ON WEAKLY SS-QUASINORMAL AND HYPERCYCLICALLY EMBEDDED PROPERTIES OF FINITE GROUPS

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Abstract. A subgroup \( H \) is said to be \( s \)-permutable in a group \( G \), if \( HP = PH \) holds for every Sylow subgroup \( P \) of \( G \). If there exists a subgroup \( B \) of \( G \) such that \( HB = G \) and \( H \) permutes with every Sylow subgroup of \( B \), then \( H \) is said to be SS-quasinormal in \( G \). In this paper, we say that \( H \) is a weakly SS-quasinormal subgroup of \( G \), if there is a normal subgroup \( T \) of \( G \) such that \( HT \) is \( s \)-permutable and \( H \cap T \) is SS-quasinormal in \( G \). By assuming that some subgroups of \( G \) with prime power order have the weakly SS-quasinormal properties, we get some new characterizations about the hypercyclically embedded subgroups of \( G \). A series of known results in the literature are unified and generalized.

1. Introduction

All groups considered in this paper will be finite and we use conventional notions and notation, as in Gorenstein [5]. Let \( \mathcal{F} \) be a formation, \( \mathcal{U} \) and \( \mathcal{N}_p \) denote the classes of all supersoluble and \( p \)-nilpotent groups, respectively. \( G^\mathcal{F} \) denotes the \( \mathcal{F} \)-residual, \( Z_\infty(G) \) is the hypercentre of \( G \). A normal subgroup \( H \) is said to be hypercyclically embedded in \( G \), if every chief factor of \( G \) below \( H \) is cyclic. The product of all hypercyclically embedded subgroups of \( G \) is denoted by \( Z_{\mathcal{U}}(G) \) and called the \( \mathcal{U} \)-hypercentre of \( G \).

Recall that a subgroup \( H \) of \( G \) is said to be \( s \)-permutable [11] (or \( s \)-quasinormal [3]) in \( G \), if \( H \) permutes with every Sylow subgroup \( P \) of \( G \). As a development, in [12] the authors introduced that: a subgroup \( H \) of \( G \) is called an SS-quasinormal (Supplement-Sylow-quasinormal) subgroup of \( G \) if there is a supplement \( B \) of \( H \) to \( G \) such that \( H \) permutes with every Sylow subgroup of \( B \). Another related concept which was investigated extensively by many scholars was called \( c \)-normal subgroup, introduced by Wang in [18]: a subgroup \( H \) is said to be \( c \)-normal in \( G \) if \( G \) has a normal subgroup \( T \) such that
$G = HT$ and $H \cap T \leq H_G$, where $H_G$ is the normal core of $H$ in $G$. There are many generalizations about these concepts, such as $s$-permutably embedded subgroup [1], $c$-supplemented subgroup [2], $c^*$-normal subgroup [21], weakly $s$-permutably embedded subgroup [14], weakly $s$-permutably embedded subgroup [14] and nearly $s$-normal [7] etc. Following Guo et al in [8], a subgroup $H$ is said to be $S$-embedded in $G$ if there exists a normal subgroup $N$ such that $HN$ is $s$-permutable in $G$ and $H \cap N \leq H_{sG}$, where $H_{sG}$ is the largest $s$-permutable subgroup of $G$ contained in $H$. By assuming that some primary subgroups of $G$ satisfying one of the above properties, many interesting results have been derived.

In this paper, we introduce the concept of weakly $SS$-quasinormal subgroup, which can cover the $SS$-quasinormal and $S$-embedded subgroups properly.

**Definition 1.1.** A subgroup $H$ is said to be weakly $SS$-quasinormal in $G$, if there exists $T \trianglelefteq G$ such that $HT$ is $s$-permutable and $H \cap T$ is $SS$-quasinormal in $G$.

Obviously, every $S$-embedded ($s$-permutable, $c$-normal) subgroup or $SS$-quasinormal subgroup of $G$ is weakly $SS$-quasinormal in $G$. But in general, a weakly $SS$-quasinormal subgroup of $G$ need not be $S$-embedded or $SS$-quasinormal in $G$. For instance:

**Example:** Let $G = S_5$ be the symmetric group of degree 5, $P \in Syl_5(G)$. Since $G = S_4P$, $S_4$ is permutable with $P$ and $S_4 \cap P = 1$, $S_4$ is weakly $SS$-quasinormal in $G$. Since $H = \langle(12)\rangle$ satisfying that $HA_5 = G$ and $H \cap A_5 = 1$, $H$ is weakly $SS$-quasinormal in $G$. But $S_4$ is not $S$-embedded, $H$ is not $SS$-quasinormal in $G$.

In this paper, we investigate the influence of some weakly $SS$-quasinormal subgroups on the structure of a finite group $G$. Some recent results are generalized.

## 2. Preliminaries

In this section we gather some results from the literature that will be used later.

**Lemma 2.1.** ([11]) Suppose that $H$ is an $s$-permutable subgroup of $G$, $H \leq G$ and $N \trianglelefteq G$.

1. If $K \leq G$, then $H \cap K$ is $s$-permutable in $K$.
2. $HN$ and $H \cap N$ are $s$-permutable in $G$, $HN/N$ is $s$-permutable in $G/N$.
3. $H$ is subnormal in $G$.
4. If $H$ is a $p$-group for some prime $p$, then $N_G(H) \geq O_p(G)$.

**Proof.** (1)-(3) are from [11], (4) is [17, Lemma A]. □

From Lemma 2.1 and Lemma 2.2 of [12], we have the following two results:

**Lemma 2.2.** Suppose that $H$ is $SS$-quasinormal in a group $G$, $K \leq G$ and $N \trianglelefteq G$.

1. If $H \leq K$, then $H$ is $SS$-quasinormal in $K$.
2. $HN/N$ is $SS$-quasinormal in $G/N$.

**Lemma 2.3.** If $P$ is a $SS$-quasinormal $p$-subgroup of $G$ and $P \leq O_p(G)$, then $P$ is $s$-permutable in $G$.

Now, we can prove that:
Lemma 2.4. Suppose that $H$ is a weakly $SS$-quasinormal subgroup of $G$, $N \trianglelefteq G$.

(1) If $H \leq K \leq G$, then $H$ is weakly $SS$-quasinormal in $K$.

(2) If $N \leq H$, then $H/N$ is weakly $SS$-quasinormal in $G/N$.

(3) Let $\pi$ be a set of primes, $H$ a $\pi$-subgroup and $N$ a normal $\pi'$-subgroup of $G$. Then $HN/N$ is weakly $SS$-quasinormal in $G/N$.

(4) If $H \leq K \trianglelefteq G$, then $G$ has a normal subgroup $L$ contained in $K$ such that $HL$ is $s$-permutable and $H \cap L$ is $SS$-quasinormal in $G$.

Proof. (1), (2) and (4) can be deduced directly by Lemma 2.1 and Lemma 2.2. Now we prove the statement (3). By hypothesis, there exists a normal subgroup $T$ of $G$ such that $HT$ is $s$-permutable and $H \cap T$ is $SS$-quasinormal in $G$. It is easy to see that $TN/N \trianglelefteq G/N$, by Lemma 2.1(2) we know $(HN/N)(TN/N) = HTN/N$ is $s$-permutable in $G/N$. Since $H$ is a $\pi$-group and $N$ a $\pi'$-group,

$$|H \cap TN| = \frac{|H| \cdot |TN|_{\pi}}{|HTN|_{\pi}} = \frac{|H| \cdot |T|_{\pi}}{|HT|_{\pi}} = |H \cap T|.$$ 

This implies that $H \cap TN = H \cap T$, so $(HN/N) \cap (TN/N) = (HN \cap TN)/N = (H \cap TN)N/N = (H \cap T)N/N$ which is $SS$-quasinormal in $G/N$ by Lemma 2.2(2). Hence we proved that $HN/N$ is weakly $SS$-quasinormal in $G/N$, as required. □

Lemma 2.5. ([10] Lemma 2.2) Let $G$ be a group and $p$ a prime dividing $|G|$ with $(|G|, p - 1) = 1$.

(1) If $N$ is normal in $G$ of order $p$, then $N$ lies in $Z(G)$.

(2) If $G$ has cyclic Sylow $p$-subgroups, then $G$ is $p$-nilpotent.

(3) If $M$ is a subgroup of $G$ with index $p$, then $M$ is normal in $G$.

Lemma 2.6. ([4] A, Lemma 1.2) Let $U$, $V$ and $W$ be subgroups of a group $G$. Then the following statements are equivalent:

(a) $U \cap VW = (U \cap V)(U \cap W)$;
(b) $UV \cap UW = U(V \cap W)$.

Lemma 2.7. ([22] I, Theorem 1.4) Let $H/K$ be a $p$-chief factor of a group $G$. Then $|H/K| = p$ if and only if $G/C_G(H/K) \in A(p - 1)$, where $A(p - 1)$ is the formation of all abelian groups of exponent dividing $p - 1$.

Lemma 2.8. ([16] Lemma 2.2) Let $E$ be a normal $p$-subgroup of a group $G$. If $E \leq Z_d(G)$, then $(G/C_G(E))^{A(p-1)} \leq O_p(G/C_G(E))$.

Lemma 2.9. ([4] A, Lemma 1.2) Let $P$ be a $p$-group, $\alpha$ a $p'$-automorphism of $P$.

(1) If $[\alpha, \Omega_2(P)] = 1$, then $\alpha = 1$.

(2) If $[\alpha, \Omega_1(P)] = 1$ and either $p$ is odd or $P$ is abelian, then $\alpha = 1$.

3. Main results

Theorem 3.1. Let $p$ be a prime divisor of $|G|$ with $(|G|, p - 1) = 1$, $P$ a Sylow $p$-subgroup of $G$. Then $G$ is $p$-nilpotent if and only if every maximal subgroup of $P$ is weakly $SS$-quasinormal in $G$. 

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Proof. The necessity is obvious, we need to prove only the sufficiency. Suppose that the result is false and let $G$ be a counterexample of minimal order. Then we have

(1) $P$ is not cyclic and $G$ is not a non-abelian simple group.

By Lemma 2.5(2), we may assume that $P$ is not cyclic. Let $P_1$ be a maximal subgroup of $P$, then by hypothesis there exists a normal subgroup $T$ of $G$ such that $P_1T$ is $s$-permutable and $P_1 \cap T$ is $SS$-quasinormal in $G$.

Assume that $G$ is a non-abelian simple group, then $T = 1$ or $G$. If $T = 1$, then $P_1 = P_1T$ is $s$-permutable in $G$. Hence $P_1$ is a proper subnormal subgroup of $G$, a contradiction. Thus $T = G$ and therefore $P_1 = P_1 \cap T$ is $SS$-quasinormal in $G$. Then there exists some supplement $B$ of $P_1$ to $G$ such that $P_1$ permutates with every Sylow subgroup of $B$. Since $|B : P_1 \cap B|_p = |G : P_1|_p = p$, $S \not\subseteq P_1$ for all $S \in \text{Syl}_p(B)$ and $P_1S = SP_1$ is a Sylow $p$-subgroup of $G$. By comparison of orders, we know that $S \cap P_1 = B \cap P_1$ holds for all $S \in \text{Syl}_p(B)$. Hence $B \cap P_1 = \bigcap_{b \in B}(S^b \cap P_1) \subseteq \bigcap_{b \in B}S^b = O_p(B)$. Since $|O_p(B) : B \cap P_1| \leq p$, we have $|B/O_p(B)|_p \leq p$. Then by Lemma 2.5(2), $B/O_p(B)$ is $p$-nilpotent and so $B$ is $p$-soluble. Hence, $B$ has a Hall $p'$-subgroup $K$. Since $P_1$ permutates with every Sylow subgroup of $K$, $P_1K = KP_1$ is a subgroup of $G$ with index $p$. Thus by Lemma 2.5(3), we know that $P_1K$ is normal in $G$, which is a contradiction. Therefore, $G$ is not a non-abelian simple group.

(2) $G$ has a unique minimal normal subgroup, $N$ say. Moreover, $G/N$ is $p$-nilpotent and $\Phi(G) = 1$.

Let $N$ be a minimal normal subgroup of $G$ and $M/N$ a maximal subgroup of $PN/N$. It is easy to see that $M = P_1N$ for some maximal subgroup $P_1$ of $P$ and $P \cap N = P_1 \cap N$ is a Sylow $p$-subgroup of $N$. Since $P_1$ is weakly $SS$-quasinormal in $G$, there exists a normal subgroup $K$ of $G$ such that $P_1K$ is $s$-permutable and $P_1 \cap K$ is $SS$-quasinormal in $G$. Clearly, $KN/N$ is a normal subgroup of $G/N$ and $P_1N/N \cdot KN/N = P_1KN/N$ is $s$-permutable in $G/N$. Moreover, since $P_1 \cap N$ is a Sylow $p$-subgroup of $N$, $|(P_1 \cap N)(KN/N)|_p = |P_1 \cap N| = |N|_p = |N \cap P_1K|_p$. Since $P_1$ is a $p$-group,

$$|P_1K \cap N|_p' = \frac{|P_1K|_p \cdot |N|_p'}{|P_1K \cap N|_p'} = \frac{|K|_p \cdot |N|_p'}{|KN|_p'} = |K \cap N|_p' = |(P_1 \cap N)(KN/N)|_p'. $$

This statement shows that $(P_1 \cap N)(KN/N) = P_1K \cap N$. Thus by Lemma 2.6 we have $P_1N \cap KN = (P_1 \cap K)N$. It follows from Lemma 2.2 that $P_1N/N \cap KN/N = (P_1 \cap K)N/N$ is $SS$-quasinormal in $G/N$. Therefore, $M/N$ is weakly $SS$-quasinormal in $G/N$ and so $G/N$ satisfies the hypothesis of the theorem. The minimal choice of $G$ implies that $G/N$ is $p$-nilpotent.

Since the class of all $p$-nilpotent groups forms a saturated formation, $N$ is the unique minimal normal subgroup of $G$ and $\Phi(G) = 1$.

(3) $O_{p'}(G) = 1$.

If $O_{p'}(G) \neq 1$, then $N \leq O_{p'}(G)$ and $G/O_{p'}(G)$ is $p$-nilpotent by (2). Hence $G$ is also a $p$-nilpotent group, a contradiction.

(4) $O_p(G) = 1$ and $N$ is not $p$-nilpotent.

If $O_p(G) \neq 1$, then $N \leq O_p(G)$ is an elementary abelian $p$-group. Since $\Phi(G) = 1$, $G$ has a maximal subgroup $M$ such that $G = MN$ and $M \cap N = 1$. Since $O_p(G) \cap M$ is normalized by $N$ and $M$, it is normalized by $G$. The uniqueness of $N$ yields that $N = O_p(G)$. Since $P = N(P \cap M)$ and $N \cap M = 1$,
$P \cap M$ is a Sylow $p$-subgroup of $M$ and there exists a maximal subgroup $P_1$ of $P$ such that $P \cap M \leq P_1$ and $P = NP_1$.

Since $P_1$ is weakly SS-quasinormal in $G$, there exist $T \unlhd G$ and $B \leq G$ such that $P_1T$ is $s$-permutable in $G$, $(P_1 \cap T)B = G$ and $(P_1 \cap T)B_q = B_q(P_1 \cap T)$ for any $B_q \in \mathrm{Syl}_q(B)$ with $q \in \pi(B)$. If $T = 1$, then $P_1P_1T$ is $s$-permutable in $G$. It follows that $P_1 \leq O_p(G) = N$ and so $P = NP_1 = N$ is a minimal normal subgroup of $G$. Since $N_G(P_1) \geq O^p(G)$ by Lemma 2.4(4) and $P_1 \leq P$, $P_1$ is a proper normal subgroup of $G$ contained in $P = O_p(G)$, a contradiction. Thus we have $T \neq 1$ and so $N \leq T$. In this case, $N \cap P_1 = N \cap P_1 \cap T = N \cap (P_1 \cap T)B_q \leq (P_1 \cap T)B_q$ for any $B_q \in \mathrm{Syl}_q(B)$ with $q \neq p$. Hence $B_q \leq N_G(N \cap P_1)$ holds for any $q \neq p$. Since $N \cap P_1 \leq P$, it is normal in $G$. Thus $N \cap P_1 = 1$ and $|N| = p$. Let $C/N$ be the normal $p$-complement of $G/N$, then $N$ is a cyclic Sylow $p$-subgroup of $C$. By Lemma 2.5(2), $C$ is $p$-nilpotent and the normal $p$-complement of $C$ is also a normal $p$-complement of $G$, a contradiction.

If $N$ is $p$-nilpotent, then $N_q = \text{char } N \leq G$, so $N_q \leq O_{p'}(G) = 1$ by (3). Thus $N$ is a $p$-group and $N \leq O_p(G) = 1$, a contradiction.

(5) $G = PN$.

By Lemma 2.4, we know that $PN$ satisfies the hypothesis of the theorem. Therefore, $PN$ is $p$-nilpotent if $PN \leq G$. It follows that $N$ is $p$-nilpotent, which contradicts with (4). Hence $G = PN$ and $N = O^p(G)$.

(6) The final contradiction.

Since $N$ is not soluble, $N = S_1 \times S_2 \times \cdots \times S_t$ is a direct product of some isomorphic non-abelian simple groups $S_i$. By (1) and (5), we know $N \leq G$ and $P \cap N \leq P$. Thus $S_p = P \cap S_1 \leq P_1$ for some maximal subgroup $P_1$ of $P$, where $S_p$ is a Sylow $p$-subgroup of $S_1$. Then there exists a normal subgroup $T$ of $G$ such that $P_1T$ is $s$-permutable in $G$, and there is a supplement $B$ of $P_1 \cap T$ to $G$ such that $P_1 \cap T$ permutes with every Sylow subgroup of $B$. If $T = 1$, then $P_1$ is $s$-permutable in $G$ and so $O_p(G) \neq 1$, contradicts with (4). Thus $T \neq 1$ and so $N \leq T$. If $P_1 \cap T = 1$, then $|T|_p \leq p$. Hence $T$ is $p$-nilpotent by Lemma 2.5(2), $N$ is also $p$-nilpotent. This contradiction shows that $P_1 \cap T \neq 1$. Let $B_q$ be a Sylow $q$-subgroup of $B$, where $q \in \pi(B)$ and $q \neq p$. Then $P_1B_q = B_qP_1$ is a subgroup of $G$ and

$$|B_q \cap P_1T| = \frac{|B_q| \cdot |P_1T|_q}{|B_qP_1T|_q} = \frac{|B_q| \cdot |T|_q}{|B_qT|_q} = |B_q \cap T| = |(B_q \cap P_1)(B_q \cap T)|.$$  

Since $(B_q \cap P_1)(B_q \cap T) \leq B_q \cap P_1T$, this equation implies that $B_q \cap P_1T = (B_q \cap P_1)(B_q \cap T)$. By Lemma 2.6, we have $B_qP_1 \cap B_qT = B_q(P_1 \cap T)$. Therefore, $N \cap P_1B_q = N \cap (P_1B_q \cap TB_q) = N \cap (P_1 \cap T)B_q$ and $S_1 \cap (P_1 \cap T) = S_1 \cap P_1 = S_p$ is a Sylow $p$-subgroup of $S_1$. This means that for any prime divisor $q \neq p$ of $|S_1|$, $S_1 \cap (P_1 \cap T)B_q$ is a Hall $\{p, q\}$-subgroup of $S_1$. Since $S_1$ is non-abelian and $(|S_1|, p - 1) = 1$, $p = 2$. Then for any prime divisor $q \neq 2$ of $|S_1|$, non-abelian simple group $S_1$ has a Hall $\{2, q\}$-subgroup, which contradicts with [13] Lemma 2.6. This contradiction completes the proof of the theorem. □

Next, we can prove that:
Theorem 3.2. Suppose that \( p \) is a prime divisor of \( |G| \) with \((|G|, p - 1) = 1\), \( P \) a Sylow \( p \)-subgroup of \( G \). Then \( G \) is \( p \)-nilpotent if and only if every cyclic subgroup of \( P \) with order \( p \) or 4 (if \( p = 2 \) and \( P \) is non-abelian) not having a \( p \)-nilpotent supplement in \( G \) is weakly SS-quasinormal in \( G \).

Proof. We need to prove only the sufficiency. Assume that the result is false and let \( G \) be a counterexample of minimal order. Then we have

1. \( P \) is not cyclic and \( G \) is a minimal non-\( p \)-nilpotent group.

By Lemma 2.5, we may assume that \( P \) is not cyclic. Let \( M \) be a proper subgroup of \( G \), \( L \) a cyclic subgroup of \( P \cap M \) with order \( p \) or 4 (if \( p = 2 \) and \( P \) is non-abelian) having no \( p \)-nilpotent supplement in \( M \), then \( L \) has no \( p \)-nilpotent supplement in \( G \). Hence \( L \) is weakly SS-quasinormal in \( G \) by hypothesis. In this case, by Lemma 2.4 we know \( L \) is a weakly SS-quasinormal subgroup of \( M \). Hence \( M \) satisfies the hypothesis of the theorem. The minimal choice of \( G \) yields that \( M \) is \( p \)-nilpotent. Hence, \( G \) is a minimal non-nilpotent group and satisfying that: (i) \( G = [P]G_q \), where \( G_q \) is a non-normal cyclic Sylow \( q \)-subgroup of \( G \) for some prime \( q \) different from \( p \); (ii) \( P/\Phi(P) \) is a chief factor of \( G \); (iii) \( P \) is of exponent \( p \) if \( p > 2 \) and exponent at most 4 if \( p = 2 \).

2. Some minimal subgroup \( X/\Phi(P) \) of \( P/\Phi(P) \) is not \( s \)-permutable in \( G/\Phi(P) \).

If every minimal subgroup of \( P/\Phi(P) \) is \( s \)-permutable in \( G/\Phi(P) \), then by [15, Lemma 2.11] we know \( P/\Phi(P) \) has a maximal subgroup which is normal in \( G/\Phi(P) \). Since \( P/\Phi(P) \) is a chief factor of \( G \), \(|P/\Phi(P)| = p \) and so \( P \) is cyclic, which contradicts with (1). Thus there exists some minimal subgroup \( X/\Phi(P) \) of \( P/\Phi(P) \) such that \( X/\Phi(P) \) is not \( s \)-permutable in \( G/\Phi(P) \).

3. The final contradiction.

Let \( x \in X \setminus \Phi(P) \), then \( \langle x \rangle \) is a cyclic group of prime order or order 4. Let \( T \) be any supplement of \( \langle x \rangle \) in \( G \). Then \( G = \langle x \rangle T \) and \( P = P \cap \langle x \rangle T = \langle x \rangle(P \cap T) \). Since \( P/\Phi(P) \) is abelian, \( (P \cap T)\Phi(P)/\Phi(P) \leq G/\Phi(P) \) and hence \((P \cap T)\Phi(P) \leq G \). Thus \( P \cap T \leq \Phi(P) \) or \( P \cap T = P \), as \( P/\Phi(P) \) is a chief factor of \( G \). If \( P \cap T \leq \Phi(P) \) for some supplement \( T \), then \( P = \langle x \rangle \) is cyclic, contradicts with (2). Now assume that \( P \cap T = P \) for any supplement \( T \), then \( T = G \) is the unique supplement of \( \langle x \rangle \) in \( G \). Since \( G \) is not \( p \)-nilpotent, \( \langle x \rangle \) is weakly SS-quasinormal in \( G \) by the hypothesis. Thus there exists a normal subgroup \( K \) of \( G \) contained in \( P \) such that \( \langle x \rangle K \) is \( s \)-permutable and \( \langle x \rangle \cap K \) is SS-quasinormal in \( G \). Since \( \langle x \rangle \cap K \leq P = O_p(G) \), \( \langle x \rangle \cap K \) is \( s \)-permutable in \( G \) by Lemma 2.3. Since \( P/\Phi(P) \) is a chief factor of \( G \), \( K \leq \Phi(P) \) or \( K = P \). If \( K \leq \Phi(P) \), then \( X/\Phi(P) = \langle x \rangle K\Phi(P)/\Phi(P) \) is \( s \)-permutable in \( G/\Phi(P) \), a contradiction. If \( K = P \), then \( \langle x \rangle = \langle x \rangle \cap K \) is \( s \)-permutable in \( G \) and so \( X/\Phi(P) = \langle x \rangle \Phi(P)/\Phi(P) \) is \( s \)-permutable in \( G/\Phi(P) \), a contradiction too. This final contradiction completes the proof of the theorem.

Now we prove our main result:

Theorem 3.3. Let \( E \) be a normal subgroup of \( G \), suppose that there exists a normal subgroup \( X \) of \( G \) such that \( F^*(E) \leq X \leq E \) and for any non-cyclic Sylow subgroup \( P \) of \( X \), either every maximal subgroup of \( P \) or every cyclic subgroup of \( P \) with order \( p \) or 4 (if \( p = 2 \) and \( P \) is non-abelian 2-group) is weakly SS-quasinormal in \( G \). Then \( E \leq Z_d(G) \), i.e., each G-chief factor below \( E \) is cyclic.
Proof. Since $F^*(E) \leq X$, if we can prove $X \leq Z_d(G)$, then we have $F^*(E) \leq Z_d(G)$ by [10, Theorem C], we can conclude that $E \leq Z_d(G)$. Thus, we only need to prove $X \leq Z_d(G)$. Assume that the result is false and let $(G, X)$ be a counterexample of minimal order. Let $p = \min \pi(|X|)$ and $P \in Syl_p(X)$, then

(1) $X$ is $p$-nilpotent.

By [10, IV, Theorem 2.8], we may assume that $P$ is not cyclic. Then the hypothesis and Lemma 2.5 imply that either every maximal subgroup of $P$ or every cyclic subgroup of $P$ with order $p$ or 4 (if $P$ is a non-abelian 2-group) is weakly $SS$-quasinormal in $X$. Therefore, by Theorem 3.1 and Theorem 3.2, we can conclude that $X$ is $p$-nilpotent.

(2) $X = P$ is not cyclic.

Let $K$ be the normal $p$-complement of $X$. Since $K$ is a characteristic subgroup of $X$ and $X \leq G$, $K$ is normal in $G$. If $K \neq 1$, then by Lemma 2.4 we know that the hypothesis still holds for $(G, K)$ and $(G/K, X/K)$. Thus the minimal choice of $(G, X)$ implies that $X/K \leq Z_d(G/K)$ and $K \leq Z_d(G)$. Hence we can get $X \leq Z_d(G)$, a contradiction. Therefore, we may assume that $K = 1$ and then $X = P$ is a normal $p$-subgroup of $G$. If $P = X$ is a cyclic group, then $X$ is hypercyclically embedded in $G$, a contradiction.

(3) $P$ is not a minimal normal subgroup of $G$ and every cyclic subgroup of $P$ with prime order or order 4 (if $P$ is a non-abelian 2-group) is $S$-embedded in $G$.

By [8, Lemma 2.2], we may assume that $P$ is not a minimal normal subgroup of $G$.

Now, we suppose that every maximal subgroup of $P$ is weakly $SS$-quasinormal in $G$. By Lemma 2.4 we know for any minimal normal subgroup $N$ of $G$ contained in $P$ the hypothesis holds for $(G/N, P/N)$. Hence the minimal choice of $(G, X)$ implies that $P/N \leq Z_d(G/N)$. Therefore, $N$ is the only minimal normal subgroup of $G$ contained in $P$ and $|N| > p$. If $\Phi(P) = 1$, then $P$ is an elementary abelian $p$-group.

Let $N_1$ be a maximal subgroup of $N$, $B$ a complement of $N$ in $P$ and $P_1 = N_1B$. Then $P_1$ is a maximal subgroup of $P$ and so it is weakly $SS$-quasinormal in $G$. Since $P_1$ is subnormal in $G$, by Lemma 2.3 we know $P_1$ is $S$-embedded in $G$. Let $T$ be a normal subgroup of $G$ contained in $P$ such that $P_1T$ is $s$-permutable in $G$ and $P_1 \cap T \leq (P_1)_{sG}$. If $T = 1$ or $T = P$, then $P_1$ is $s$-permutable in $G$ and hence $N_1 = N_1(B \cap N) = N_1B \cap N = P_1 \cap N$ is $s$-permutable in $G$. Now we suppose that $1 < T < P$, in this case $N \leq T$. Hence $N_1 \leq T$, which implies that $N_1 \leq P_1 \cap T \leq (P_1)_{sG}$. Since $N \cap P_1 = N_1$, $N_1 = N \cap (P_1)_{sG}$ is $s$-permutable in $G$. Thus, we have proved that every maximal subgroup of $N$ is $s$-permutable in $G$. Hence by [15, Lemma 2.11], we know some maximal subgroup of $N$ is normal in $G$, a contradiction. Therefore, $\Phi(P) \neq 1$ and $N \leq \Phi(P)$. Since $P/N \leq Z_d(G/N)$, we have $P/\Phi(P) \leq Z_d(G/\Phi(P))$. Then by Lemma 2.8 $(G/C_G(P/\Phi(P)))^{\pi(p-1)}$ is a $p$-group. Hence $(G/C_G(P))^{\pi(p-1)}$ is a $p$-group by [5, Theorem 5.1.4]. Since $O_p(G/C_G(N)) = 1$ by [22] Appendices, Corollary 6.4], $G/C_G(N) \in A^2(p-1)$. Therefore $|N| = p$ by Lemma 2.7 a contradiction.

By hypothesis, we may assume that every cyclic subgroup $H$ of $P$ with prime order or order 4 (if $P$ is a non-abelian 2-group) is weakly $SS$-quasinormal in $G$. By Lemma 2.3 we know they are $S$-embedded in $G$. 


(4) $G$ has a nontrivial normal subgroup $R \leq P$ such that $P/R$ is a non-cyclic chief factor of $G$, $R \leq Z_{ul}(G)$ and $V \leq R$ for any normal subgroup $V \neq P$ of $G$ contained in $P$.

Let $P/R$ be a chief factor of $G$, then by (3) we know that $R \neq 1$ and the hypothesis holds for $(G,R)$. Hence $R \leq Z_{ul}(G)$ and by the choice of $(G,X)$, we know $P/R$ is not cyclic. Let $V$ be any normal subgroup of $G$ with $V < P$, then $V \leq Z_{ul}(G)$. If $V \not\leq R$, then from the $G$-isomorphism $P/R = VR/R \cong V/(V \cap R)$ we can deduce that $P \leq Z_{ul}(G)$, which is contrary to the choice of $(G,P)$.

If $P$ is a non-abelian 2-group, we use $\Omega$ to denote the subgroup $\Omega_2(P)$. Otherwise, let $\Omega = \Omega_1(P)$. Then

(5) $C_G(\Omega)/C$ is a $p$-group.

This follows from Lemma 2.8.

(6) $\Omega \not\leq Z_{ul}(G)$.

Suppose that $\Omega \leq Z_{ul}(G)$, then Lemma 2.8 implies that $(G/C_G(\Omega))^{A(p-1)}$ is a $p$-group. Hence by (5), $G/C_G(P/R) \in A(p-1)$ and so $|P/R| = p$, a contradiction.

(7) The final contradiction.

By (4) and (6), we know $\Omega = P$. Let $V_1, V_2, \ldots, V_t$ be the set of all cyclic subgroups of $P$ with order $p$ or 4 which is not contained in $R$, then each $V_i$ is $S$-embedded in $G$.

Let $T_i$ be a normal subgroup of $G$ contained in $P$ such that $V_i T_i$ is $s$-permutable in $G$ and $T_i \cap V_i \leq (V_i)_{SG}$. If $V_i$ is $s$-permutable in $G$, then $T_i R/R$ is $s$-permutable in $G/R$. Now we assume that $V_i$ is not $s$-permutable in $G$, then $1 < T_i < P$. Thus $T_i \leq R$ and $V_i R = V_i T_i R$ is $s$-permutable in $G$. Therefore, $V_i R/R$ is $s$-permutable in $G/R$ in any case. Since $P/R = (V_1 R/R)(V_2 R/R) \cdots (V_t R/R)$ is an elementary abelian group and $V_i R/R \cong V_i/\langle V_i \cap R \rangle$ is cyclic for any $1 \leq i \leq t$, every minimal subgroup of $P/R$ is $s$-permutable in $G/R$. Hence by [15 Lemma 2.11], we know that $P/R$ is a cyclic group of order $p$, a contradiction. This contradiction completes the proof of the theorem.\hfill $\Box$

From Theorem 3.3, we can immediately deduce that:

**Theorem 3.4.** Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$, the classes of all supersoluble groups. Then $G \in \mathcal{F}$ if and only if $G$ has a normal subgroup $E$ such that $E/E \in \mathcal{F}$ and there exists a normal subgroup $X$ of $G$ such that $P^*(E) \leq X \leq E$ and for any non-cyclic Sylow subgroup $P$ of $X$, either every maximal subgroup of $P$ or every cyclic subgroup of $P$ with order $p$ or 4 (if $P$ is a non-abelian 2-group) is weakly SS-quasinormal in $G$.

**Proof.** The necessity is obvious, we need to prove only the sufficiency. By Theorem 3.3, we know that $E \leq Z_{ul}(G)$ in both cases. Since $Z_{ul}(G) \leq Z_{\mathcal{F}}(G)$, $E \leq Z_{\mathcal{F}}(G)$ and so $G/Z_{\mathcal{F}}(G) \cong (G/E)/(Z_{\mathcal{F}}(G)/E) \in \mathcal{F}$. It follows that $G \in \mathcal{F}$, as required.\hfill $\Box$

**Remarks:** Since every $s$-permutable, $c$-normal, SS-quasinormal, nearly $s$-normal and $S$-embedded subgroup of $G$ is weakly SS-quasinormal in $G$, our results in Section 3 generalize many known results in the literature.
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REFERENCES


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