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# A NOTE ON THE NORMALIZER OF SYLOW 2-SUBGROUP OF SPECIAL LINEAR GROUP $\mathrm{SL}_2(p^f)$

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Abstract. Let $G = \mathrm{SL}_2(p^f)$ be a special linear group and $P$ be a Sylow 2-subgroup of $G$, where $p$ is a prime and $f$ is a positive integer such that $p^f > 3$. By $N_G(P)$ we denote the normalizer of $P$ in $G$. In this paper, we show that $N_G(P)$ is nilpotent (or 2-nilpotent, or supersolvable) if and only if $p^{2f} \equiv 1 \pmod{16}$.

## 1. Introduction

Let $G = \mathrm{SL}_2(p^f)$ be a special linear group and $P$ be a Sylow 2-subgroup of $G$. By $N_G(P)$ we denote the normalizer of $P$ in $G$.

If $p^f = 2$. One has $G \cong S_3$, the symmetric group of degree 3. It follows that $N_G(P) = P$ is nilpotent.

If $p^f = 3$. One has $G \cong Q_8 \rtimes \mathrm{Z}_3$, the semidirect product of the quaternion group $Q_8$ and cyclic group $\mathrm{Z}_3$. It follows that $N_G(P) = G$. It is easy to see that $N_G(P)$ is non-nilpotent. Moreover, $N_G(P)$ is not only non-2-nilpotent but also non-supersolvable.

In this paper, our main goal is to investigate the case $p^f > 3$. We have:

**Theorem 1.1.** *Let $G = \mathrm{SL}_2(p^f)$ and $P$ be a Sylow 2-subgroup of $G$, where $p$ is a prime and $f$ is a positive integer such that $p^f > 3$. Then*

(1) *$N_G(P)$ is nilpotent if and only if $p^{2f} \equiv 1 \pmod{16}$.*
(2) *$N_G(P)$ is 2-nilpotent if and only if $p^{2f} \equiv 1 \pmod{16}$.*
(3) *$N_G(P)$ is supersolvable if and only if $p^{2f} \equiv 1 \pmod{16}$.*

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## 2. On the Normalizer of Sylow 2-Subgroup of $\mathrm{PSL}_2(p^f)$

**Proposition 2.1.** *Let $G = \mathrm{PSL}_2(p^f)$ and $P$ be a Sylow 2-subgroup of $G$, where $p$ is a prime and $f$ is a positive integer such that $p^f > 3$. Then*

(1) $N_G(P) = P \rtimes \mathrm{Z}_{(p^f-1)}$ *if* $p = 2$;

(2) $N_G(P) = P$ *if* $p^{2f} \equiv 1 \,(\mathrm{mod}\, 16)$;

(3) $N_G(P) = A_4$ *if* $p > 2$ *and* $p^{2f} \not\equiv 1 \,(\mathrm{mod}\, 16)$.

*Proof.* (1) Suppose $p = 2$. By [1, II, Theorem 8.27], we can easily get that $N_G(P) = P \rtimes \mathrm{Z}_{(p^f-1)}$.

(2) Suppose $p > 2$ and $p^{2f} \equiv 1 \,(\mathrm{mod}\, 16)$. We claim that

$$\text{either } 8 \mid p^f + 1 \text{ or } 8 \mid p^f - 1.$$

Otherwise, assume that $8 \nmid p^f + 1$ and $8 \nmid p^f - 1$. Since $p^{2f} \equiv 1 \,(\mathrm{mod}\, 16)$, we must have that $4 \mid p^f + 1$ and $4 \mid p^f - 1$. It follows that $4 \mid 2$, a contradiction. Therefore, we have that either $8 \mid p^f + 1$ or $8 \mid p^f - 1$.

($i$) Suppose $p^f + 1 = 8m$ and $p^f - 1 = 2n$, where $m$ and $n$ are positive integers. We claim that

$$n \text{ is an odd integer.}$$

Otherwise, assume that $n$ is even. Then $4 \mid p^f - 1$. Since $8 \mid p^f + 1$, we have $4 \mid 2$, a contradiction. Therefore, $n$ is an odd integer.

It is obvious that $N_G(P) < G$ since $G$ is non-solvable. By [1, II, Theorem 8.27], one has that $N_G(P)$ must be contained in a subgroup $M$ of $G$ that is isomorphic to a dihedral group of order $p^f + 1$. Then $N_G(P) = N_G(P) \cap M = N_M(P) = P$.

($ii$) Suppose $p^f - 1 = 8m$ and $p^f + 1 = 2n$, where $m$ and $n$ are positive integers. Arguing as above, we also have $N_G(P) = P$.

Thus $N_G(P) = P$ for $p^{2f} \equiv 1 \,(\mathrm{mod}\, 16)$.

(3) Suppose $p > 2$ and $p^{2f} \not\equiv 1 \,(\mathrm{mod}\, 16)$, which implies that $8 \nmid p^f + 1$ and $8 \nmid p^f - 1$. Since $p^f > 3$, one has that $G$ is non-solvable. It follows that $G$ is non-2-nilpotent and then $P$ is non-cyclic by [2, Theorem 10.1.9]. So $|P| \geq 4$.

We claim that

$$|P| = 4.$$

Otherwise, assume $|P| > 4$. Since $|G| = \frac{1}{2}p^f(p^f + 1)(p^f - 1)$, we must have $4 \mid p^f + 1$ and $4 \mid p^f - 1$, which implies that $4 \mid 2$, a contradiction. Thus $|P| = 4$.

So by [1, II, Theorem 8.27], we have $N_G(P) = A_4$. □

## 3. On the Normalizer of Sylow 2-Subgroup of $\mathrm{SL}_2(p^f)$

**Proposition 3.1.** *Let $G = \mathrm{SL}_2(p^f)$ and $P$ be a Sylow 2-subgroup of $G$, where $p$ is a prime and $f$ is a positive integer such that $p^f > 3$. Then*

(1) $N_G(P) = P \rtimes \mathrm{Z}_{(p^f-1)}$ *if* $p = 2$;

(2) $N_G(P) = P$ *if* $p^{2f} \equiv 1 \, (\mathrm{mod}\, 16)$;

(3) $N_G(P) = \mathrm{SL}_2(3)$ *if* $p > 2$ *and* $p^{2f} \not\equiv 1 \, (\mathrm{mod}\, 16)$.

*Proof.* (1) Suppose $p = 2$. Then $\mathrm{SL}_2(p^f) \cong \mathrm{PSL}_2(p^f)$. By Proposition 2.1, we have $N_G(P) = P \rtimes \mathrm{Z}_{(p^f-1)}$.

(2) Suppose $p > 2$. We have $G/Z(G) = \mathrm{SL}_2(p^f)/\mathrm{Z}_2 \cong \mathrm{PSL}_2(p^f)$. It is obvious that $P/Z(G)$ is a Sylow 2-subgroup of $G/Z(G)$. Moreover, one has $P/Z(G) \leq N_G(P)/Z(G) = N_{G/Z(G)}(P/Z(G))$.

$(i)$ Suppose $p^{2f} \equiv 1 \, (\mathrm{mod}\, 16)$. By Proposition 2.1, we have $N_{G/Z(G)}(P/Z(G)) = P/Z(G)$. It follows that $N_G(P) = P$.

$(ii)$ Suppose $p^{2f} \not\equiv 1 \, (\mathrm{mod}\, 16)$. By Proposition 2.1, one has $N_{G/Z(G)}(P/Z(G)) \cong A_4$. That is, $N_G(P)/Z(G) = N_G(P)/\mathrm{Z}_2 \cong A_4$. It implies that $N_G(P)$ is a group of order 24 in which the Sylow 2-subgroup is normal but the Sylow 3-subgroup is not normal. By classification of groups of order 24, we have $N_G(P) \cong \mathrm{SL}_2(3)$ or $A_4 \times \mathrm{Z}_2$. However, by [1, II, Theorem 8.10], one has that $P$ is isomorphic to a generalized quaternion group, which implies that $N_G(P) \not\cong A_4 \times \mathrm{Z}_2$. Thus we have $N_G(P) \cong \mathrm{SL}_2(3)$. □

## 4. A Corollary of Propositions 2.1 and 3.1

By $C_G(P)$ we denote the centralizer of $P$ in $G$. The following corollary is a direct consequence of Propositions 2.1 and 3.1.

**Corollary 4.1.** *Let* $G = \mathrm{SL}_2(p^f)$ *or* $\mathrm{PSL}_2(p^f)$ *and* $P$ *be a Sylow* 2*-subgroup of* $G$*, where* $p$ *is a prime and* $f$ *is a positive integer such that* $p^f > 3$*. Then*

(1) *Suppose* $p = 2$*, then* $C_G(P) = P$;

(2) *Suppose* $p^{2f} \equiv 1 \, (\mathrm{mod}\, 16)$*, then* $C_G(P) = Z(P) \cong \mathrm{Z}_2$;

(3) *Suppose* $p > 2$ *and* $p^{2f} \not\equiv 1 \, (\mathrm{mod}\, 16)$*, then* $C_G(P) = C_{\mathrm{SL}_2(3)}(P) \cong \mathrm{Z}_2$ *if* $G = \mathrm{SL}_2(p^f)$ *and* $C_G(P) = C_{A_4}(P) \cong {\mathrm{Z}_2}^2$ *if* $G = \mathrm{PSL}_2(p^f)$.

## 5. Proof of Theorem 1.1

*Proof.* $(i)$ Theorem 1.1 (1) and (2) are direct consequences of Proposition 3.1.

$(ii)$ For proof of Theorem 1.1 (3), we only need to prove that $N_G(P)$ in Proposition 3.1 (1) and (3) are non-supersolvable.

$(a)$ Suppose $N_G(P) = P \rtimes \mathrm{Z}_{(p^f-1)}$, where $p = 2$. Assume that $N_G(P)$ is supersolvable. Let $M$ be any maximal subgroup of $N_G(P)$. It follows that $|N_G(P) : M|$ is a prime.

If $|N_G(P) : M| = 2$. It is clear that $M \trianglelefteq N_G(P)$.

If $|N_G(P) : M| = q \neq 2$. We have $q \mid p^f - 1$. In particular, we have $M = P \rtimes \mathrm{Z}_{(\frac{p^f-1}{q})}$. It is easy to see that $M \trianglelefteq N_G(P)$.

It follows that every maximal subgroup of $N_G(P)$ is normal and then $N_G(P)$ is nilpotent, a contradiction. Therefore, $N_G(P) = P \rtimes \mathrm{Z}_{(p^f-1)}$ is non-supersolvable.

($b$) Suppose $N_G(P) = \mathrm{SL}_2(3)$. It is obvious that $\mathrm{SL}_2(3)$ is non-supersolvable.

Thus we have that $N_G(P)$ is supersolvable if and only if $p^{2f} \equiv 1 \pmod{16}$. □

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