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ON WEAKLY SS-QUASINORMAL AND HYPERCYCLICALLY EMBEDDED PROPERTIES OF FINITE GROUPS

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Abstract. A subgroup H is said to be s-permutable in a group G, if $HP = PH$ holds for every Sylow subgroup P of G. If there exists a subgroup B of G such that $HB = G$ and H permutes with every Sylow subgroup of B, then H is said to be SS-quasinormal in G. In this paper, we say that H is a weakly SS-quasinormal subgroup of G, if there is a normal subgroup T of G such that HT is s-permutable and $H \cap T$ is SS-quasinormal in G. By assuming that some subgroups of G with prime power order have the weakly SS-quasinormal properties, we get some new characterizations about the hypercyclically embedded subgroups of G. A series of known results in the literature are unified and generalized.

1. Introduction

All groups considered in this paper will be finite and we use conventional notions and notation, as in Gorenstein [5]. Let $\mathcal{F}$ be a formation, $\mathcal{U}$ and $\mathcal{N}_p$ denote the classes of all supersoluble and p-nilpotent groups, respectively. $G^{\mathcal{F}}$ denotes the $\mathcal{F}$-residual, $Z_\infty(G)$ is the hypercentre of G. A normal subgroup H is said to be hypercyclically embedded in G, if every chief factor of G below H is cyclic. The product of all hypercyclically embedded subgroups of G is denoted by $Z_{\mathcal{U}}(G)$ and called the $\mathcal{U}$-hypercentre of G.

Recall that a subgroup H of G is said to be s-permutable [11] (or s-quasinormal [3]) in G, if H permutes with every Sylow subgroup P of G. As a development, in [12] the authors introduced that: a subgroup H of G is called an SS-quasinormal (Supplement-Sylow-quasinormal) subgroup of G if there is a supplement B of H to G such that H permutes with every Sylow subgroup of B. Another related concept which was investigated extensively by many scholars was called c-normal subgroup, introduced by Wang in [18]: a subgroup H is said to be c-normal in G if G has a normal subgroup T such that

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$G = HT$ and $H \cap T \leq H_G$, where H_G is the normal core of H in G. There are many generalizations about these concepts, such as s-permutably embedded subgroup [1], c-supplemented subgroup [2], c^*-normal subgroup [21], weakly s-permutable subgroup [15], weakly s-permutably embedded subgroup [14] and nearly s-normal [7] etc. Following Guo et al in [8], a subgroup H is said to be S-embedded in G if there exists a normal subgroup N such that HN is s-permutable in G and $H \cap N \leq H_{sG}$, where H_{sG} is the largest s-permutable subgroup of G contained in H. By assuming that some primary subgroups of G satisfying one of the above properties, many interesting results have been derived.

In this paper, we introduce the concept of weakly SS-quasinormal subgroup, which can cover the SS-quasinormal and S-embedded subgroups properly.

Definition 1.1. *A subgroup H is said to be weakly SS-quasinormal in G, if there exists $T \trianglelefteq G$ such that HT is s-permutable and $H \cap T$ is SS-quasinormal in G.*

Obviously, every S-embedded (s-permutable, c-normal) subgroup or SS-quasinormal subgroup of G is weakly SS-quasinormal in G. But in general, a weakly SS-quasinormal subgroup of G need not be S-embedded or SS-quasinormal in G. For instance:

Example: Let $G = S_5$ be the symmetric group of degree 5, $P \in Syl_5(G)$. Since $G = S_4P$, S_4 is permutable with P and $S_4 \cap P = 1$, S_4 is weakly SS-quasinormal in G. Since $H = \langle (12) \rangle$ satisfying that $HA_5 = G$ and $H \cap A_5 = 1$, H is weakly SS-quasinormal in G. But S_4 is not S-embedded, H is not SS-quasinormal in G.

In this paper, we investigate the influence of some weakly SS-quasinormal subgroups on the structure of a finite group G. Some recent results are generalized.

2. Preliminaries

In this section we gather some results from the literature that will be used later.

Lemma 2.1. *([11]) Suppose that H is an s-permutable subgroup of G, $H \leq G$ and $N \trianglelefteq G$.*

(1) *If $K \leq G$, then $H \cap K$ is s-permutable in K.*
(2) *HN and $H \cap N$ are s-permutable in G, HN/N is s-permutable in G/N.*
(3) *H is subnormal in G.*
(4) *If H is a p-group for some prime p, then $N_G(H) \geq O^p(G)$.*

Proof. (1)-(3) are from [11], (4) is [17, Lemma A]. □

From Lemma 2.1 and Lemma 2.2 of [12], we have the following two results:

Lemma 2.2. *Suppose that H is SS-quasinormal in a group G, $K \leq G$ and $N \trianglelefteq G$.*

(1) *If $H \leq K$, then H is SS-quasinormal in K.*
(2) *HN/N is SS-quasinormal in G/N.*

Lemma 2.3. *If P is a SS-quasinormal p-subgroup of G and $P \leq O_p(G)$, then P is s-permutable in G.*

Now, we can prove that:

Lemma 2.4. *Suppose that H is a weakly SS-quasinormal subgroup of G, $N \trianglelefteq G$.*

(1) *If $H \leq K \leq G$, then H is weakly SS-quasinormal in K.*
(2) *If $N \leq H$, then H/N is weakly SS-quasinormal in G/N.*
(3) *Let π be a set of primes, H a π-subgroup and N a normal π'-subgroup of G. Then HN/N is weakly SS-quasinormal in G/N.*
(4) *If $H \leq K \trianglelefteq G$, then G has a normal subgroup L contained in K such that HL is s-permutable and $H \cap L$ is SS-quasinormal in G.*

Proof. $(1), (2)$ and (4) can be deduced directly by Lemma 2.1 and Lemma 2.2. Now we prove the statement (3). By hypothesis, there exists a normal subgroup T of G such that HT is s-permutable and $H \cap T$ is SS-quasinormal in G. It is easy to see that $TN/N \trianglelefteq G/N$, by Lemma 2.1(2) we know $(HN/N)(TN/N) = HTN/N$ is s-permutable in G/N. Since H is a π-group and N a π'-group,

$$|H \cap TN| = \frac{|H| \cdot |TN|_\pi}{|HTN|_\pi} = \frac{|H| \cdot |T|_\pi}{|HT|_\pi} = |H \cap T|.$$

This implies that $H \cap TN = H \cap T$, so $(HN/N) \cap (TN/N) = (HN \cap TN)/N = (H \cap TN)N/N = (H \cap T)N/N$ which is SS-quasinormal in G/N by Lemma 2.2(2). Hence we proved that HN/N is weakly SS-quasinormal in G/N, as required. □

Lemma 2.5. ([20, Lemma 2.2]) *Let G be a group and p a prime dividing $|G|$ with $(|G|, p-1) = 1$.*

(1) *If N is normal in G of order p, then N lies in $Z(G)$.*
(2) *If G has cyclic Sylow p-subgroups, then G is p-nilpotent.*
(3) *If M is a subgroup of G with index p, then M is normal in G.*

Lemma 2.6. ([4, A, Lemma 1.2]) *Let U, V and W be subgroups of a group G. Then the following statements are equivalent:*

(a) *$U \cap VW = (U \cap V)(U \cap W)$;*
(b) *$UV \cap UW = U(V \cap W)$.*

Lemma 2.7. ([22, I, Theorem 1.4]) *Let H/K be a p-chief factor of a group G. Then $|H/K| = p$ if and only if $G/C_G(H/K) \in \mathcal{A}\ (p-1)$, where $\mathcal{A}\ (p-1)$ is the formation of all abelian groups of exponent dividing $p-1$.*

Lemma 2.8. ([16, Lemma 2.2]) *Let E be a normal p-subgroup of a group G. If $E \leq Z_{\mathcal{U}}(G)$, then $(G/C_G(E))^{A(p-1)} \leq O_p(G/C_G(E))$.*

Lemma 2.9. ([4, A, Lemma 1.2]) *Let P be a p-group, α a p'-automorphism of P.*

(1) *If $[\alpha, \Omega_2(P)] = 1$, then $\alpha = 1$.*
(2) *If $[\alpha, \Omega_1(P)] = 1$ and either p is odd or P is abelian, then $\alpha = 1$.*

3. Main results

Theorem 3.1. *Let p be a prime divisor of $|G|$ with $(|G|, p-1) = 1$, P a Sylow p-subgroup of G. Then G is p-nilpotent if and only if every maximal subgroup of P is weakly SS-quasinormal in G.*

Proof. The necessity is obvious, we need to prove only the sufficiency. Suppose that the result is false and let G be a counterexample of minimal order. Then we have

(1) P is not cyclic and G is not a non-abelian simple group.

By Lemma 2.5(2), we may assume that P is not cyclic. Let P_1 be a maximal subgroup of P, then by hypothesis there exists a normal subgroup T of G such that P_1T is s-permutable and $P_1 \cap T$ is SS-quasinormal in G.

Assume that G is a non-abelian simple group, then $T = 1$ or G. If $T = 1$, then $P_1 = P_1T$ is s-permutable in G. Hence P_1 is a proper subnormal subgroup of G, a contradiction. Thus $T = G$ and therefore $P_1 = P_1 \cap T$ is SS-quasinormal in G. Then there exists some supplement B of P_1 to G such that P_1 permutes with every Sylow subgroup of B. Since $|B : P_1 \cap B|_p = |G : P_1|_p = p$, $S \nsubseteq P_1$ for all $S \in Syl_p(B)$ and $P_1S = SP_1$ is a Sylow p-subgroup of G. By comparison of orders, we know that $S \cap P_1 = B \cap P_1$ holds for all $S \in Syl_p(B)$. Hence $B \cap P_1 = \bigcap_{b\in B}(S^b \cap P_1) \leq \bigcap_{b\in B} S^b = O_p(B)$. Since $|O_p(B) : B \cap P_1| \leq p$, we have $|B/O_p(B)|_p \leq p$. Then by Lemma 2.5(2), $B/O_p(B)$ is p-nilpotent and so B is p-soluble. Hence, B has a Hall p'-subgroup K. Since P_1 permutes with every Sylow subgroup of K, $P_1K = KP_1$ is a subgroup of G with index p. Thus by Lemma 2.5(3), we know that P_1K is normal in G, which is a contradiction. Therefore, G is not a non-abelian simple group.

(2) G has a unique minimal normal subgroup, N say. Moreover, G/N is p-nilpotent and $\Phi(G) = 1$.

Let N be a minimal normal subgroup of G and M/N a maximal subgroup of PN/N. It is easy to see that $M = P_1N$ for some maximal subgroup P_1 of P and $P \cap N = P_1 \cap N$ is a Sylow p-subgroup of N. Since P_1 is weakly SS-quasinormal in G, there exists a normal subgroup K of G such that P_1K is s-permutable and $P_1 \cap K$ is SS-quasinormal in G. Clearly, KN/N is a normal subgroup of G/N and $P_1N/N \cdot KN/N = P_1KN/N$ is s-permutable in G/N. Moreover, since $P_1 \cap N$ is a Sylow p-subgroup of N, $|(P_1 \cap N)(K \cap N)|_p = |P_1 \cap N| = |N|_p = |N \cap P_1K|_p$. Since P_1 is a p-group,

$$|P_1K \cap N|_{p'} = \frac{|P_1K|_{p'} \cdot |N|_{p'}}{|P_1KN|_{p'}} = \frac{|K|_{p'} \cdot |N|_{p'}}{|KN|_{p'}} = |K \cap N|_{p'} = |(P_1 \cap N)(K \cap N)|_{p'}.$$

This statement shows that $(P_1 \cap N)(K \cap N) = P_1K \cap N$. Thus by Lemma 2.6, we have $P_1N \cap KN = (P_1 \cap K)N$. It follows from Lemma 2.2 that $P_1N/N \cap KN/N = (P_1 \cap K)N/N$ is SS-quasinormal in G/N. Therefore, M/N is weakly SS-quasinormal in G/N and so G/N satisfies the hypothesis of the theorem. The minimal choice of G implies that G/N is p-nilpotent.

Since the class of all p-nilpotent groups forms a saturated formation, N is the unique minimal normal subgroup of G and $\Phi(G) = 1$.

(3) $O_{p'}(G) = 1$.

If $O_{p'}(G) \neq 1$, then $N \leq O_{p'}(G)$ and $G/O_{p'}(G)$ is p-nilpotent by (2). Hence G is also a p-nilpotent group, a contradiction.

(4) $O_p(G) = 1$ and N is not p-nilpotent.

If $O_p(G) \neq 1$, then $N \leq O_p(G)$ is an elementary abelian p-group. Since $\Phi(G) = 1$, G has a maximal subgroup M such that $G = MN$ and $M \cap N = 1$. Since $O_p(G) \cap M$ is normalized by N and M, it is normalized by G. The uniqueness of N yields that $N = O_p(G)$. Since $P = N(P \cap M)$ and $N \cap M = 1$,

$P \cap M$ is a Sylow p-subgroup of M and there exists a maximal subgroup P_1 of P such that $P \cap M \leq P_1$ and $P = NP_1$.

Since P_1 is weakly SS-quasinormal in G, there exist $T \trianglelefteq G$ and $B \leq G$ such that P_1T is s-permutable in G, $(P_1 \cap T)B = G$ and $(P_1 \cap T)B_q = B_q(P_1 \cap T)$ for any $B_q \in Syl_q(B)$ with $q \in \pi(B)$. If $T = 1$, then $P_1 = P_1T$ is s-permutable in G. It follows that $P_1 \leq O_p(G) = N$ and so $P = P_1N = N$ is a minimal normal subgroup of G. Since $N_G(P_1) \geq O^p(G)$ by Lemma 2.1(4) and $P_1 \trianglelefteq P$, P_1 is a proper normal subgroup of G contained in $P = O_p(G)$, a contradiction. Thus we have $T \neq 1$ and so $N \leq T$. In this case, $N \cap P_1 = N \cap P_1 \cap T = N \cap (P_1 \cap T)B_q \trianglelefteq (P_1 \cap T)B_q$ for any $B_q \in Syl_q(B)$ with $q \neq p$. Hence $B_q \leq N_G(N \cap P_1)$ holds for any $q \neq p$. Since $N \cap P_1 \trianglelefteq P$, it is normal in G. Thus $N \cap P_1 = 1$ and $|N| = p$. Let C/N be the normal p-complement of G/N, then N is a cyclic Sylow p-subgroup of C. By Lemma 2.5(2), C is p-nilpotent and the normal p-complement of C is also a normal p-complement of G, a contradiction.

If N is p-nilpotent, then $N_{p'}$ *char* $N \trianglelefteq G$, so $N_{p'} \leq O_{p'}(G) = 1$ by (3). Thus N is a p-group and $N \leq O_p(G) = 1$, a contradiction.

(5) $G = PN$.

By Lemma 2.4, we know that PN satisfies the hypothesis of the theorem. Therefore, PN is p-nilpotent if $PN < G$. It follows that N is p-nilpotent, which contradicts with (4). Hence $G = PN$ and $N = O^p(G)$.

(6) The final contradiction.

Since N is not soluble, $N = S_1 \times S_2 \times \cdots \times S_t$ is a direct product of some isomorphic non-abelian simple groups S_i. By (1) and (5), we know $N < G$ and $P \cap N < P$. Thus $S_p = P \cap S_1 \leq P_1$ for some maximal subgroup P_1 of P, where S_p is a Sylow p-subgroup of S_1. Then there exists a normal subgroup T of G such that P_1T is s-permutable in G, and there is a supplement B of $P_1 \cap T$ to G such that $P_1 \cap T$ permutes with every Sylow subgroup of B. If $T = 1$, then P_1 is s-permutable in G and so $O_p(G) \neq 1$, contradicts with (4). Thus $T \neq 1$ and so $N \leq T$. If $P_1 \cap T = 1$, then $|T|_p \leq p$. Hence T is p-nilpotent by Lemma 2.5(2), N is also p-nilpotent. This contradiction shows that $P_1 \cap T \neq 1$. Let B_q be a Sylow q-subgroup of B, where $q \in \pi(B)$ and $q \neq p$. Then $P_1B_q = B_qP_1$ is a subgroup of G and

$$|B_q \cap P_1T| = \frac{|B_q| \cdot |P_1T|_q}{|B_qP_1T|_q} = \frac{|B_q| \cdot |T|_q}{|B_qT|_q} = |B_q \cap T| = |(B_q \cap P_1)(B_q \cap T)|.$$

Since $(B_q \cap P_1)(B_q \cap T) \subseteq B_q \cap P_1T$, this equation implies that $B_q \cap P_1T = (B_q \cap P_1)(B_q \cap T)$. By Lemma 2.6, we have $B_qP_1 \cap B_qT = B_q(P_1 \cap T)$. Therefore, $N \cap P_1B_q = N \cap (P_1B_q \cap TB_q) = N \cap (P_1 \cap T)B_q$ and $S_1 \cap (P_1 \cap T) = S_1 \cap P_1 = S_p$ is a Sylow p-subgroup of S_1. This means that for any prime divisor q $(\neq p)$ of $|S_1|$, $S_1 \cap (P_1 \cap T)B_q$ is a Hall $\{p, q\}$-subgroup of S_1. Since S_1 is non-abelian and $(|S_1|, p-1) = 1$, $p = 2$. Then for any prime divisor $q \neq 2$ of $|S_1|$, non-abelian simple group S_1 has a Hall $\{2, q\}$-subgroup, which contradicts with [13, Lemma 2.6]. This contradiction completes the proof of the theorem. □

Next, we can prove that:

Theorem 3.2. *Suppose that p is a prime divisor of $|G|$ with $(|G|, p-1) = 1$, P a Sylow p-subgroup of G. Then G is p-nilpotent if and only if every cyclic subgroup of P with order p or 4 (if $p = 2$ and P is non-abelian) not having a p-nilpotent supplement in G is weakly SS-quasinormal in G.*

Proof. We need to prove only the sufficiency. Assume that the result is false and let G be a counterexample of minimal order. Then we have

(1) P is not cyclic and G is a minimal non-p-nilpotent group.

By Lemma 2.5, we may assume that P is not cyclic. Let M be a proper subgroup of G, L a cyclic subgroup of $P \cap M$ with order p or 4 (if $p = 2$ and P is non-abelian) having no p-nilpotent supplement in M, then L has no p-nilpotent supplement in G. Hence L is weakly SS-quasinormal in G by hypothesis. In this case, by Lemma 2.4 we know L is a weakly SS-quasinormal subgroup of M. Hence M satisfies the hypothesis of the theorem. The minimal choice of G yields that M is p-nilpotent. Hence, G is a minimal non-nilpotent group and satisfying that: (i) $G = [P]G_q$, where G_q is a non-normal cyclic Sylow q-subgroup of G for some prime q different from p; (ii) $P/\Phi(P)$ is a chief factor of G; (iii) P is of exponent p if $p > 2$ and exponent at most 4 if $p = 2$.

(2) Some minimal subgroup $X/\Phi(P)$ of $P/\Phi(P)$ is not s-permutable in $G/\Phi(P)$.

If every minimal subgroup of $P/\Phi(P)$ is s-permutable in $G/\Phi(P)$, then by [15, Lemma 2.11] we know $P/\Phi(P)$ has a maximal subgroup which is normal in $G/\Phi(P)$. Since $P/\Phi(P)$ is a chief factor of G, $|P/\Phi(P)| = p$ and so P is cyclic, which contradicts with (1). Thus there exists some minimal subgroup $X/\Phi(P)$ of $P/\Phi(P)$ such that $X/\Phi(P)$ is not s-permutable in $G/\Phi(P)$.

(3) The final contradiction.

Let $x \in X \backslash \Phi(P)$, then $\langle x \rangle$ is a cyclic group of prime order or order 4. Let T be any supplement of $\langle x \rangle$ in G. Then $G = \langle x \rangle T$ and $P = P \cap \langle x \rangle T = \langle x \rangle (P \cap T)$. Since $P/\Phi(P)$ is abelian, $(P \cap T)\Phi(P)/\Phi(P) \trianglelefteq G/\Phi(P)$ and hence $(P \cap T)\Phi(P) \trianglelefteq G$. Thus $P \cap T \leq \Phi(P)$ or $P \cap T = P$, as $P/\Phi(P)$ is a chief factor of G. If $P \cap T \leq \Phi(P)$ for some supplement T, then $P = \langle x \rangle$ is cyclic, contradicts with (2). Now assume that $P \cap T = P$ for any supplement T, then $T = G$ is the unique supplement of $\langle x \rangle$ in G. Since G is not p-nilpotent, $\langle x \rangle$ is weakly SS-quasinormal in G by the hypothesis. Thus there exists a normal subgroup K of G contained in P such that $\langle x \rangle K$ is s-permutable and $\langle x \rangle \cap K$ is SS-quasinormal in G. Since $\langle x \rangle \cap K \leq P = O_p(G)$, $\langle x \rangle \cap K$ is s-permutable in G by Lemma 2.3. Since $P/\Phi(P)$ is a chief factor of G, $K \leq \Phi(P)$ or $K = P$. If $K \leq \Phi(P)$, then $X/\Phi(P) = \langle x \rangle K\Phi(P)/\Phi(P)$ is s-permutable in $G/\Phi(P)$, a contradiction. If $K = P$, then $\langle x \rangle = \langle x \rangle \cap K$ is s-permutable in G and so $X/\Phi(P) = \langle x \rangle \Phi(P)/\Phi(P)$ is s-permutable in $G/\Phi(P)$, a contradiction too. This final contradiction completes the proof of the theorem. □

Now we prove our main result:

Theorem 3.3. *Let E be a normal subgroup of G, suppose that there exists a normal subgroup X of G such that $F^*(E) \leq X \leq E$ and for any non-cyclic Sylow subgroup P of X, either every maximal subgroup of P or every cyclic subgroup of P with order p or 4 (if P is a non-abelian 2-group) is weakly SS-quasinormal in G. Then $E \leq Z_{\mathcal{U}}(G)$, i.e., each G-chief factor below E is cyclic.*

Proof. Since $F^*(E) \leq X$, if we can prove $X \leq Z_{\mathcal{U}}(G)$, then we have $F^*(E) \leq Z_{\mathcal{U}}(G)$. By [16, Theorem C], we can conclude that $E \leq Z_{\mathcal{U}}(G)$. Thus, we only need to prove $X \leq Z_{\mathcal{U}}(G)$. Assume that the result is false and let (G, X) be a counterexample of minimal order. Let p =min $\pi(|X|)$ and $P \in Syl_p(X)$, then

(1) X is p-nilpotent.

By [10, IV, Theorem 2.8], we may assume that P is not cyclic. Then the hypothesis and Lemma 2.5 imply that either every maximal subgroup of P or every cyclic subgroup of P with order p or 4 (if P is a non-abelian 2-group) is weakly SS-quasinormal in X. Therefore, by Theorem 3.1 and Theorem 3.2, we can conclude that X is p-nilpotent.

(2) $X = P$ is not cyclic.

Let K be the normal p-complement of X. Since K is a characteristic subgroup of X and $X \trianglelefteq G$, K is normal in G. If $K \neq 1$, then by Lemma 2.4 we know that the hypothesis still holds for (G, K) and $(G/K, X/K)$. Thus the minimal choice of (G, X) implies that $X/K \leq Z_{\mathcal{U}}(G/K)$ and $K \leq Z_{\mathcal{U}}(G)$. Hence we can get $X \leq Z_{\mathcal{U}}(G)$, a contradiction. Therefore, we may assume that $K = 1$ and then $X = P$ is a normal p-subgroup of G. If $P = X$ is a cyclic group, then X is hypercyclically embedded in G, a contradiction.

(3) P is not a minimal normal subgroup of G and every cyclic subgroup of P with prime order or order 4 (if P is a non-abelian 2-group) is S-embedded in G.

By [8, Lemma 2.2], we may assume that P is not a minimal normal subgroup of G.

Now, we suppose that every maximal subgroup of P is weakly SS-quasinormal in G. By Lemma 2.4, we know for any minimal normal subgroup N of G contained in P the hypothesis holds for $(G/N, P/N)$. Hence the minimal choice of (G, X) implies that $P/N \leq Z_{\mathcal{U}}(G/N)$. Therefore, N is the only minimal normal subgroup of G contained in P and $|N| > p$. If $\Phi(P) = 1$, then P is an elementary abelian p-group.

Let N_1 be a maximal subgroup of N, B a complement of N in P and $P_1 = N_1B$. Then P_1 is a maximal subgroup of P and so it is weakly SS-quasinormal in G. Since P_1 is subnormal in G, by Lemma 2.3 we know P_1 is S-embedded in G. Let T be a normal subgroup of G contained in P such that P_1T is s-permutable in G and $P_1 \cap T \leq (P_1)_{sG}$. If $T = 1$ or $T = P$, then P_1 is s-permutable in G and hence $N_1 = N_1(B \cap N) = N_1B \cap N = P_1 \cap N$ is s-permutable in G. Now we suppose that $1 < T < P$, in this case $N \leq T$. Hence $N_1 \leq T$, which implies that $N_1 \leq P_1 \cap T \leq (P_1)_{sG}$. Since $N \cap P_1 = N_1$, $N_1 = N \cap (P_1)_{sG}$ is s-permutable in G. Thus, we have proved that every maximal subgroup of N is s-permutable in G. Hence by [15, Lemma 2.11], we know some maximal subgroup of N is normal in G, a contradiction. Therefore, $\Phi(P) \neq 1$ and $N \leq \Phi(P)$. Since $P/N \leq Z_{\mathcal{U}}(G/N)$, we have $P/\Phi(P) \leq Z_{\mathcal{U}}(G/\Phi(P))$. Then by Lemma 2.8, $(G/C_G(P/\Phi(P)))^{A(p-1)}$ is a p-group. Hence $(G/C)^{A(p-1)}$ is a p-group by [5, Theorem 5.1.4]. Since $O_p(G/C_G(N)) = 1$ by [22, Appendices, Corollary 6.4], $G/C_G(N) \in \mathcal{A}\ (p-1)$. Therefore $|N| = p$ by Lemma 2.7, a contradiction.

By hypothesis, we may assume that every cyclic subgroup H of P with prime order or order 4 (if P is a non-abelian 2-group) is weakly SS-quasinormal in G. By Lemma 2.3, we know they are S-embedded in G.

(4) G has a nontrivial normal subgroup $R \leq P$ such that P/R is a non-cyclic chief factor of G, $R \leq Z_{\mathcal{U}}(G)$ and $V \leq R$ for any normal subgroup $V \neq P$ of G contained in P.

Let P/R be a chief factor of G, then by (3) we know that $R \neq 1$ and the hypothesis holds for (G, R). Hence $R \leq Z_{\mathcal{U}}(G)$ and by the choice of (G, X), we know P/R is not cyclic. Let V be any normal subgroup of G with $V < P$, then $V \leq Z_{\mathcal{U}}(G)$. If $V \nleq R$, then from the G-isomorphism $P/R = VR/R \cong V/(V \cap R)$ we can deduce that $P \leq Z_{\mathcal{U}}(G)$, which is contrary to the choice of (G, P).

If P is a non-abelian 2-group, we use Ω to denote the subgroup $\Omega_2(P)$. Otherwise, let $\Omega = \Omega_1(P)$. Then

(5) $C_G(\Omega)/C$ is a p-group.

This follows from Lemma 2.9.

(6) $\Omega \nleq Z_{\mathcal{U}}(G)$.

Suppose that $\Omega \leq Z_{\mathcal{U}}(G)$, then Lemma 2.8 implies that $(G/C_G(\Omega))^{A(p-1)}$ is a p-group. Hence by (5), $G/C_G(P/R) \in \mathcal{A}\ (p-1)$ and so $|P/R| = p$, a contradiction.

(7) The final contradiction.

By (4) and (6), we know $\Omega = P$. Let $V_1, V_2, \cdots, V_t$ be the set of all cyclic subgroups of P with order p or 4 which is not contained in R, then each V_i is S-embedded in G.

Let T_i be a normal subgroup of G contained in P such that V_iT_i is s-permutable in G and $T_i \cap V_i \leq (V_i)_{sG}$. If V_i is s-permutable in G, then V_iR/R is s-permutable in G/R. Now we assume that V_i is not s-permutable in G, then $1 < T_i < P$. Thus $T_i \leq R$ and $V_iR = V_iT_iR$ is s-permutable in G. Therefore, V_iR/R is s-permutable in G/R in any case. Since $P/R = (V_1R/R)(V_2R/R) \cdots (V_tR/R)$ is an elementary abelian group and $V_iR/R \cong V_i/V_i \cap R$ is cyclic for any $1 \leq i \leq t$, every minimal subgroup of P/R is s-permutable in G/R. Hence by [15, Lemma 2.11], we know that P/R is a cyclic group of order p, a contradiction. This contradiction completes the proof of the theorem. □

From Theorem 3.3, we can immediately deduce that:

Theorem 3.4. *Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$, the classes of all supersoluble groups. Then $G \in \mathcal{F}$ if and only if G has a normal subgroup E such that $G/E \in \mathcal{F}$ and there exists a normal subgroup X of G such that $F^*(E) \leq X \leq E$ and for any non-cyclic Sylow subgroup P of X, either every maximal subgroup of P or every cyclic subgroup of P with order p or 4 (if P is a non-abelian 2-group) is weakly SS-quasinormal in G.*

Proof. The necessity is obvious, we need to prove only the sufficiency. By Theorem 3.3, we know that $E \leq Z_{\mathcal{U}}(G)$ in both cases. Since $Z_{\mathcal{U}}(G) \leq Z_{\mathcal{F}}(G)$, $E \leq Z_{\mathcal{F}}(G)$ and so $G/Z_{\mathcal{F}}(G) \cong (G/E)/(Z_{\mathcal{F}}(G)/E) \in \mathcal{F}$. It follows that $G \in \mathcal{F}$, as required. □

Remarks: Since every s-permutable, c-normal, SS-quasinormal, nearly s-normal and S-embedded subgroup of G is weakly SS-quasinormal in G, our results in Section 3 generalize many known results in the literature.

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