnormal. But a residually nilpotent group with this property is nilpotent if it is locally finite, by (for example) Theorem 3 of [2]. The lemma is thus proved. □

Proposition 3.2. *Suppose that $G = AH$, where A is an elementary abelian p-group containing G' and $H \in \mathfrak{NC}$. Then $G \in \mathfrak{NC}$.*

Proof. There is a normal nilpotent subgroup H_0 of H with H/H_0 Chernikov. If $AH_0 \in \mathfrak{NC}$ then there is a normal subgroup K_0 of H_0 such that H_0/K_0 is Chernikov and AK_0 is nilpotent, and since G/AK_0

is Chernikov the result follows; thus we may replace H by H_0 and assume that H is nilpotent. If c is the nilpotency class of H then $[H \cap A,_c G] = [H \cap A,_c H] = 1$, so $H \cap A \leq Z_c(G)$ and we may factor and hence assume that $H \cap A = 1$, that is, H is abelian and $G = A \rtimes H$. Again factoring if necessary, we may assume that $C_H(A) = 1$. If R denotes the nilpotent residual of G then G/R is nilpotent, by Lemma 3.1, and so $R = [A,_t G]$ for some integer t. We may replace A by R and hence assume that $[A, H] = A$.

Suppose that $K \leq H$ and that B is a proper K-invariant subgroup of A. If $BK \notin \mathfrak{NC}$ then BK is subnormal in G and so, for some integer r, $[A,_r K] \leq BK \cap A = B$. In particular, $[A, K] < A$, and since $[A, H] = A$ we have $[A, K]H$ non-subnormal and hence in $\mathfrak{NC}$; thus there is a subgroup M of H with H/M Chernikov and $[A, K]M$ nilpotent. Let $L = M \cap K$; then $[A, L]L$ is nilpotent and so AL is nilpotent, and since K/L is Chernikov we have AK/AL Chernikov and hence $AK \in \mathfrak{NC}$, which gives the contradiction that $BK \in \mathfrak{NC}$. Thus $BK \in \mathfrak{NC}$ for every subgroup K of H and proper K-invariant subgroup B of A.

Let K be a large subgroup of H and suppose that $[A, K] < A$. By the above, there is a subgroup K_0 of K with K/K_0 Chernikov and $[A, K]K_0$ nilpotent. Then AK_0 is nilpotent and, by Lemma 2.1(c), $K_0^{p^t}$ centralizes A for some integer t. Since $C_K(A) = 1$ we have $K_0^{p^t} = 1$ and hence the contradiction that K is small. This shows that $[A, K] = A$ for every large subgroup K of H.

Now let h be a nontrivial element of H. Since G is Baer we have $D := [A, \langle h \rangle] < A$; also, $D \lhd G$ and $DH \in \mathfrak{NC}$. So there is a subgroup K of H with H/K Chernikov and DK nilpotent, and this implies that K^{p^m} centralizes D, for some $m \in \mathbb{N}$. By the three-subgroup lemma it follows that $[A, K^{p^m}, \langle h \rangle] = 1$. Again because $C_H(A) = 1$ we see that $[A, K^{p^m}] < A$, and by what was shown above we now have K^{p^m} small and hence H small. By Lemma 2.1(b) we deduce that G is nilpotent-by-Chernikov, as required. $\square$

We return now to our general hypothesis: let G be a locally finite Baer group in the class $\mathfrak{X}$, and suppose for a contradiction that $G \notin \mathfrak{NC}$. Our first aim is to reduce to the case where G is a metabelian p-group. By Lemma 2 of [13] there is exactly one primary component of G that is not in $\mathfrak{NC}$, so we may assume that G is a p-group for some prime p. By Proposition 2 of [13] G is soluble, and assuming that G is a counterexample of minimal derived length there is a normal abelian subgroup B of G such that $G/B \in \mathfrak{NC}$, and by means of Lemma 2.1(a) we may pass to an appropriate subgroup of G and assume that G/B is nilpotent. By minimality we have $G' \in \mathfrak{NC}$, and so by a second application of Lemma 2.1(a) there is a normal nilpotent subgroup A of G such that G/A is Chernikov-by-abelian; we may assume that $B \leq A$ and, since G is a Baer group, we may choose A so that K/A is divisible (Chernikov) for some K that contains G'.

By Lemma 2.2 we may now assume that G' is nilpotent. By an easy application of Theorem 7 of [6], $G/G'' \notin \mathfrak{NC}$ and, factoring, we may assume that G is metabelian. Re-labelling if necessary, *we may suppose that G/A is abelian, where A is a normal abelian subgroup of G.*

Suppose next that $A^p = 1$, and let U be an arbitrary $\mathfrak{NC}$-subgroup of G. By Proposition 3.2, $AU \in \mathfrak{NC}$, so there is a normal subgroup V of U with U/V Chernikov and AV nilpotent. By

Lemma 2.1(c), some finite power of V lies in the centralizer C of A, and so UC/C is small. Conversely, if U is a subgroup of G with UC/C small then we may apply Lemma 2.1(b) to deduce that $U \in \mathfrak{NC}$. We now proceed to obtain the contradiction that (on the assumption that $A^p = 1$) G is in $\mathfrak{NC}$. Here is an outline of our argument; it proceeds just like the proof of Lemma 9 of [2], to which the reader is referred for further detail.

Denote by Θ the set of subgroups X of G such that XC/C is large (equivalently, the set of non-$\mathfrak{NC}$-subgroups of G). Then Θ is non-empty since it contains G, and by Lemma 2 of [7] (see also Lemma 3 of [2]) there is a positive integer d, a Θ-subgroup H of G and a finite subgroup F of H such that every Θ-subgroup of H containing F is subnormal of defect at most d in H.

If *every* subgroup of H that contains F is subnormal of defect at most d in H then, by Theorem 0.2 of [4], we obtain the contradiction that H is nilpotent. So there is a subgroup K of H and a finite subgroup V of K containing F such that $[H,_d V] \nleq K$; choose $a \in [H,_d V] \setminus K$. For each $i \geq 0$, define H_i by setting $H_i/(H \cap C) = \Omega_i(H/(H \cap C))$, and let m be least such that $V \leq H_m$. Then each H_i is nilpotent, and by repeated application of Theorem 6 of [7] we may construct a union Y of finite subgroups $V = V_0 \leq V_1 \leq \cdots$ of H such that $a \notin Y$ and $|V_{i+1}/(V_{i+1} \cap H_{m+i})| = p^{i+1}$ for each $i \geq 0$. It is routine to show that YC/C is large, and we thus have $a \in [H,_d V] \leq [H,_d Y] \leq Y$, which is a contradiction. As pointed out above, the outcome of these considerations is that *G/A^p is in $\mathfrak{NC}$.*

In the general case, denote by C_0 the centralizer in G of A/A^p. There is a nilpotent subgroup G_0/A^p of G/A^p such that $A \leq G_0$ and G/G_0 is divisible Chernikov. By Lemma 2.1(c) there is a positive integer k such that $G_0^{p^k} \leq C_0$, so G/C_0 is small and, by Lemma 2.1(b), C_0 is not nilpotent. Let $c \in C_0$. Since G is a Baer group $A\langle c\rangle$ is nilpotent; also $A\langle c\rangle/A^{p^i}$ has finite exponent for each i. Denoting by X^ω the intersection of all subgroups X^{p^i} of the arbitrary group X, we therefore have $(A\langle c\rangle)^\omega = A^\omega$. But $(A\langle c\rangle)^\omega$ is centralized by c (see, for example, Lemma 1 of [2]), and since c was arbitrary we see that $A^\omega \leq Z(C_0)$. Since $[A^{p^i}, C_0] \leq A^{p^{i+1}}$ for all i, C_0/A^ω is residually nilpotent and hence, by Lemma 3.1, nilpotent. This gives our final contradiction that C_0 is nilpotent, and the proof of the theorem is complete.

References

[1] A.O. Asar, Locally nilpotent p-groups whose proper subgroups are hypercentral or nilpotent-by-Chernikov, *J. London Math. Soc. (2)* **61** (2000), no. 2, 412–422.

[2] C. Casolo, On the structure of groups with all subgroups subnormal, *J. Group Theory* **5** (2002), no. 3, 293–300.

[3] G. Cutolo and H. Smith, Locally finite groups with all subgroups subnormal or nilpotent-by-Chernikov. *Centr. Eur. J. Math.* (to appear).

[4] E. Detomi, Groups with many subnormal subgroups, *J. Algebra* **264** (2003), no. 2, 385–396.

[5] M.R. Dixon, M.J. Evans and H. Smith, Groups with all proper subgroups nilpotent-by-finite rank, *Arch. Math. (Basel)* **75** (2000), no. 2, 81–91.

[6] P. Hall, Some sufficient conditions for a group to be nilpotent, *Illinois J. Math.* **2** (1958), 787–801.

[7] W. Möhres, Torsionsgruppen, deren Untergruppen alle subnormal sind, *Geom. Dedicata* **31** (1989), no. 2, 237–244.

[8] W. Möhres, Auflösbarkeit von Gruppen, deren Untergruppen alle subnormal sind, *Arch. Math. (Basel)* **54** (1990), no. 3, 232–235.

[9] F. Napolitani and E. Pegoraro, On groups with nilpotent by Černikov proper subgroups, *Arch. Math. (Basel)* **69** (1997), no. 2, 89–94.

[10] H. Smith, Hypercentral groups with all subgroups subnormal, *Bull. London Math. Soc.* **15** (1983), no. 3, 229–234.

[11] H. Smith, Groups with all non-nilpotent subgroups subnormal, Topics in infinite groups, *Quad. Mat.*, vol. 8, Dept. Math., Seconda Univ. Napoli, Caserta, 2001, pp. 309–326.

[12] H. Smith, On non-nilpotent groups with all subgroups subnormal, *Ricerche Mat.* **50** (2001), no. 2, 217–221.

[13] H. Smith, Groups with all subgroups subnormal or nilpotent-by-Chernikov, *Rend. Sem. Mat. Univ. Padova* 126 (2011), 245–253.

[14] H. Smith, Groups that involve finitely many primes and have all subgroups subnormal, *J. Algebra* **347** (2011), no. 1, 133-142

Giovanni Cutolo

Dipartimento di Matematica e Applicazioni "R. Cacciopoli", Università degli Studi di Napoli,
Via Cintia, Monte S. Angelo, Napoli, Italy

`Email: cutolo@unina.it`

Howard Smith

Department of Mathematics, Bucknell University, Lewisburg, PA 17837, U.S.A.

`Email: howsmith@bucknell.edu`