ON ONE CLASS OF MODULES OVER GROUP RINGS WITH FINITENESS RESTRICTIONS

OLGA YU. DASHKOVA

Communicated by Victor Bovdi

ABSTRACT. The author studies the $\mathbf{R}G$-module A such that $\mathbf{R}$ is an associative ring, a group G has infinite section p-rank (or infinite 0-rank), $C_G(A) = 1$, and for every proper subgroup H of infinite section p-rank (or infinite 0-rank respectively) the quotient module $A/C_A(H)$ is a finite $\mathbf{R}$-module. It is proved that if the group G under consideration is locally soluble then G is a soluble group and $A/C_A(G)$ is a finite $\mathbf{R}$-module.

1. Introduction

Let A be a vector space over a field F. The subgroups of the group $GL(F, A)$ of all automorphisms of A are called linear groups. If A has a finite dimension over F then $GL(F, A)$ can be considered as the group of non-singular $(n \times n)$-matrices, where $n = dim_F A$. Finite dimensional linear groups have played an important role in various fields of mathematics, physics and natural sciences, and have been studied many times. When A is infinite dimensional over F, the situation is totally different. Infinite dimensional linear groups have been investigated little. The study of this class of groups requires additional restrictions.

It was introduced the definition of the central dimension of an infinite dimensional linear group [7]. Let H be a subgroup of $GL(F, A)$. H acts on the quotient space $A/C_A(H)$ in a natural way. The authors define $centdim_F H$ to be $dim_F(A/C_A(H))$. The subgroup H is said to have a finite central dimension if $centdim_F H$ is finite and H has infinite central dimension otherwise.

If $G \leq GL(F, A)$ then A can be considered as an FG-module. The natural generalization of this case is the consideration of an $\mathbf{R}G$-module A, where $\mathbf{R}$ is an associative ring whose structure is near

to a field. At this point the generalization of the notion of the central dimension of a subgroup of a linear group is the notion of the cocentralizer of a subgroup [12].

Let A be an $\mathbf{R}G$-module where $\mathbf{R}$ is a ring, G is a group. If $H \leq G$ then the quotient module $A/C_A(H)$ considered as an $\mathbf{R}$-module is called the cocentralizer of a subgroup H in the module A.

Modules over group rings of finite groups have been considered by many authors. Recently this class of modules was investigated in [9]. Study of modules over group rings of infinite groups requires some additional restrictions as in the case of infinite dimensional linear groups. In [6] it was considered linear groups of infinite central dimension satisfying certain restrictions on the ranks of their subgroups. Recall that a group G is said to have finite 0-rank (or finite torsion-free rank), $r_0(G) = r$, if G has a finite subnormal series with exactly r infinite cyclic factors, all other factors being periodic. It is well known that the 0-rank is independent of the chosen series. It is also well known that if G is a group of finite 0-rank, $H \leq G$ and L is a normal subgroup of G, then H and G/L also have finite 0-rank. Furthermore $r_0(H) \leq r_0(G)$ and $r_0(G) = r_0(L) + r_0(G/L)$.

Now let p be a prime. A group G has finite section p-rank, $r_p(G) = r$, if every elementary abelian p-section of G is finite of order at most p^r and there is an elementary abelian p-section U/V such that $|U/V| = p^r$. It is known that if G is a group of finite section p-rank, $H \leq G$ and L is a normal subgroup of G, then H and G/L also have finite section p-rank. Furthermore $r_p(H) \leq r_p(G)$ and $r_p(G) \leq r_p(L) + r_p(G/L)$. Recall that a group G has finite abelian section rank if $r_p(G)$ is finite for all prime p. We note that R. Baer and H. Heineken [1] have shown that for soluble (and even hyperabelian) groups finite abelian section rank is equivalent to finite abelian subgroup rank. Furthermore, a group G has a finite special rank $r(G) = r$ if every finitely generated subgroup of G can be generated by r elements and r is the least positive integer with this property. This notion is due to Mal'cev [13].

In [6] we have studied linear groups G of infinite central dimension and infinite section p-rank such that every proper subgroup of infinite section p-rank has finite central dimension. It was proved that a locally soluble linear group G under consideration is soluble and described the structure of G. The analogous results were obtained in the cases where G has infinite abelian section rank or infinite special rank.

In [2] we have investigated an $\mathbf{R}G$-module A where $\mathbf{R}$ is a dedekind domain, G is a locally soluble group of infinite rank (for different ranks), $C_G(A) = 1$, $A/C_A(G)$ is not an artinian $\mathbf{R}$-module and for every proper subgroup H of infinite rank the cocentralizer of H in A is an artinian $\mathbf{R}$-module. The similar problems in the cases of the noetherian or minimax condition for commutative noetherian rings and commutative rings of scalars were considered in [3]–[5]. In all cases the structure of the groups under consideration were described.

The subject of the investigation of this paper is an $\mathbf{R}G$-module A where $\mathbf{R}$ is an associative ring, G is a locally soluble group of infinite rank (for different ranks), $C_G(A) = 1$, and for every proper subgroup H of infinite rank $A/C_A(H)$ is a finite $\mathbf{R}$-module. As appeared, unlike the previous, $A/C_A(G)$ cannot be infinite $\mathbf{R}$-module.

The main results of the work are theorems 4.7–4.9.

2. **Preliminary results**

In this section it was established some of the elementary properties of the groups of this type.

Later on we consider the $\mathbf{R}G$-module A such that $C_G(A) = 1$, $\mathbf{R}$ is an associative ring.

Lemma 2.1. *Let A be an $\mathbf{R}G$-module, $r_p(G)$ be infinite for some $p \geq 0$ and $A/C_A(G)$ be an infinite $\mathbf{R}$-module. Suppose that for every proper subgroup M such that $r_p(M)$ is infinite the cocentralizer of M in A is a finite $\mathbf{R}$-module. The following statements are valid:*

(i) If U, V are proper subgroups of G and $G = \langle U, V \rangle$ then at least one of $r_p(U)$, $r_p(V)$ is finite.

(ii) If H is a proper subgroup of G such that $r_p(H)$ is infinite then the cocentralizer of every subgroup of H and the cocentralizer of every subgroup of G containing H in A are finite $\mathbf{R}$-modules.

(iii) If K, L are proper subgroups of G containing H, then $\langle K, L \rangle$ is a proper subgroup of G.

Lemma 2.2. *Let A be an $\mathbf{R}G$-module, $r_p(G)$ be infinite for some $p \geq 0$ and $A/C_A(G)$ be an infinite $\mathbf{R}$-module. Suppose that for every proper subgroup M such that $r_p(M)$ is infinite the cocentralizer of M in A is a finite $\mathbf{R}$-module. If K is a proper normal subgroup of G such that $r_p(K)$ is infinite and G/K is finitely generated, then G/K is a cyclic q-group for some prime q.*

Proof. Suppose that $G = \langle K, S \rangle$ for some finite set S with the property that if T is a proper subset of S then $G \neq \langle K, T \rangle$. Let $S = \langle x_1, x_2, \ldots, x_n \rangle$. If $n > 1$, then $\langle K, x_1, x_2, \ldots, x_{n-1} \rangle$ and $\langle K, x_n \rangle$ are proper, and Lemma 2.1 provides a contradiction. It follows that G/K is cyclic. If G/K is infinite, or if G/K is finite but $|\pi(G/K)| > 1$, then G is a product of two proper subgroups G_1 and G_2 such that $r_p(G_1)$ and $r_p(G_2)$ are infinite. Lemma 2.1 again gives a contradiction. Hence G/K is a cyclic q-group for some prime q. $\square$

Lemma 2.3. [6] *Let G be a group and let q be a prime. Suppose that A is an infinite normal elementary abelian q-subgroup of G such that G/A is finite. Then G is generated by two proper subgroups having infinite section q-rank.*

Lemma 2.4. [6] *Let G be a group and q be a prime. Let A be a normal divisible abelian q-subgroup of G such that G/A is finite. If A has infinite section q-rank, then G is generated by two proper subgroups having infinite section q-rank.*

We shall also use the following result.

Lemma 2.5. [6] *Let G be a group and suppose that A is a normal subgroup of G such that G/A is infinite, periodic and abelian-by-finite. If $|\pi(G/A)| > 1$ then G is a product of two proper subgroups containing A.*

We apply these results in the next lemma.

Lemma 2.6. *Let A be an $\mathbf{R}G$-module, $r_p(G)$ be infinite for some $p \geq 0$ and $A/C_A(G)$ be an infinite $\mathbf{R}$-module. Suppose that for every proper subgroup M such that $r_p(M)$ is infinite the cocentralizer of M in A is a finite $\mathbf{R}$-module. If K is a normal subgroup of H, $H \leq G$, H/K is abelian-by-finite and $r_p(H/K)$ is infinite then the cocentralizer of H in A is a finite $\mathbf{R}$-module.*

Proof. Suppose first that $p = 0$. Let L be a normal subgroup of H such that H/L is finite and L/K is abelian. $r_0(L/K)$ is infinite so L/K contains a free abelian subgroup B/K such that $r_0(B/K)$ is infinite and L/B is periodic. Since H/L is finite, B has only finitely many conjugates in H, $B_1, \ldots, B_m$ say, and if $C = core_H B$ then there is an embedding $L/C \leq L/B_1 \times L/B_2 \times \cdots \times L/B_m$. It follows that L/C is periodic and hence $r_0(C/K)$ is infinite as is $r_0(C)$. Note also that C/K is free abelian. If H/C is finite or $|\pi(H/C)| = 1$, then choose a prime $q \notin \pi(H/C)$ and set $D/K = (C/K)^q$. If H/C is infinite and $|\pi(H/C)| > 1$, then let $D = C$. Then, in each case, H/D is infinite, periodic and $|\pi(H/D)| > 1$. Also $r_0(D)$ is infinite. Applying Lemmas 2.5 and 2.1 we see that the cocentralizer of H in A is a finite **R**-module.

Now suppose that $p > 0$ and let L be defined as before. Choose a free abelian subgroup B/K of L/K such that L/B is periodic. If $r_0(B/K)$ is infinite then we can proceed as in the case $p = 0$, so suppose that $r_0(B/K)$ is finite. As above, if $C = core_H B$ then L/C is periodic and $r_p(L/C)$ is infinite. Factoring by the Sylow p'-subgroup of L/C, if necessary, we may assume that L/C is a p-group. If $L/L^p C$ is infinite then $H/L^p C$ satisfies the hypotheses of Lemma 2.3 so H is the product of two proper subgroups, each of infinite section p-rank and hence the cocentralizer of each in A is a finite **R**-module. Thus the cocentralizer of H in A is a finite **R**-module in this case. If $L/L^p C$ is finite then using properties of basic subgroups, we have that $L/C = E/C \times D/C$ for some finite subgroup E/C and divisible subgroup D/C. Since H/L is finite and L/C is abelian, $F/C = (E/C)^{H/C}$ is also finite. Furthermore L/F is a divisible abelian p-group of infinite section p-rank. Now we can apply Lemma 2.4 to H/F to deduce that H is a product of two proper subgroups, each of infinite section p-rank, and hence the cocentralizer of each in A is a finite **R**-module. Thus the cocentralizer of H in A is a finite **R**-module. $\square$

Lemma 2.7. *Let A be an **R**G-module, $r_p(G)$ be infinite for some $p \geq 0$ and $A/C_A(G)$ be an infinite **R**-module. Suppose that for every proper subgroup M such that $r_p(M)$ is infinite the cocentralizer of M in A is a finite **R**-module. If H is a normal subgroup of G and G/H is abelian-by-finite, then G/H is isomorphic to a subgroup of C_{q^∞}, for some prime q.*

Proof. We may assume that $G \neq H$. If $r_p(G/H)$ is infinite, then the cocentralizer of G in A is a finite **R**-module by Lemma 2.6. Thus $r_p(G/H)$ is finite and therefore $r_p(H)$ is infinite. Moreover, if G/H is finite then the result follows by Lemma 2.2. Thus we may suppose that G/H is infinite.

We first suppose that G/H is an abelian group. By Lemma 2.1(iii) G/H cannot be free abelian. Let B/H be a free abelian subgroup of G/H such that G/B is periodic. Since $r_p(H)$ is infinite, $r_p(B)$ is also infinite. If $|\pi(G/B)| > 1$, then clearly G is a product of proper subgroups G_1 and G_2 such that $r_p(G_1)$ and $r_p(G_2)$ are infinite. Lemma 2.1(iii) gives a contradiction. Thus G/B is a q-group, for some prime q. If B/H is nontrivial let r be a prime distinct from q and let $C/H = (B/H)^r \neq B/H$. Then G/C is periodic, $\pi(G/C) = \{q, r\}$ and we now obtain a contradiction. It follows that G/H is a periodic q-group. If G/H is divisible then it is a direct product of copies of Prüfer q-groups and Lemma 2.1(iii) implies that $G/H \simeq C_{q^\infty}$. Otherwise $(G/H)/(G/H)^q$ is a nontrivial elementary abelian q-group and

Lemma 2.1(iii) shows that $|(G/H)/(G/H)^q| = q$. It is then easy to see that $G/H = (E/H) \times (D/H)$ where D/H is divisible and $|E/H| = q$ and Lemma 2.1(iii) again gives a contradiction.

For the general case let L/H be a normal abelian subgroup of G/H such that G/L is finite. Let U/H be an arbitrary subgroup of finite index in G/H. If $V/H = core_{G/H}U/H$ then G/V is also finite and $r_p(V)$ is infinite. By Lemma 2.2 G/V is a cyclic q-group for some prime q so $G' \le V \le U$. Thus if W/H is the finite residual of L/H then G/W is abelian, $r_p(W)$ is infinite and, by the first part of the argument, G/W is finite, since it is residually finite. Thus $G = WK$ for some subgroup K containing H such that K/H is finitely generated. Since G/H is infinite Lemma 2.2 implies that $G \ne K$. Then from Lemma 2.1(iii) we deduce that $G = W$, so G/H is abelian and the result follows by the first part of the proof. $\qquad\square$

3. Soluble groups

In this section we apply the results of Section 2 to soluble groups.

Theorem 3.1. *Let A be a $\mathbf{R}G$-module, G be a soluble group and $r_p(G)$ be infinite for some $p \ge 0$. Suppose that for every proper subgroup M such that $r_p(M)$ is infinite the cocentralizer of M in A is a finite $\mathbf{R}$-module. Then $A/C_A(G)$ is a finite $\mathbf{R}$-module.*

Proof. Suppose that $A/C_A(G)$ is an infinite $\mathbf{R}$-module. If $G = D_0 \ge D_1 \ge \cdots \ge D_{n-1} \ge D_n = E$ is the derived series of G then there is a natural number m such that G/D_m is finite, but D_m/D_{m+1} is infinite. Let $K = D_m$. By Lemma 2.7 $G/K' \simeq C_{q^\infty}$, for some prime q. Since $r_p(K')$ is infinite the cocentralizer of K' in A is a finite $\mathbf{R}$-module. Let $C = C_A(K')$. Hence A/C is a finite $\mathbf{R}$-module. Now $K' \le C_G(C)$ and, since the cocentralizer of G in A is an infinite $\mathbf{R}$-module, $G/C_G(C) \simeq C_{q^\infty}$. Since K' is a normal subgroup of G, C is an $\mathbf{R}G$-submodule of A. Since A/C is a finite $\mathbf{R}$-module, $G/C_G(A/C)$ is finite. Let $H = C_G(C) \cap C_G(A/C)$. Each element of H acts trivially on every factor of the series $\langle 0 \rangle \le C \le A/C$. By Kaluzhnin Theorem (p. 144 [10]) H is abelian. By Remak's Theorem

$$G/H \le G/C_G(C) \times G/C_G(A/C).$$

Therefore G/H is abelian-by-finite. It follows from Lemma 2.7 that G/H is isomorphic to C_{q^∞}, for some prime q. Let K/H be any finite subgroup of G/H. Since $r_p(H)$ is infinite $r_p(K)$ is infinite also. Therefore $A/C_A(K)$ is a finite $\mathbf{R}$-module and $G/C_G(A/C_A(K))$ is finite. Put $L = C_G(A/C_A(K))$. Let $L \ne G$. If LH is a proper subgroup of G then LH/H is a proper infinite subgroup of G/H. Contradiction. It follows that $LH = G$. Contradiction with Lemma 2.1. Hence $G = C_G(A/C_A(K))$. Therefore $[G, A] \le C_A(K)$. From the choice of K it follows that $[G, A] \le C_A(G)$ and so G acts trivially on every factor of the series $0 \le C_A(G) \le A$. By Kaluzhnin Theorem (p. 144 [10]) G is abelian. It follows that we can choose the proper subgroups G_1 and G_2 of G such that $G = G_1G_2$ and $r_p(G_1)$ and $r_p(G_2)$ are infinite. Contradiction with Lemma 2.1. It follows that $A/C_A(G)$ is a finite $\mathbf{R}$-module. $\quad\square$

Theorem 3.2. *Let A be an $\mathbf{R}G$-module, G be a soluble group of infinite abelian section rank. Suppose that the cocentralizer of every proper subgroup of infinite abelian section rank in A is a finite $\mathbf{R}$-module. Then $A/C_A(G)$ is a finite $\mathbf{R}$-module.*

Proof. Since G is soluble and has infinite abelian section rank, there is a prime p such that $r_p(G)$ is infinite. For this prime, if H is a proper subgroup and $r_p(H)$ is infinite then H has infinite abelian section rank and therefore the cocentralizer of H in A is a finite $\mathbf{R}$-module. We may now apply Theorem 3.1. $\qquad\square$

Theorem 3.3. *Let A be an $\mathbf{R}G$-module, G be a soluble group of infinite special rank. Suppose that the cocentralizer of every proper subgroup of infinite special rank in A is a finite $\mathbf{R}$-module. Then $A/C_A(G)$ is a finite $\mathbf{R}$-module.*

Proof. First we consider the case when $A/C_A(G)$ is an infinite $\mathbf{R}$-module. If G has infinite abelian section rank and X is a proper subgroup of infinite abelian section rank then X has infinite special rank, so the cocentralizer of X in A is a finite $\mathbf{R}$-module. The result then follows from Theorem 3.2. Therefore we suppose that G has finite abelian section rank.

Let U be a normal subgroup of G such that G/U is infinite abelian-by-finite and let V/U be a normal abelian subgroup of G/U such that G/V is finite. Since $r_0(G)$ is finite V/U contains a finitely generated subgroup B/U such that V/B is periodic. If $C/U = (B/U)^{G/U}$, then C/U is finitely generated also. Suppose that G/U has infinite special rank. Since G has finite abelian section rank it follows that p-subgroups of V/C are Chernikov, for each prime p. Thus $\pi(V/C)$ is infinite. If D/C is the Sylow $\pi(G/V)$-subgroup of V/C then V/D has infinite special rank, $G/D = (V/D)(W/D)$, where V/D is a normal subgroup of G/D, $(V/D) \cap (W/D) = E$, W/D is finite, and $\pi(V/D) \cap \pi(W/D)$ is empty. Then V/D is a product of two G-invariant subgroups of infinite special rank, so that G/D is a product of two proper subgroups of infinite special rank. Thus the cocentralizer of G in A is a finite $\mathbf{R}$-module. Thus G/U has finite special rank and hence U has infinite special rank. As in Section 2 we deduce that $G/U \simeq C_{q^\infty}$ for some prime q. As in Theorem 3.1, G is abelian. It follows that we can choose the proper subgroups G_1 and G_2 of G such that $G = G_1 G_2$ and $r_p(G_1)$ and $r_p(G_2)$ are infinite. Contradiction with Lemma 2.1. It follows that $A/C_A(G)$ is a finite $\mathbf{R}$-module. $\qquad\square$

4. **Locally soluble groups**

In this section we show that locally soluble groups of the type under discussion are actually soluble.

Lemma 4.1. *Let A be a $\mathbf{R}G$-module, G be a locally soluble group. Suppose that the cocentralizer of G in A is a finite $\mathbf{R}$-module. Then G is almost abelian.*

Proof. Let $C = C_A(G)$. Then A has the series of $\mathbf{R}G$-submodules $\langle 0 \rangle \leq C \leq A$, where A/C is a finite $\mathbf{R}$-module. Since $G \leq C_G(C)$ then $G/C_G(C)$ is trivial. As A/C is a finite $\mathbf{R}$-module then $G/C_G(A/C)$ is finite.

Let $H = C_G(C) \cap C_G(A/C)$. Each element of H acts trivially on every factor of the series $\langle 0 \rangle \leq C \leq A/C$. By Kaluzhnin Theorem (p. 144 [10]) H is abelian. By Remak's Theorem

$$G/H \leq G/C_G(C) \times G/C_G(A/C).$$

It follows that G/H is finite. Then G is an almost abelian group.

$\square$

Lemma 4.2. *Let A be a $\mathbf{R}G$-module, G be a locally soluble group, $r_p(G)$ be infinite for some $p \geq 0$ and $A/C_A(G)$ be an infinite $\mathbf{R}$-module. Suppose that for every proper subgroup M such that $r_p(M)$ is infinite the cocentralizer of M in A is a finite $\mathbf{R}$-module. If G is not soluble then G is perfect.*

Proof. Note that H is a proper normal subgroup of G of finite index then $r_p(H)$ is infinite and hence the cocentralizer of H in A is a finite $\mathbf{R}$-module. It follows from Lemma 4.1 that H is soluble and from Lemma 2.2 that G/H is abelian. Thus G is soluble which is a contradiction. Hence if $G \neq G'$ then G/G' is divisible. It follows that G contains normal subgroup H such that $G/H \simeq C_{q^\infty}$, for some prime q. Then $r_p(H)$ is infinite, hence the cocentralizer of H in A is a finite $\mathbf{R}$-module. By Lemma 4.1 H is soluble. Then G is soluble. Contradiction. $\square$

Let $d(G)$ denote the derived length of the soluble group G and let $T(G)$ denote the maximal normal periodic subgroup of G.

Lemma 4.3. [6] *Let p be a prime, let G be a soluble group and suppose that $r_p(G) = r$. There are functions $s_p : \mathbf{N} \longrightarrow \mathbf{N}$, $f_p : \mathbf{N} \longrightarrow \mathbf{N}$ such that $d(G/T(G)) \leq s_p(r)$ and $G/T(G)$ has finite special rank at most $f_p(r)$.*

We shall use the notation established in Lemma 4.3 in the next result.

Theorem 4.4. [6]. *Let p be a prime and let G be a locally soluble group such that $r_p(G) = r$. Then $G/T(G)$ is a soluble group such that $d(G/T(G)) \leq s_p(r)$ and $G/T(G)$ has finite special rank at most $f_p(r)$.*

Lemma 4.5. *Let A be a $\mathbf{R}G$-module, G be a locally soluble group, $r_p(G)$ be infinite for some $p \geq 0$ and $A/C_A(G)$ be an infinite $\mathbf{R}$-module. Suppose that for every proper subgroup M such that $r_p(M)$ is infinite the cocentralizer of M in A is a finite $\mathbf{R}$-module. If G is not soluble, H is a normal subgroup of G and $r_p(H)$ is finite, then $H/T(H)$ is G-central.*

Proof. Let $r_p(H) = r$. If $p = 0$, we apply Lemma 2.12 [8] and if $p > 0$ then we apply Theorem 4.4 to H and, in either case, we see that $H/T(H)$ is soluble and has finite special rank, which is a function of r. Let $n = r_0(H/T(H))$, which is dependent upon r only. Then H has a finite series of G-invariant subgroups

$$T(H) = H_0 \leq H_1 \leq \cdots \leq H_d = H,$$

each of whose factors is abelian.

We see that $H_1/T(H)$ is torsion-free and of finite rank at most n so that $Aut(H_1/T(H))$ is isomorphic to a subgroup of $GL(n, \mathbf{Q})$. Hence $G/C_G(H_1/T(H))$ is a locally soluble group isomorphic to a subgroup of $GL(n, \mathbf{Q})$. It follows from Corollary 3.8 [15] that $G/C_G(H_1/T(H))$ is soluble and hence trivial by Lemma 4.2. Thus $[G, H_1] \le T(H)$.

Suppose inductively that $[G, H_{d-1}] \le T(H)$. Then $H_{d-1}/T(H) \le Z(G/T(H))$ so that $H/T(H)$ is nilpotent of class at most 2. Let $K/T(H) = Z(H/T(H))$. Then, as above, since $K/T(H)$ and H/K are torsion-free abelian and of rank at most n we have $[G, K] \le T(H)$ and $[G, H] \le K$. It follows by the three subgroups lemma and Lemma 4.2 that $[G, H] = [G, G, H] \le T(H)$ and the result follows by induction on d. $\square$

Lemma 4.6. *Let A be a $\mathbf{R}G$-module, G be an insoluble, locally soluble group, $r_p(G)$ be infinite for some $p \ge 0$ and $A/C_A(G)$ be an infinite $\mathbf{R}$-module. Suppose that for every proper subgroup M such that $r_p(M)$ is infinite the cocentralizer of M in A is a finite $\mathbf{R}$-module. Then G contains a proper normal subgroup V such that if U is a normal subgroup of G and $V \le U \le G$, $U \ne G$, then U is soluble and the cocentralizer of U in A is a finite $\mathbf{R}$-module.*

Proof. Let $T = T(G)$ and suppose first that $T \ne G$ and $r_p(T)$ is finite (if $p = 0$ these conditions are automatically satisfied). By Lemma 4.2, G/T is not soluble and hence is not simple by Corollary 1 to Theorem 5.27 [14]. Hence G contains a proper normal subgroup $L \ge T$, $L \ne T$. If $r_p(L)$ is finite Lemma 4.5 implies that L/T is G-central and hence G/T contains a nontrivial maximal normal abelian subgroup V/T. Certainly $V \ne G$. If U is a normal subgroup of G and $V \le U \le G$, $V \ne U, U \ne G$, then $r_p(U)$ is infinite, by Lemma 4.5 and, by hypothesis, the cocentralizer of U in A is a finite $\mathbf{R}$-module. Then U is soluble by Lemma 4.1. If there is no such subgroup L then we set $V = T$ and, as above, V has required property.

Next suppose that $p > 0$. Suppose first that $r_p(T)$ is infinite. If $T \ne G$ then the cocentralizer of T in A is a finite $\mathbf{R}$-module. We deduce from Lemma 4.1 that T is soluble. Then G/T is not soluble and hence is not simple by Corollary 1 to Theorem 5.27 [14]. If U is a normal subgroup of G and $T \le U \le G$, $U \ne G$ then $r_p(U)$ is infinite so the cocentralizer of U in A is a finite $\mathbf{R}$-module. By Lemma 4.1 U is soluble. Thus we may set $T = V$ in this case.

Now let $T = G$. Suppose first that the Sylow p-subgroups of G are all of finite section p-rank. Then G has the minimal condition on p-subgroups, by Lemma 3.1 [11]. We obtain a contradiction, since in this case the p-subgroups are Chernikov and hence of finite, bounded ranks. Thus G contains some p-subgroup P of infinite rank and hence G contains an infinite elementary abelian p-subgroup A by Corollary 2 to Theorem 6.36 [14]. Certainly A is a proper subgroup of G and if U is a normal subgroup of G then $UA \ne G$; otherwise G/U is abelian, contrary to Lemma 4.2. Thus UA is a proper subgroup of G of infinite section p-rank and hence cocentralizer of UA in A is a finite $\mathbf{R}$-module. Consequently cocentralizer of U in A is a finite $\mathbf{R}$-module. By Lemma 4.1 U is soluble. We set $V = 1$ in this case. $\square$

Theorem 4.7. *Let A be a $\mathbf{R}G$-module, G be a locally soluble group, $r_p(G)$ be infinite for some $p \geq 0$. Suppose that for every proper subgroup M such that $r_p(M)$ is infinite the cocentralizer of M in A is a finite $\mathbf{R}$-module. Then G is soluble and $A/C_A(G)$ is a finite $\mathbf{R}$-module.*

Proof. Suppose, on the contrary, that G is not soluble. By Lemma 4.6 G contains a normal subgroup V with the property that if U is a normal subgroup of G and $V \leq U \leq G, U \neq G$ then U is soluble and the cocentralizer of U in A is a finite $\mathbf{R}$-module. Set $V = U_0$ and $d(U_0) = d_0$. Assume that we have constructed normal soluble subgroups $U_0 \leq U_1 \leq \cdots \leq U_n$ such that $d(U_i) = d_i$ for $i = 0, 1, \ldots, n$, and that $d_i < d_{i+1}$ for $i = 0, 1, \cdots, n - 1$. Since G is not soluble, there exists a normal subgroup U_{n+1}, containing U_n, such that $d(U_{n+1}) = d_{n+1} > d(U_n)$ and we therefore obtain an ascending chain of soluble normal subgroups of increasing derived lengths. Let $W = \bigcup_{n \geq 1} U_n$. By construction W is not soluble and $V \leq W$. It follows that $W = G$.

Now let $C_A(U_n) = C_n$ for each $n \in \mathbf{N}$. Since U_n is a normal subgroup of G, C_n is an $\mathbf{R}G$-submodule for each n. Since the cocentralizer of U_n in A is a finite $\mathbf{R}$-module, A/C_n is a finite $\mathbf{R}$-module. Thus $G/C_G(A/C_n)$ is finite. We deduce from Lemma 4.2 that $G = C_G(A/C_n)$, for each $n \in \mathbf{N}$. Since $G = \bigcup_{n \geq 1} U_n$ it follows that $C_A(G) = \bigcap_{n \geq 1} C_A(U_n) = \bigcap_{n \geq 1} C_n$ and hence G acts trivially on the factors of the series $0 \leq C_A(G) \leq A$. In this case G is clearly abelian. This contradiction proves that G is actually soluble. By Theorem 3.1 $A/C_A(G)$ is a finite $\mathbf{R}$-module. $\qquad\square$

Using a method analogous to the proof of Theorem 3.2 and applying Theorem 4.7 we obtain the following result.

Theorem 4.8. *Let A be an $\mathbf{R}G$-module, G be a locally soluble group of infinite abelian section rank. Suppose that the cocentralizer of every proper subgroup of infinite abelian section rank in A is a finite $\mathbf{R}$-module. Then G is soluble and $A/C_A(G)$ is a finite $\mathbf{R}$-module.*

We just sketch the following theorem.

Theorem 4.9. *Let A be an $\mathbf{R}G$-module, G be a locally soluble group of infinite special rank. Suppose that the cocentralizer of every proper subgroup of infinite special rank in A is a finite $\mathbf{R}$-module. Then G is soluble and $A/C_A(G)$ is a finite $\mathbf{R}$-module.*

Proof. Let G be a counterexample to the theorem. If N is a proper normal subgroup of infinite special rank then the cocentralizer of N in A is a finite $\mathbf{R}$-module and it follows from Lemma 4.1 that N is soluble. If N is of finite special rank then by Lemma 10.39 [14] N is hyperabelian. Let $\{N_\alpha\}$ be a family of all proper normal subgroups of G. Then subgroup $J = \prod N_\alpha$ is also hyperabelian. Since a simple locally soluble group is cyclic it follows that G is also hyperabelian. By Theorem 7.1 [1] G contains a subgroup K that is either an elementary abelian q-subgroup of infinite special rank, for some prime q, or a torsion-free abelian subgroup of infinite special rank. Let N be a proper normal subgroup of G of finite special rank. By Lemma 10.39 [14] there exists a natural number d such that $N^{(d)}$ is a direct product of Chernikov p-groups, for different primes p. If $N^{(d)}K \neq G$ then N is soluble. If $N^{(d)}K = G$ let r be some prime different from q, $r \in \pi(N^{(d)})$, and let X be the Sylow

$\{q,r\}'$-subgroup of $N^{(d)}$. Then $XK \neq G$ and, as above, X, and hence N, is soluble. Thus every proper normal subgroup of G is soluble and the cocentralizer of it in A is a finite **R**-module. The proof now proceeds as in Theorem 4.7. □

Example.

Let $A = Dr_{n \in \mathbb{N}} \langle a_n \rangle$ be a vector space over a field F, $charF = p$, p be a prime, $G = Dr_{k \in \mathbb{N}} \langle g_k \rangle$, $|g_k| = p, k \in \mathbb{N}$.

Let $a_1 g_i = a_1 + a_{i+1}$, $a_l g_i = a_l, l = 2, 3, \ldots, i \in \mathbb{N}$. Then $G \leq GL(F, A)$ and $C_A(G) = Dr_{j \geq 2} \langle a_j \rangle$. It follows that $|A/C_A(G)| = p$.

References

[1] R. Baer and H. Heineken, Radical groups of finite abelian subgroup rank, *Illinois J. Math.*, **16** no. 4 (1972) 533–580.

[2] O. Yu. Dashkova, On modules over group rings of locally soluble groups with rank restrictions on some systems of subgroups, *Asian-Eur. J. Math.*, **3** no. 1 (2010) 45–55.

[3] O. Yu. Dashkova, On one class of modules that are close to Noetherian, *J. Math. Sci. (N. Y.)*, **169** no. 5 (2010) 636–643.

[4] O. Yu. Dashkova, On modules over integer-valued group rings of locally soluble groups with rank restrictions imposed on subgroups, *Ukrainian Math. J.*, **63** no. 9 (2012) 1379–1389.

[5] O. Yu. Dashkova, Modules over group rings of solvable groups with rank restrictions on subgroups, *Siberian Adv. Math.*, **23** no. 2 (2013) 77–83.

[6] O. Yu. Dashkova, M. R. Dixon and L. A. Kurdachenko, Linear groups with rank restrictions on the subgroups of infinite central dimension, *J. Pure Appl. Algebra*, **208** no. 3 (2007) 785-795.

[7] M. R. Dixon, M. J. Evans and L. A. Kurdachenko, Linear groups with the minimal condition on subgroups of infinite central dimension, *J. Algebra*, **277** no. 1 (2004) 172–186.

[8] S. Franciosi, F. De Giovanni and L. A. Kurdachenko, The Schur property and groups with uniform conjugacy classes, *J. Algebra*, **174** (1995) 823-847.

[9] P. M. Gudivok, V. P. Rudko and V. A. Bovdi, *Crystallographic groups*, (in Ukrainian), Uzhgorod: Uzhgorodskii Natsionalnii Universitet, 2006 174 pp.

[10] M. I. Kargapolov and Yu. I. Merzlyakov, Bases of the Theory of Groups, (in Russian), Moscow, Nauka, 1975.

[11] O. H. Kegel and B. A. F. Wehrfritz, *Locally Finite Groups*, North-Holland Mathematical Library, North-Holland Publishing Co., Amsterdam, London, **3** 1973.

[12] L. A. Kurdachenko, On groups with minimax classes of conjugate elements, (Russian), *Infinite groups and adjoining algebraic structures (Russian)*, Akad. Nauk Ukrainy, Inst. Mat., Kiev, (1993) 160–177.

[13] A. I. Mal'cev, On groups of finite rank, (Russian), *Mat. Sbornik N.S.*, **22** no. 2 (1948) 351–352.

[14] D. J. R. Robinson, *Finiteness Conditions and Generalized Soluble Groups*, Ergebnisse der Mathematik und ihrer Grenzgebiete, Springer-Verlag, New York-Berlin, **1,2** 1972.

[15] B. A. F. Wehrfritz, *Infinite Linear Groups*, Ergebnisse der Mathematik und ihrer Grenzgebiete, Springer-Verlag, New York, Heidelberg, Berlin, 1973.

Olga Yu. Dashkova

Department of Mathematics, The Branch of Moscow State University in Sevastopol, Sevastopol, Ukraine

Email: odashkova@yandex.ru