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ON THE RIGHT n-ENGEL GROUP ELEMENTS

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Abstract. In this paper we study right n-Engel group elements. By modifying a group constructed by Newman and Nickel, we construct, for each integer $n \geq 5$, a 2-generator group $G = \langle a, b \rangle$ with the property that b is a right n-Engel element but where $[b^k,_n a]$ is of infinite order when $k \notin \{0, 1\}$.

1. Introduction

Let G be any group and n a non-negative integer. For any two elements a and b of G, we define inductively $[a,_n b]$ the n-Engel commutator of the pair (a, b), as follows:

$$[a,_0 b] := a, \; [a, b] = [a,_1 b] := a^{-1}b^{-1}ab \text{ and } [a,_n b] = [[a,_{n-1} b], b]$$

for all $n > 0$. An element x of G is called right (left) n-Engel if $[x,_n g] = 1$ ($[g,_n x] = 1$) for all $g \in G$. We denote by $R_n(G)$ ($L_n(G)$), the set of all right (left) n-Engel elements of G. A group G is called n-Engel if $G = L_n(G)$ or equivalently $G = R_n(G)$. It is clear that $R_1(G) = Z(G)$ the center of G, and by a result of Kappe [5], $R_2(G)$ is a characteristic subgroup of G. Kappe and Ratchford [6] have shown that if G is a metabelian group, if $n \geq 2$, or G is a center-by-metabelian group such that $[\gamma_k(G), \gamma_j(G)] = 1$ for some $k, j \geq 2$ with $k + j - 2 \leq n$ and $n \geq 3$ then $R_n(G)$ is a subgroup of G. Macdonald [7] has shown that the inverse or square of a right 3-Engel element need not be right 3-Engel. Nickel [10] generalized this result for $n \geq 3$. In fact he constructed a group with a right n-Engel element a where neither a^{-1} nor a^2 is a right n-Engel element. The construction of Nickel's example was guided by computer experiments and the arguments are based on commutator calculus. Although Macdonald's example shows that $R_3(G)$ is not in general a subgroup of G, Heineken [3] has shown that if A is the subset of a group G consisting of all elements a such that $a^{\pm 1} \in R_3(G)$, then

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A is a subgroup if either G has no element of order 2 or A consists only of elements having finite odd order. Abdollahi and Khosravi [2] have shown that in a group G without elements of order 2, $R_3(G)$ is a subgroup. Furthermore they proved that $R_4(G)$ is a subgroup for locally nilpotent groups G without elements of orders 2, 3 or 5. Newman and Nickel [8] have shown that for every $n \geq 5$ there exists a nilpotent group G of class $n+2$ containing a right n-Engel element a and an element b such that $[b,_n a]$ has infinite order. As we mentioned above, Nickel [10] has shown that for every $n \geq 5$ there exists a nilpotent group of class n+2 having a right n-Engel element a and element b such that $[a^{-1},_n b] = [a^2,_n b] \neq 1$. The construction follows the ideas in the above example by working in a free nilpotent group with certain commutators made trivial to simplify the calculations involved. We have checked that the latter element in Nickels example is of finite order whenever $n \in \{5, 6, 7, 8\}$. In this paper we modify the construction given by Newman and Nickel to show that there exists a nilpotent group G of class $n+2$ such that $b \in R_n(G)$ but $[b^{-1},_n a] = [b^2,_n a]$ has infinite order. Furthermore $[b^{-k},_n a] = [b^{-1},_n a]^{\binom{k}{2}}$ has infinite order, for every integer $k \geq 2$.

In [1] the following question has been proposed

Question 1.1. *Let n be a positive integer. Is there a set of prime numbers π_n depending only on n such that the set of right n-Engel elements in any nilpotent or finite π_n'-group forms a subgroup?*

We will see that our construction gives a negative answer to Question 1.1.

2. Right n-Engel elements for $n \geq 5$

In this section we show that for every $n \geq 5$, there is a nilpotent 2-generator group $G = \langle a, b\rangle$ of class $n+2$ with the property that b is right n-Engel but where $[b^k,_n a]$ is of infinite order when $k \notin \{1, 0\}$.

Note that by Nickel's example [10], for every $n \geq 3$ we have already had a nilpotent group K of class $n+2$ containing a right n-Engel element x such that $[x^{-1},_n y] = [x^2,_n y] \neq 1$ for some $y \in K$ i.e, neither x^2 nor x^{-1} are right n-Engel. We have checked by nq package of Nickel [9] in GAP [11] that $[x^{-1},_n y] = [x^2,_n y]$ is of finite order whenever $n \in \{5, 6, 7, 8\}$. In fact,

(1) $o([x^{-1},_5 y]) = 3$, NqRuntime=1.7 Sec
(2) $o([x^{-1},_6 y]) = 7$, NqRuntime=54.8 Sec
(3) $o([x^{-1},_7 y]) = 4$, NqRuntime=1702 Sec
(4) $o([x^{-1},_8 y]) = 9$, NqRuntime=56406 Sec

All given timings were obtained on an Intel Pentium 4-1.70GHz processor with 512 MB running Red Hat Enterprise Linux 5.

Newman and Nickel [8] construct the group H as follows. Let F be the relatively free group, generated by $\{a, b\}$ with nilpotency class $n+2$ and $\gamma_4(F)$ abelian. Let M be the (normal) subgroup of F generated by all commutators in a, b with at least 3 entries b and the commutators $[b,_{n+1} a]$ and $[b,_n a, b]$. Then $H = \dfrac{F}{M}$. Note that the normal closure of b in H is nilpotent of class 2.

We denote the generators of H by a and b again. Put

$$t=[b,_n a],\ u_j=[b,_{n-1-j}a,b,_j u],\ 0\le j\le n-2,$$

$$u=\prod_{j=0}^{n-2}u_j,\ v=[u_{n-2},a],\ w=\prod_{j=0}^{n-3}[u_j,a]$$

and let N be the subgroup $\langle tuw,t^2w,vw\rangle$. Then aN is right n-Engel in $\frac{H}{N}$ and $[b,_n a]N$ has infinite order in $\frac{H}{N}$.

We modify the construction above by replacing N by the subgroup $N_0:=\langle u,vw,vt^{-1}\rangle$. First, note that N_0 is a normal subgroup of H. For, clearly $t,v,w\in Z(H)$ and $u^b=u$. Also direct calculations show that $u^a=uvw$. This means that $N_0^a=N_0$ and so N_0 is a normal subgroup of H. Now we can state our main result of this section:

Theorem 2.1. *$[b,_n a]N_0=[b^{-2},_n a]N_0$ has infinite order in $\frac{H}{N_0}$ and $[b^{-1},_n h]\in N_0$ for all $h\in H$. Furthermore $[b^{-k},_n a]N_0=v^{\binom{k}{2}}N_0$ for all $k\ge 2$.*

Proof of Theorem 2.1. As H is nilpotent of class $n+2$, we may assume that h is of the form $a^{\alpha}b^{\beta}[b,a]^{\gamma}$. Since

$$\begin{aligned}\prod_{j=0}^{n-1}[b^{-1},_j a,[b,a],_{n-1-j}a] &= [b,[b,a],_{n-1}a][b,a,[b,a],_{n-2}a]\\ &\quad\cdots[b,_{n-1}a,[b,a]]\\ &= [b,b,_n a][b,a,b,_{n-1}a]^{-1}[b,a,b,_{n-1}a]\\ &\quad [b,_2 a,b,_{n-2}a]^{-1}\cdots[b,_{n-1}a,b,a][b,_n a,b]^{-1}\\ &= [b,_n a,b]^{-1}\\ &= 1,\end{aligned}$$

we have that $[b^{-1},_n a^{\alpha}b^{\beta}[b,a]^{\gamma}]=[b^{-1},_n a^{\alpha}b^{\beta}]$. We have

$$\begin{aligned}[b^{-1},_n a^{\alpha}b^{\beta}] &= [b^{-1},_n a^{\alpha}](\prod_{j=0}^{n-1}[b^{-1},_j a^{\alpha},b^{\beta},_{n-1-j}a^{\alpha}])\\ &\quad(\prod_{j=0}^{n-1}[b^{-1},_j a^{\alpha},a^{\alpha},b^{\beta},_{n-1-j}a^{\alpha}])\\ &= (vt^{-1})^{\alpha^n}(vw)^{\alpha^n\beta}u^{-\alpha^{n-1}\beta}(vw)^{-(n-2)\binom{\alpha}{2}\alpha^{n-2}\beta}\end{aligned}$$

where the last expression is clearly in N_0. Therefore $b^{-1}N_0$ is a right n-Engel element in $\frac{H}{N_0}$. This complete the second part of the theorem.

Now $K = \langle t, u, v, w\rangle$ is a free abelian group of rank 4 and $\frac{K}{N_0}$ is thus an infinite cyclic group generated by $tN_0 = vN_0 = w^{-1}N_0$. In particular $[b,_n a]N_0 = tN_0$ has infinite order. We also have

$$\begin{aligned}[b^{-k},_n a]N_0 &= [b^{-1},_n a]^k [b, a, b,_{n-1} a]^{\binom{k}{2}} N_0 \\ &= (vN_0)^{\binom{k}{2}} \\ &= (tN_0)^{\binom{k}{2}}\end{aligned}$$

In particular $[b^{-2},_n a]N_0 = tN_0 = [b,_n a]N_0$ and $[b^{-k},_n a]N_0$ is of infinite order if k is an integer different from 1 and 0. This complete the proof. □

Now we answer negatively Question 1.1 which has been proposed in [1]. Let T be the torsion subgroup of H/N_0 and $x = bN_0T$ and $y = aN_0T$. Then the group $\mathcal{M} = H/N_0T = \langle x, y\rangle$ is a torsion free, nilpotent of class n + 2, $x \in R_n(\mathcal{M})$ and $[x^k,_n y]$ is of infinite order if $k \notin \{0, 1\}$. Since, for any given prime number p, a finitely generated torsion-free nilpotent group is residually finite p-group, it follows that for any prime number p and integer $k \geq 2$, there is a finite p-group $G(p, k)$ of class $n + 2$ containing a right n-Engel element t such that t^k is not right n-Engel when $k \notin \{0, 1\}$. This answers negatively Question 1.1.

As we mentioned in Section 1, Newman and Nickel [8] have shown that $R_n(G) \subseteq L_n(G)$ if $n = 3$ and G is a $2'$-group or $n = 4$ and G is a locally nilpotent $\{2, 3, 5\}'$-group. Also they have shown that $R_n(G)$ is not necessarily a subgroup of $L_n(G)$ for $n \geq 5$. Under the same condition, in [2] it is shown that $R_n(G)$ is a subgroup of G if $n = 3$ or $n = 4$. By Theorem 2.1 $R_n(G)$ is not necessarily a subgroup of G for $n \geq 5$. According to the above results the following question arise

Question 2.2. *Is there any relation between the statements $R_n(G) \subseteq L_n(G)$ and $R_n(G) \leq G$?*

Kappe and Ratchford [6] have shown that if G is a metabelian group or G is a center-by-metabelian group such that $[\gamma_k(G), \gamma_j(G)] = 1$ for some $k, j \geq 2$ with $k + j - 2 \leq n$ and $n \geq 3$ then

$$R_n(G, g) = \{a \in G \mid [a,_n g] = 1\}$$

is a subgroup of G. Thus $R_n(G) = \bigcap_{g \in G} R_n(G, g)$ is a subgroup. Kappe [4] has shown that

Theorem (Kappe) *Let G be a metabelian group and $g \in G$ a right n-Engel element. Then g is a left n-Engel element if $\gamma_{n+1}(G)$ does not contain elements of prime order $p \leq n - 1$. However, for each $n \geq 3$ and each prime $p \leq n - 1$ there exist groups G, even finite p-group, such that $\gamma_{n+1}(G)$ contains elements of order p and G contains an element which is right n-Engel but not left n-Engel.*

Now let G be a metabelian group. Then by Kappe and Ratchford's result it is clear that $R_n(G)$ is a subgroup of G, for every $n \in \mathbb{N}$. On the other hand, by the above Theorem there is a metabelian group G and $n \in \mathbb{N}$ such that $R_n(G) \nsubseteq L_n(G)$. Therefore, to answer the Question 2.2 it is sufficient to answer the following question.

Question 2.3. *Is it true that if $R_n(G) \subseteq L_n(G)$ then $R_n(G)$ is a subgroup of G?*

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