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ON ZERO PATTERNS OF CHARACTERS OF FINITE GROUPS

J. ZHANG*, G. ZENG AND Z. SHEN

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Abstract. The aim of this note is to characterize the finite groups in which all non-linear irreducible characters have distinct zero entries number.

1. Introduction

It is well known that the set of values $\mathrm{cd}(G) = \{\chi(1) : \chi \in \mathrm{Irr}(G)\}$ has a strong influence on the group structure of G, where $\mathrm{Irr}(G)$ denotes the set of irreducible complex characters of G. The aim of this paper is to provide some evidence that also the zeros of irreducible characters encode non-trivial information of G.

Let G be a finite group and $\upsilon(\chi) := \{g \in G \mid \chi(g) = 0\}$, where χ is an irreducible complex character of G. Denote $\mathrm{Irr}_1(G)$ the set of non-linear irreducible complex characters of G. Let $\mathrm{Irr}_1(G) = \{\chi_1, \ldots, \chi_r\}$ and let $k_G(\upsilon(\chi_i)) = n_i$, where $k_G(\upsilon(\chi_i))$ denotes the number of conjugacy classes of G contained in $\upsilon(\chi_i)$. We may assume that $n_1 \leq \cdots \leq n_r$. Then the vector $(n_1, \ldots, n_r)$ is called the zero pattern vector of G. A group with the zero pattern vector $(n_1, \ldots, n_r)$ is said to be of zero pattern $(n_1, \ldots, n_r)$. This paper shows that if the multiplicity of zero entries number of all non-linear irreducible characters is given then the structure of a finite group is restricted.

Berkovich, Chillag and Herzog [1] study all finite groups in which all the non-linear irreducible characters have distinct degrees. Such groups are called D-groups. Motivated by this work, we define

Definition 1.1. A group G is called a DZ-group if G is of zero pattern $(n_1, \ldots, n_r)$ with $n_1 < \cdots < n_r$.

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*Corresponding author.

We should prove that DZ-groups are in accord with D-groups, which is the following result.

Theorem 1.2. *Let G be a finite non-abelian group. Then G is a DZ-group if and only if one of the following occurs:*

(1) G is an extra-special 2-group.

(2) G is a 2-transitive Frobenius group having a cyclic complement.

(3) G is the 2-transitive Frobenius group of order 72 having a quaternion complement.

In this paper, G always denotes a finite group. Notation is standard and taken from [4]. In particular, if G is a nilpotent group, then we denote $\mathrm{C}(G)$ the nilpotent class of G. For $N \lhd G$, set $\mathrm{Irr}(G|N) = \mathrm{Irr}(G)\backslash\mathrm{Irr}(G/N)$.

2. Proof of Theorem 1.2

Let $\mathrm{Lin}(G)$ be the group of linear characters of G so that $\mathrm{Lin}(G)$ acts on $\mathrm{Irr}_1(G)$ by multiplication. If χ is a non-linear irreducible character of G, then for any linear character γ both characters χ and $\gamma\chi$ have the same set of zeros, and so $k_G(\upsilon(\chi)) = k_G(\upsilon(\gamma\chi))$. For a group G, if $G' < G$ and $|C_G(g)| = |C_{G/G'}(gG')|$ holds for any $g \in G - G'$, then (G, G') is called a Camina pair.

We will frequently use the the following lemma (see [7, Theorem 2.2]).

Lemma 2.1. *Let $1 < G' < G$. Then (G, G') is a Camina pair with $k_G(G') = r + 1$ if and only if $|\mathrm{Irr}_1(G)| = r$ and the action of Lin(G) on Irr$_1(G)$ is trivial, that is, $\lambda\chi = \chi$ for every $\lambda \in$ Lin(G) and every $\chi \in \mathrm{Irr}_1(G)$.*

Lemma 2.2. *Let G be a DZ-group. Then*

(1) If χ is a non-linear irreducible character of G, then χ is rational-valued.

(2) G is not a perfect group, that is, $G' < G$.

(3) (G, G') is a Camina pair.

Proof. (1) Let ε be a $|G|$th root of unity and $\sigma \in \mathrm{Gal}(Q(\varepsilon)/Q)$. Then $\chi^\sigma \in \mathrm{Irr}(G)$. As G is a DZ-group, $\chi^\sigma = \chi$ and so $\chi^\sigma(x) = (\chi(x))^\sigma = \chi(x)$ for all $x \in G$. Hence, $\chi(x)$ is rational for all $x \in G$.

(2) Suppose that $G' = G$. Then by (1), G is a rational group. Let N be a normal subgroup of G such that G/N is a composition factor and so G/N is a non-abelian simple group. Recall that G/N is a rational group; then by [2, Corollary B1], $G/N \cong \mathrm{Sp}_6(2)$ or $\mathrm{O}_8^+(2)$.

Suppose that $G/N \cong \mathrm{Sp}_6(2)$. Looking at the character table of $\mathrm{Sp}_6(2)$ from [8], take $\chi_{17}, \chi_{18} \in \mathrm{Irr}(\mathrm{Sp}_6(2))$ with $\chi_{17}(1) = \chi_{18}(1) = 189$. Note that both χ_{17} and χ_{18} vanish on the same elements of G/N. Hence, both χ_{17} and χ_{18} (as members of $\mathrm{Irr}(G)$) vanish on the same elements of G, and thus we obtain a contradiction.

Similarly, if $G/N \cong \mathrm{O}_8^+(2)$, then arguing as the above paragraph, we also obtain a contradiction.

(3) As G is a DZ-group, observe that the action of $\mathrm{Lin}(G)$ on $\mathrm{Irr}_1(G)$ is trivial. Then, by Lemma 2.1, we have that (G, G') is a Camina pair. □

Lemma 2.3. *([9, Lemma 2.4]). Let (G, G') be a Camina pair. Then one of the following holds:*

(1) G is a p-group.

(2) G is a Frobenius group with kernel G'.

(3) G is a Frobenius group with complement Q_8.

Following [5], we shall say that an element x of G is a vanishing element if there exists $\chi \in \mathrm{Irr}(G)$ such that $\chi(x) = 0$. If this is not the case, we shall call x a non-vanishing element.

We will also make use of the following result, which is Theorem A of [5].

Lemma 2.4. *Suppose that G has a normal Sylow p-subgroup P. Then all elements of $Z(P)$ are non-vanishing in G.*

The following result comes from [3], which is the key to the proof of Theorem 1.2.

Proposition 2.5. *Let G be a finite non-abelian group which has an irreducible character χ such that χ does not vanish on exactly two conjugacy classes of G. Let N denote the set consisting of such two conjugacy classes and let $N = \{1, x^G\}$. Then the following results hold:*

(1) χ is unique and is, moreover, the unique faithful irreducible character of G, G contains a unique minimal normal subgroup N which is necessarily an elementary abelian p-group for some prime p. The character χ vanishes on $G \backslash N$ and is non-zero on N.

(2) let λ be an irreducible constituent of χ_N. Define $m = (\chi_N, \lambda)$ and $T = \mathrm{I}_G(\lambda)$ (the inertia group of λ in G). Then λ^T has a unique irreducible constituent, say θ. Moreover, $\theta^G = \chi$, $\theta_N = m\lambda$ and $|T : N| = m^2$ so θ is fully ramified over N.

(3) Both $C_G(x)$ and $T = \mathrm{I}_G(\lambda)$ are Sylow p-subgroups of G.

The groups with exactly two non-linear irreducible characters have been classified by Pálfy (see [6]).

Lemma 2.6. *Let G be a group with exactly two non-linear irreducible characters. Then G is isomorphic to one of the following:*

(1) G is an extra-special 3-group.

(2) G is a 2-group of order 2^{2m+2}, $|G'| = 2$, $|Z(G)| = 4$, and G has two non-linear irreducible characters with equal degree 2^m.

(3) $G = (Q_8, E(3^2))$, where Q_8 is the quaternion group of order 8.

(4) $G = (C((q^m - 1)/2), E(q^m))$, where q is an odd prime.

(5) $G/Z(G) = (C(q^m - 1), E(q^m))$, $cd(G) = \{1, q^m - 1\}$ and $|Z(G)| = 2$, where q is any prime.

Now we prove Theorem 1.2.

Proof. Let G be of zero pattern $(n_1, \ldots, n_r)$ with $n_1 < \cdots < n_r$. Set $\mathrm{Irr}_1(G) = \{\chi_1, \ldots, \chi_r\}$ with $k_G(\upsilon(\chi_i)) = n_i$. Then by Lemma 2.1 and Lemma 2.2, we may assume that $G' = \{1\} \cup x_r^G \cup \cdots \cup x_1^G$.

It follows from the hypothesis that χ_r vanishes on all but two conjugacy classes. Let N denote the set consisting of such two conjugacy classes and let $N = \{1, x_r^G\}$.

Hence by Proposition 2.5(1), N is the unique minimal normal subgroup of G, and so N is an elementary abelian p-group for some prime p. Moreover, χ_r vanishes on $G\backslash N$ (note that $G\backslash N = (G\backslash G') \cup x_1^G \cup ... \cup x_{r-1}^G$). Then by Lemma 2.2 and Lemma 2.3, we have to consider the following three cases:

Case 1. G is a p-group.

By Lemma 2.2(2), G has a real-valued non-principal irreducible character, so that G is even (see [4, (3.16), p. 46]). It follows that $p = 2$.

Recall that χ_r vanishes on $G\backslash N$, then by Lemma 2.4, we easily see that $N = Z(G)$ and so $|Z(G)| = 2$. Note that χ_r is faithful, thus by [4, Corollary 2.30], we get

$$(\chi_r(1))^2 = |G : Z(G)|,$$

and so we may assume that $|G : Z(G)| = 2^{2m}$.

We claim that $r = 1$. Otherwise, $r \geq 2$. Recall that χ_r is the unique faithful irreducible character of G and $Z(G)$ is the unique minimal normal subgroup of G. Observe that $\mathrm{Irr}_1(G/Z(G)) = \{\chi_1, ..., \chi_{r-1}\}$. Now it follows by the hypothesis that χ_{r-1} vanishes on all but three conjugacy classes. Let χ_{r-1} vanish on $(G\backslash G') \cup x_1^G \cup ... \cup x_{r-2}^G$. Consider the group $G/Z(G)$. Observe that χ_{r-1} (as members of $\mathrm{Irr}(G/Z(G))$) vanishes on all but two conjugacy classes of $G/Z(G)$. From the proof above, we obtain

$$(\chi_{r-1}(1))^2 = |G/Z(G) : Z(G/Z(G)| \text{ and } |Z(G/Z(G))| = 2.$$

Hence

$$(\chi_{r-1}(1))^2 = |G/Z(G) : Z(G/Z(G)| = 2^{2m-1}.$$

Then we obtain a contradiction, which yields that $r = 1$. Thus $|Z(G)| = |G'| = 2$. That is, G is an extra-special 2-group.

Case 2. G is a Frobenius group with kernel G' and cyclic complement H.

In this case, by Proposition 2.5(3), $C_G(x_r)$ is a Sylow p-subgroups of G. Since G' is nilpotent, we have that G' is a p-group. Observe that $C_G(x_r) = C_{G'}(x_r) = G'$, and so we easily see that $N = Z(G')$. If G' is abelian, then $N = G'$, and thus we easily see that G is of type (2) in Theorem 1.2.

Now we show that G' is abelian. Assume that G' is not abelian. We seek a contradiction. Without loss of generality, we may assume that $\mathrm{C}(G') = 2$. By proposition 2.5(2), we easily get

$$\chi_r(1) = |H|\sqrt{|G' : Z(G')|}$$

Consider the group $G/Z(G')$. Observe that $G/Z(G')$ is a DZ-group. Note that $(G/Z(G'))' = G'/Z(G')$ is abelian and that $G/Z(G')$ is a Frobenius group with kernel $G'/Z(G')$ and cyclic complement $HZ(G')/Z(G')$. Then $G/Z(G')$ satisfies (2) of Theorem 1.2. Hence $r = 2$ and $\chi_1(1) = |H|$. Thus G has exactly two non-linear irreducible characters. Then by Lemma 2.6, we obtain a contradiction, which implies that G' is abelian, and we are done.

Case 3. G is a Frobenius group with kernel M and complement Q_8.

By Proposition 2.5(3), $C_G(x_r)$ is a Sylow p-subgroups of G. Since M is abelian, we have that M is a p-group. Then by Lemma 2.4, we get $N = M$. Hence we easily see that G satisfies (3) of Theorem 1.2. The proof is complete. □

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Jinshan Zhang

School of Science, Sichuan University of Science and Engineering, 64300 Zigong, China

Email: `zjscdut@163.com`

Guangju Zeng

School of Science, Sichuan University of Science and Engineering, 64300 Zigong, China

Email: `weiwei@suse.edu.cn`

Zhencai Shen

School of Science, China Agricultural University, 100083 Beijing, China

Email: `zhencai688@sina.com`