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QUASIRECOGNITION BY PRIME GRAPH OF FINITE SIMPLE GROUPS $^2D_n(3)$

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Abstract. Let G be a finite group. In [Ghasemabadi et al., characterizations of the simple group $^2D_n(3)$ by prime graph and spectrum, Monatsh Math., 2011] it is proved that if n is odd, then $^2D_n(3)$ is recognizable by prime graph and also by element orders. In this paper we prove that if n is even, then $D = {}^2D_n(3)$ is quasirecognizable by prime graph, i.e. every finite group G with $\Gamma(G) = \Gamma(D)$ has a unique nonabelian composition factor and this factor is isomorphic to D.

1. Introduction

If n is an integer, then we denote by $\pi(n)$ the set of all prime divisors of n. If G is a finite group, then $\pi(|G|)$ is denoted by $\pi(G)$. Also the set of orders of elements of G is denoted by $\omega(G)$. This set is closed under divisibility relation. The prime graph $\Gamma(G)$ of a finite group G is defined as a graph with vertex set $\pi(G)$ in which two distinct primes $p, p' \in \pi(G)$ are adjacent iff G contains an element of order pp'. Denote by $t(G)$ the maximal number of primes in $\pi(G)$ pairwise nonadjacent in $\Gamma(G)$. Also we denote by $t(p, G)$ the maximal number of vertices in the independent sets of $\Gamma(G)$ containing p.

A finite nonabelian simple group P is called quasirecognizable by prime graph, if every finite group G with $\Gamma(G) = \Gamma(P)$ has a unique nonabelian composition factor and this factor is isomorphic to P. Also P is called recognizable by prime graph if $\Gamma(G) = \Gamma(P)$ implies that $G \cong P$.

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In [5, 6, 7, 8] finite groups with the same prime graph as $L_n(2)$, $U_n(2)$, $B_p(3)$ and $D_n(3)$ are obtained. In [1, 2] it is proved that if $n = 2^k + 1$, then ${}^2D_n(3)$ is recognizable by prime graph. Then in [4] it is proved that if n is odd, then ${}^2D_n(3)$ is recognizable by prime graph. In this paper, we continue this work and as the main result we prove that the simple group ${}^2D_n(3)$ is quasirecognizable by prime graph. Therefore these simple groups are quasirecognizable by element orders. We note that the main tool in [4] is 2-independent number, since $t(2, {}^2D_n(3)) = 3$ if n is odd. But if n is even, then $t(2, {}^2D_n(3)) = 2$. Therefore we use a different method in our proof. In this paper as the main result we prove the following theorem:

Theorem 1.1. *Let $n \geq 4$ be even. Then the simple group ${}^2D_n(3)$ is quasirecognizable by prime graph.*

As a consequence of this result we get that if $n \geq 4$ is even, then the simple group ${}^2D_n(3)$ is quasirecognizable by $\omega(G)$.

Throughout the paper we use the classification of finite simple groups, also all groups are finite and by simple groups we mean nonabelian simple groups. We denote by (a, b) the greatest common divisor of positive integers a and b. Let m be a positive integer and p be a prime number. Then m_p denotes the p-part of m. In other words, $m_p = p^k$ if $p^k \mid m$ but $p^{k+1} \nmid m$. Let q be a prime power. We denote by $L_n^+(q)$ the simple group $L_n(q)$, and by $L_n^-(q)$ the simple group $U_n(q)$. Also we denote by $D_n^+(q)$ the simple group $D_n(q)$, and by $D_n^-(q)$ the simple group ${}^2D_n(q)$. Sometimes we use both notations. All further unexplained notations are standard and refer to [3].

2. Preliminary Results

Lemma 2.1. [9, Theorem 1] *Let G be a finite group with $t(G) \geq 3$ and $t(2, G) \geq 2$. Then the following hold:*

(1) *there exists a finite nonabelian simple group S such that $S \leq \bar{G} = G/N \leq \mathrm{Aut}(S)$ for the maximal normal soluble subgroup N of G;*

(2) *for every independent subset ρ of $\pi(G)$ with $|\rho| \geq 3$ at most one prime in ρ divides the product $|N| \cdot |\bar{G}/S|$. In particular, $t(S) \geq t(G) - 1$;*

(3) *one of the following holds:*

  (a) *every prime $r \in \pi(G)$ nonadjacent to 2 in $\Gamma(G)$ does not divide the product $|N| \cdot |\bar{G}/S|$; in particular, $t(2, S) \geq t(2, G)$;*

  (b) *there exists a prime $r \in \pi(N)$ nonadjacent to 2 in $\Gamma(G)$; in which case $t(G) = 3$, $t(2, G) = 2$, and $S \cong \mathrm{Alt}_7$ or $L_2(q)$ for some odd q.*

Remark 2.2. *In Lemma 2.1, for every odd prime $p \in \pi(S)$, we have $t(p, S) \geq t(p, G) - 1$.*

If q is a natural number, r is an odd prime and $(q, r) = 1$, then by $e(r, q)$ we denote the smallest natural number m such that $q^m \equiv 1 \pmod r$. Given an odd q, put $e(2, q) = 1$ if $q \equiv 1 \pmod 4$ and put $e(2, q) = 2$ if $q \equiv -1 \pmod 4$. We can see that if r is an odd prime such that $r \mid (q^n - 1)$, then $e(r, q) \mid n$.

Lemma 2.3. [12, Proposition 2.5] *Let $G = D_n^{\varepsilon}(q)$ be a finite simple group of Lie type over a field of characteristic p. Define*

$$\eta(m) = \begin{cases} m, & \text{if } m \text{ is odd;} \\ m/2, & \text{otherwise.} \end{cases}$$

Suppose r, s are odd primes and $r, s \in \pi(D_n^{\varepsilon}(q)) \setminus \{p\}$. Put $k = e(r, q)$, $l = e(s, q)$, and $1 \leq \eta(k) \leq \eta(l)$. Then r and s are nonadjacent if and only if $2\eta(k) + 2\eta(l) > 2n - (1 - \varepsilon(-1)^{k+l})$ and k and l satisfy the following condition:

$$\frac{l}{k} \text{ is not an odd integer,}$$

and, if $\varepsilon = +$, then the chain of equalities:

$$n = l = 2\eta(l) = 2\eta(k) = 2k$$

is not true.

Lemma 2.4. [11, Proposition 2.1] *Let $G = L_n(q)$ be a finite simple group of Lie type over a field of characteristic p. Let r and s be odd primes and $r, s \in \pi(G) \setminus \{p\}$. Put $k = e(r, q)$ and $l = e(s, q)$ and assume that $2 \leq k \leq l$. Then r and s are nonadjacent if and only if $k + l > n$ and k does not divide l.*

Lemma 2.5. [11, Proposition 2.2] *Let $G = U_n(q)$ be a finite simple group of Lie type over a field of characteristic p. Define*

$$\nu(m) = \begin{cases} m, & m \equiv 0 \pmod 4; \\ m/2, & m \equiv 2 \pmod 4; \\ 2m, & m \equiv 1 \pmod 2. \end{cases}$$

Let r and s be odd primes and r, $s \in \pi(G) \setminus \{p\}$. Put $k = e(r, q)$ and $l = e(s, q)$ and suppose that $2 \leq \nu(k) \leq \nu(l)$. Then r and s are nonadjacent if and only if $\nu(k) + \nu(l) > n$ and $\nu(k)$ does not divide $\nu(l)$.

For using Lemmas 2.4 and 2.5, simultaneously, we define the following function:

$$\nu'(m) = \begin{cases} m, & \varepsilon = +; \\ \nu(m), & \varepsilon = -. \end{cases}$$

Lemma 2.6. *(Zsigmondy's theorem)* [13] *Let p be a prime and let n be a positive integer. Then one of the following holds:*

1. *there is a primitive prime p' for $p^n - 1$, that is , $p' \mid (p^n - 1)$ but $p' \nmid (p^m - 1)$, for every $1 \leq m < n$,*
2. *$p = 2$, $n = 1$ or 6,*
3. *p is a Mersenne prime and $n = 2$.*

We note that a prime r with $e(r, q) = m$ is called a primitive prime divisor of $q^m - 1$ (obviously, $q^m - 1$ can have more than one primitive prime divisor).

3. **Proof of The Main Theorem**

We prove the main theorem throughout the following propositions. In this section we denote by r_i, a primitive prime divisor of $3^i - 1$. Let $b = 3^\alpha$. We note that a primitive prime divisor of $3^{i\alpha} - 1$ is a primitive prime divisor of $b^i - 1$, but the converse is not true in general.

Throughout this paper, for any $q = p^\alpha$, where $q \neq 3$, we denote by r'_i an arbitrary primitive prime divisor of $q^i - 1$.

Proposition 3.1. *Let $D = {}^2D_n(3)$, where $n \geq 4$ is even. Let G be a finite group such that $\Gamma(G) = \Gamma(D)$. Then there exists a nonabelian simple group S such that $S \leq \bar{G} = G/N \leq \mathrm{Aut}(S)$, where N is the maximal normal soluble subgroup of G. Moreover S is not isomorphic to an alternating group or a sporadic simple group.*

Proof. By assumption, from [11, Tables 6, 8] we deduce that $t(D) \geq 4$, since $n \geq 4$. Also $\rho(2, D) = \{2, r_{2n}\}$ and so $t(2, D) \geq 2$. Now by Lemma 2.1, it follows that there exists a finite nonabelian simple group S such that $S \leq \bar{G} = G/N \leq \mathrm{Aut}(S)$, where N is the maximal normal soluble subgroup of G. Also, $t(S) \geq t(G) - 1$ and $t(2, S) \geq t(2, G)$ by Lemma 2.1.

Let $n \geq 15$. Then $t(G) \geq 12$ and $t(2, G) \geq 2$. Therefore $t(S) \geq 11$ and $t(2, S) \geq 2$, since n is even. Now using Table 8 in [11] we get an independent set with 11 elements for D. Since $n \geq 15$,

$$A = \{r_{2i} \mid n - 7 \leq i \leq n\} \cup \{r_i \mid n - 6 \leq i \leq n - 1, \ i \equiv 1 \pmod 2\}$$

is an independent set for $D = {}^2D_n(3)$. Now we prove that the simple group S is not isomorphic to an alternating group. Suppose that $S \cong \mathrm{Alt}_m$, then $m \geq 83$. Since $m \geq 83$, we get that $23 \in \pi(S)$. Now we prove that there exist at most 7 prime numbers in $M = [m - 23, m] \cap \mathbb{Z}$. Obviously, there exist 12 elements of M, which are even. Also, there exist 8 elements of M, which are divisible by 3 and there exist 4 elements of M, which are divisible by 2 and 3. Hence there exist 16 elements of M, which are divisible by 2 or 3. On the other hand there exist at least 4 elements of M, which are divisible by 5, namely $\alpha_1, \alpha_2, \alpha_3$ and α_4, where $\alpha_1 < \alpha_2 < \alpha_3 < \alpha_4$ and $\alpha_{i+1} = \alpha_i + 5$, for $1 \leq i \leq 3$. If α_j and α_{j+2} are odd, then obviously one of them is not divisible by 3. Therefore there exists α_k such that $2 \nmid \alpha_k$ and $3 \nmid \alpha_k$. Finally, we get that there exist at least 17 elements of M, which are divisible by 2, 3 or 5. Hence there exist at most 7 primes in M. Therefore by Proposition 1.1 in [11] we conclude that 23 is not adjacent to at most 7 primes in $\Gamma(S) = \Gamma(\mathrm{Alt}_m)$ and so $t(23, S) \leq 8$. Now we determine $t(23, G)$. We know that $e(23, 3) = 11$ and we can suppose that $r_{11} = 23$. If $23 = r_{11} \in A$, then $t(23, G) \geq |A| = 11$, which is a contradiction by Remark 2.2. Therefore $r_{11} \notin A$ and so $n \geq 18$, which implies that $\eta(e(23, 3)) = 11 \leq \eta(e(x, 3))$, for each $x \in A$. Therefore using Lemma 2.3, we get that 23 is not adjacent in $\Gamma(G)$ to at least 9 elements of A, since 11 can divide at most two numbers of $\{n - 6, n - 5, \ldots, n - 1, 2n - 14, 2n - 12, \ldots, 2n\}$ and $\eta(e(x, 3)) + 11 > n$, for each $x \in A$. So there exists an independent set in $\Gamma(G)$ with at least 10 elements containing 23. Therefore $t(23, G) \geq 10$, which is impossible by Remark 2.2.

Now let $n \leq 14$. Since $31 \notin \pi(G)$, we get that $m \leq 30$. On the other hand by Lemma 2.1(3-a), $r_{2n} \in \pi(S)$. Now for each $n \leq 14$, easily we can see that there exists at least one primitive prime divisor of $3^{2n} - 1$ which is greater than 31, and this is a contradiction. So S is not isomorphic to an alternating group.

Let S be isomorphic to a sporadic group. We have $t(S) \leq 11$ and so by Lemma 2.1, we get that $t(G) \leq 12$. Hence $n \leq 14$. For $n \geq 6$, there exists a primitive prime divisor, say r_{2n}, such that $r_{2n} \notin \pi(S)$, which is a contradiction by Lemma 2.1(3). If $n = 4$, then $r_8 = 41 \in \pi(S)$. Therefore $S = F_1$, whereas $\pi(F_1) \nsubseteq \pi(G)$, which is a contradiction. □

Proposition 3.2. *Let $D = {}^2D_n(3)$, where $n \geq 4$ is even, and let G and S be as in the claim of Proposition 3.1. If S is isomorphic to a simple group of Lie type over a field of characteristic 3, then $S \cong D$.*

Proof. From [11, Tables 6, 8] we deduce that $t(D) \geq 4$ and $t(2, D) = 2$. Therefore $t(G) \geq 4$ and $t(2, G) = 2$. Now by Proposition 3.1, it follows that there exists a finite nonabelian simple group S such that $S \leq \bar{G} = G/N \leq \mathrm{Aut}(S)$, where N is the maximal normal soluble subgroup of G. Also, $t(S) \geq t(G) - 1$ and $t(2, S) \geq t(2, G)$, by Lemma 2.1. Therefore $t(S) \geq 3$ and $t(2, S) \geq 2$. Let S be isomorphic to a simple group of Lie type over $\mathrm{GF}(q)$, where $q = 3^\alpha$.

We know that $\rho(3, D) = \{3, r_{n-1}, r_{2n-2}, r_{2n}\}$ and so if S is isomorphic to a simple group of Lie type, then $|\rho(3, D) \cap \pi(S)| \geq 3$, by Remark 2.2. Also $3, r_{2n} \in \pi(S)$ and consequently, $r_{n-1} \in \pi(S)$ or $r_{2n-2} \in \pi(S)$.

Case 1. Let $r_{2n-2} \in \pi(S)$. Now consider primitive prime divisors r_{2n-2} and r_{2n} of $3^{2n-2} - 1$ and $3^{2n} - 1$, respectively. Therefore r_{2n-2} and r_{2n} divide the order of S and so r_{2n-2} and r_{2n} are primitive prime divisors of $q^s - 1$ and $q^t - 1$, respectively, where $s = e(r_{2n-2}, 3^\alpha)$ and $t = e(r_{2n}, 3^\alpha)$. Consequently, we get that $(2n - 2) \mid s\alpha$ and $2n \mid t\alpha$. By Zsigmondy's theorem, we conclude that $2n - 2 = s\alpha$ and $2n = t\alpha$, since $\pi(S) \subseteq \pi(D)$. Therefore $\alpha \mid 2$ and obviously $s < t$. Now we consider each possibility for S, separately.

Using Propositions 3.1 and 3.2 in [11] we see that for each simple group of Lie type S, the vertex 3 is adjacent to each primitive prime r'_i, where $r'_i \notin \rho(3, S)$.

Let $S \cong L_m(q)$. We know that $\rho(3, S) = \{3, r'_m, r'_{m-1}\}$. Since $s < t$, we conclude that $m = t$ and $m - 1 = s$. Hence $2n = m\alpha$ and $2n - 2 = (m - 1)\alpha$. Consequently, we get that $m = n$ and $\alpha = 2$. By the assumption, n is even. Then S has a maximal torus of order $(3^{2n} - 1)/((3^2 - 1)(n, 3^2 - 1))$, say T. Obviously, $r_n, r_{2n} \in \pi(T)$. Therefore $r_n \sim r_{2n}$ in $\Gamma(L_n(3^2))$, whereas $r_n \nsim r_{2n}$ in $\Gamma(G)$, by Lemma 2.3, which is a contradiction.

Let $S \cong U_m(q)$, where $m \geq 3$.

If $m = 3$, then $t(S) \leq 3$ and so $t(G) \leq 4$, which implies that $n = 4$. Therefore $r_8 \in \pi(S)$ and this implies that $\alpha \neq 1$. If $\alpha = 2$, then $\pi(S) \nsubseteq \pi(G)$, a contradiction.

Suppose $m \equiv 0 \pmod 4$. We know that $\rho(3, S) = \{3, r'_m, r'_{2m-2}\}$. Hence we conclude that $s = m$ and $t = 2m - 2$. Therefore $2n = (2m - 2)\alpha$ and $2n - 2 = m\alpha$, which implies that $n = 1 + m/(m - 2)$,

which is impossible, since $m \geq 4$ and $n \geq 4$ is an integer.
If $m \equiv 1 \pmod 4$, then we conclude that $t = 2m$ and $s = m-1$ and so $2n = 2m\alpha$ and $2n-2 = (m-1)\alpha$, which implies that $n = 2 - \alpha$, which is impossible.
If $m \equiv 2 \pmod 4$ or $m \equiv 3 \pmod 4$, then we get a contradiction, similarly.

Let $S \cong B_m(q)$ or $S \cong C_m(q)$, where m is odd. Since $\rho(3, S) = \{3, r'_m, r'_{2m}\}$, we get that $2n = 2m\alpha$ and $2n - 2 = m\alpha$, which implies that $n = 2$, a contradiction.

Since $t(3, B_m(3^\alpha)) = t(3, C_m(3^\alpha)) = t(3, {}^3D_4(3^\alpha)) = 2$, where m is even and $t(3, D) = 4$, by Lemma 2.1(3), we get that S can not be isomorphic to these groups.

Let $S \cong D_m(q)$, where m is even. We know that $\rho(3, S) = \{3, r'_{m-1}, r'_{2m-2}\}$, hence $2n = (2m-2)\alpha$ and $2n - 2 = (m-1)\alpha$. But these equations imply that $n = 2$, which is a contradiction.
If $S \cong D_m(q)$, where $m > 4$ is odd, then we get that $2n = (2m-2)\alpha$ and $2n - 2 = m\alpha$, which is impossible.

Let $S \cong E_7(q)$. We know that $\rho(3, S) = \{3, r'_7, r'_9, r'_{14}, r'_{18}\}$. Hence using $\rho(2, S)$, we conclude that $2n \mid 18\alpha$ and by Zsigmondy's theorem, we get that $n = 9\alpha$, since $\pi(S) \subseteq \pi(G)$. We have $s \in \{7, 9, 14\}$. If $s = 14$, then $2n-2 \mid 14\alpha$, and so $(2n-2) = 14\alpha$. But these equations imply that $n = 9/2$, which is impossible. If $s = 9$, then we have $2n - 2 = 9\alpha$, which implies that $n = 2$ and this is a contradiction. Similarly and checking other cases lead us to a contradiction. We have a similar argument for $E_8(q)$, $G_2(q)$, $E_6(q)$, ${}^2E_6(q)$ and $F_4(q)$, where $q = 3^\alpha$.

Let $S \cong {}^2G_2(q)$, where $q = 3^{2\alpha+1}$. Since $|S| = q^3(q^3 + 1)(q - 1)$, we have $6(2\alpha + 1) \leq 2n$. On the other hand $r_{2n} \in \pi(S)$, which implies that $6(2\alpha + 1) = 2n$, which is impossible, since n is even.

Let $S \cong {}^2D_m(3^\alpha)$, where m is odd. By Zsigmondy's theorem we get that $2m\alpha = 2n$, since $r_{2n} \in \pi(S)$. Therefore $n = m\alpha$, which implies that $m \leq n/2$, since m is odd. On the other hand, $t(S) \geq t(D) - 1$, implies that $n/2 \geq m \geq n - 2$. Easily we see that this is impossible.

Therefore $S \cong {}^2D_m(3^\alpha)$, where m is even. We have $n = m\alpha$ and $2n - 2 = (2m - 2)\alpha$. Hence $m = n$ and $\alpha = 1$, and we conclude that $S \cong D$.

Case 2. Let $r_{2n-2} \notin \pi(S)$. Hence $r_{n-1} \in \pi(S)$. Consider primitive prime divisors r_{n-1} and r_{2n} of $3^{n-1} - 1$ and $3^{2n} - 1$, respectively. Therefore r_{n-1} and r_{2n} divide the order of S and so r_{n-1} and r_{2n} are primitive prime divisors of $q^s - 1$ and $q^t - 1$, respectively, where $s = e(r_{n-1}, 3^\alpha)$ and $t = e(r_{2n}, 3^\alpha)$. Now we conclude that $(n - 1) \mid s\alpha$ and $2n \mid t\alpha$. On the other hand using Zsigmondy theorem, we conclude that $t\alpha \leq 2n$ and so $t\alpha = 2n$. Also if $n - 1 < s\alpha$, then we have $s\alpha = 2n - 2$, which implies that $r_{2n-2} \in \pi(S)$, a contradiction. Therefore $s\alpha = n - 1$. Therefore $s < t$ and $\alpha \mid 2$, since $\alpha \mid 2n$ and $\alpha \mid 2n - 2$. Now we consider each possibility for S, separately. We note that Propositions 3.1 and 3.2 in [12] show that for each simple group of Lie type S, the vertex 3 is adjacent to each primitive prime r'_i where $r'_i \notin \rho(3, S)$.

Let $S \cong L_m(q)$. We know that $\rho(3, S) = \{3, r'_m, r'_{m-1}\}$ and for each r'_i, where $i \neq m - 1$ and $i \neq m$, we have $3 \sim r'_i$. Therefore since $s < t$, we conclude that $m = t$ and $m - 1 = s$. Hence $2n = m\alpha$ and $n - 1 = (m - 1)\alpha$. So we get that $n = \alpha - 1$, which is a contradiction, since $\alpha \leq 2$.

Let $S \cong U_m(q)$, where $m \geq 4$.
Suppose $m \equiv 1 \pmod 4$. We know that $\rho(3, S) = \{3, r'_{m-1}, r'_{2m}\}$, so we conclude that $2m = t$ and

$m-1=s$. Hence $n-1=(m-1)\alpha$ and $2n=2m\alpha$. But these equations imply that $n=m$ and $\alpha=1$, which is impossible, since m is odd and n is even. A similar argument can be applied for $U_m(q)$, where $m\not\equiv 1 \pmod 4$.

Let $S\cong B_m(q)$ or $S\cong C_m(q)$, where m is odd. Since $\rho(3,S)=\{3,r'_m,r'_{2m}\}$, we get that $n-1=m\alpha$ and $2n=2m\alpha$, which is a contradiction. If $S\cong D_m(q)$ or $S\cong {}^2D_m(q)$, where m is odd, then a similar proof leads us to a contradiction.

Let $S\cong E_7(q)$. We know that $\rho(3,S)=\{3,r'_7,r'_9,r'_{14},r'_{18}\}$. Hence $2n\mid 18\alpha$ and by Zsigmondy's theorem, we conclude that $n=9\alpha$. We have $s\in\{7,9,14\}$. If $s=14$, then $(n-1)\mid 14\alpha$, and so $n-1=14\alpha$. But these equations imply that $n=-9/5$, which is impossible. If $s=9$, then we have $n-1=9\alpha$, which implies that $n-1=9\alpha=n$ and this is a contradiction.

Let $S\cong {}^2G_2(q)$, where $q=3^{2\alpha+1}$. Since $|S|=q^3(q^3+1)(q-1)$, we have $6(2\alpha+1)\leq 2n$. On the other hand $r_{2n}\in\pi(S)$, which implies that $6(2\alpha+1)=2n$, which is impossible, since n is even.

For other simple groups of Lie type similarly we get a contradiction and we omit the proof for convenience.

So $S\cong {}^2D_m(3^\alpha)$, where $m\geq 4$ and m is even. We know that $\rho(3,S)=\{3,r'_{m-1},r'_{2m-2},r'_{2m}\}$. If $s=2m-2$ and $t=2m$, then we get that $n=2\alpha-1$, which is impossible. Hence $(m-1)\alpha=n-1$ and $2m\alpha=2n$ and these equations imply that $\alpha=1$ and $m=n$, which implies that $S\cong D$. □

Proposition 3.3. *Let $D={}^2D_n(3)$, where $n\geq 4$ is even, and let G and S be as in the claim of Proposition 3.1. The simple group S is not isomorphic to any simple group of Lie type over* $\mathrm{GF}(p^\alpha)$, *where $p\neq 3$.*

Proof. Let $n\geq 8$. Then $t(S)\geq 6$. Therefore $S\cong L^{\varepsilon}_m(q)$, where $m\geq 11$ or $S\cong B_m(q)$, where $m\geq 7$ or $S\cong C_m(q)$, where $m\geq 7$ or $S\cong D^{\varepsilon}_m(q)$, where $m\geq 7$, $\varepsilon\in\{-,+\}$ and $q\neq 3^\alpha$.
Using Table 8 in[11], consider the independent set

$$B=\{r_{2i}\mid n-4\leq i\leq n\}\cup\{r_i\mid n-4\leq i\leq n-1,\ i\equiv 1 \pmod 2\}$$

in ${}^2D_n(3)$. We note that B has 7 elements. By Lemma 2.1(2), we know that $|B\cap\pi(S)|\geq 6$. We note that in each case $t(p,S)\leq 4$ and so $p\notin B$. Therefore $\eta(e(p,3))\leq\eta(e(x,3))$, for each $x\in B$. On the other hand, $t(p,S)\leq 4$ implies that $t(p,G)\leq 5$, by Remark 2.2. Therefore p is adjacent to at least two elements of B in $\Gamma(G)$. In the sequel, for each case we get a contradiction by Lemma 2.3. For convenience, we state the details of the proof for one of these cases.

For example let $r_i\sim p\sim r_j$ in $\Gamma(G)$. Let $t=e(p,3)$. Therefore one of the following occurs:

(1) $2\eta(t)+2\eta(i)\leq 2n-(1+(-1)^{t+i})$ and
$2\eta(t)+2\eta(j)\leq 2n-(1+(-1)^{t+j})$;
(2) $2\eta(t)+2\eta(i)\leq 2n-(1+(-1)^{t+i})$ and j/t is odd;
(3) i/t is odd and $2\eta(t)+2\eta(j)\leq 2n-(1+(-1)^{t+j})$;
(4) i/t and j/t are odd.

Easily we can see that these relations imply that $\eta(t)\leq 4$. Therefore $p\in\{2,5,7,13,41\}$. Now we consider each possibility for p.

(i) Let $p = 2$. We know that $t(S) \geq 6$. If $S \cong L^{\varepsilon}_m(q)$, then $t(S) = [(m+1)/2]$, which implies that $m \geq 11$. Similarly if $S \cong D_m(q)$ or $S \cong {}^2D_m(q)$ or $S \cong B_m(q)$ or $S \cong C_m(q)$, then $m \geq 7$. Hence $31 \in \pi(S)$. Therefore $31 \in \pi(G)$. Since $e(31,3) = 30$, we get that $n \geq 15$ and so $t(G) \geq 12$, which implies that $t(S) \geq 11$. Hence if $S \cong L^{\varepsilon}_m(2)$, then $m \geq 21$ and for other cases $m \geq 13$, which implies that $127 \in \pi(S)$, since $e(127,2) = 7$ and $(2^7 - 1) \mid (q^7 - 1)$. Therefore $127 \in \pi(G)$.

Now we have $\nu'(e(127,2)) \leq 14$ and $\eta(e(127,2)) = 7$. Therefore $\nu'(e(127,2^{\alpha})) \leq 14$ and $\eta(e(127,2^{\alpha})) \leq 7$. Let $x \in \pi(S)$. By Lemmas 2.4 and 2.5, we conclude that if $S \cong L^{\varepsilon}_m(q)$ and $\nu'(e(x,2^{\alpha})) \leq m - 14$, then x is adjacent to 127 in $\Gamma(S)$ and by Lemma 2.3 we get that if $S \cong D^{\varepsilon}_m(q)$ and $\eta(e(x,2^{\alpha})) \leq m-8$, then x is adjacent to 127 in $\Gamma(S)$ and if $S \cong B_m(q)$ or $S \cong C_m(q)$ and $\eta(e(x,2^{\alpha})) \leq m - 8$, then x is adjacent to 127 in $\Gamma(S)$, by [11, Proposition 2.3]. Therefore in each case, $t(127,S) \leq 15$. On the other hand, $\eta(e(127,3)) = 63$ and so $n \geq 63$. By Lemma 2.3,

$$\{r_{2i} \mid n-12 \leq i \leq n\} \cup \{r_i \mid n-12 \leq i \leq n-1,\ i \equiv 1 \pmod 2\} \setminus \{127\}$$

contains at least 16 independent elements nonadjacent to 127. Hence $t(127,G) > 16$, which is a contradiction by Remark 2.2.

(ii) If $p = 5$, then similarly to (i) we conclude that $31 \in \pi(S)$ and we have $\nu'(e(31,5)) \leq 6$ and $\eta(e(31,5)) = 3$. Similarly to (i), we have $t(31,S) \leq 6$. On the other hand, $\eta(e(31,3)) = 15$ and by Lemma 2.3 we conclude that $t(31,G) > 7$, which is a contradiction by Remark 2.2.

(iii) If $p = 7$, then $2801 \in \pi(S)$. We note that $\nu'(e(2801,7)) \leq 10$ and $\eta(e(2801,7)) = 5$. But $\eta(e(2801,3)) = 1400$, and we get a contradiction similarly to the above cases.

(iv) If $p = 13$, then $30941 \in \pi(S)$ and we have $\nu'(e(30941,13)) \leq 10$ and $\eta(e(30941,13)) = 5$. On the other hand, $\eta(e(30941,3)) = 15470$, which is impossible.

(v) For $p = 41$, we get a contradiction similarly.

If $S \cong G_2(q)$, where $q > 2$, $S \cong F_4(q)$, $S \cong E^{\varepsilon}_6(q)$, ${}^3D_4(q)$, ${}^2B_2(2^{2\alpha+1})$, ${}^2F_4(2^{2\alpha+1})$, where $\alpha \geq 2$, ${}^2F_4(2)'$ or ${}^2F_4(8)$, then $t(S) \leq 5$ and so $t(G) \leq 6$, which implies that $n \leq 7$, a contradiction.

Let $S \cong E_8(q)$ or $S \cong E_7(q)$, where $q = p^{\alpha}$ and $p \neq 3$. In this case $t(S) \leq 12$. By Lemma 2.1, $t(G) - 1 \leq t(S)$ and so $n \leq 17$. Therefore $p \in \pi(G) \setminus \{3\} \subseteq \{2, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 61, 67, 73, 103, 193, 271, 307, 547, 661, 757, 1021, 1093, 1181, 3851, 4561, 6481, 16493, 398581, 797161, 21523361\}$. But for each p in this set, we get that $\pi(p^7 - 1) \not\subseteq \pi(G)$, which is a contradiction, since $\pi(p^7 - 1) \subseteq \pi(q^7 - 1) \subseteq \pi(S) \subseteq \pi(G)$.

Now we consider $n \leq 6$. By assumption, S is isomorphic to a finite simple group of Lie type over $\mathrm{GF}(p^{\alpha})$, where $p \neq 3$.

Let $n = 6$, and so $D = {}^2D_6(3)$. Then $p \in \pi({}^2D_6(3)) \setminus \{3\}$. We know that $73 = r_{2n} \in \pi(S)$. If $p \neq 2$ and $p \neq 73$, then in each case there exists a prime number t such that $e(t,p) = e(r_{2n},p)$ and $t \notin \pi({}^2D_6(3))$, which is a contradiction. For example if $p = 5$ then $e(73,5) = e(543097,5) = 72$ and so $543097 \in \pi(S) \setminus \pi(G)$, which is a contradiction. Therefore let $p = 73$. Since in a Lie type group over $\mathrm{GF}(73^{\alpha})$, $q^2 - 1$ divides the order of the group, we conclude that $37 \in \pi(S)$, since $e(37,73) = 2$ and $\pi(p^2 - 1) \subseteq \pi(q^2 - 1)$. Hence $37 \in \pi(S) \setminus \pi(G)$, which is a contradiction.

Let $p = 2$. Since $|\{41, 61, 73\} \cap \pi(S)| \geq 2$ and $73 \in \pi(S)$, we conclude that $41 \in \pi(S)$ or $61 \in \pi(S)$. If $61 \in \pi(S)$, then $e(61, 2) = e(1321, 2) = 60$ and this implies that $1321 \in \pi(S) \setminus \pi(G)$, which is a contradiction and so $61 \notin \pi(S)$ and $41 \in \pi(S)$.

We note that $e(41, 2) = 20$ and $e(73, 2) = 9$. Hence $20 \mid t\alpha$ and $9 \mid s\alpha$, where $s, t \geq 1$. Using Zsigmondy's theorem, we conclude that $t\alpha = 20$ and $s\alpha = 9$ or $s\alpha = 18$, since $\pi(S) \subseteq \pi(G)$. If $s\alpha = 18$, then $19 \in \pi(2^{18} - 1) \subseteq \pi(S) \setminus \pi(G)$, which is impossible. Therefore $s\alpha = 9$ and so $\alpha = 1$. If $S \cong L_m(2)$, $U_m(2)$, $B_m(2)$, $C_m(2)$ or $D^{\varepsilon}_m(2)$, then $m \geq 10$, since $41 \in \pi(S)$. Therefore $31 = r'_5 \in \pi(S)$, which is a contradiction. Since $41 = r'_{20} \in \pi(S)$, we conclude that S only can be isomorphic to $E_8(2)$ among exceptional finite simple groups of Lie type. On the other hand $31 = r'_5 \in \pi(E_8(2))$, which is a contradiction.

Let $n = 4$, that is $D = {}^2D_4(3)$. We note that $p \in \pi(G) \backslash \{3\} = \{2, 5, 7, 13, 41\}$. Since $41 = r_{2n} \in \pi(S)$ and $e(41, 5) = e(9161, 5) = 20$, $e(41, 7) = e(810221830361, 7) = 40$ and $e(41, 13) = e(29881, 13) = 40$ we get that $p \notin \{5, 7, 13\}$.

Let $p = 41$. Using Zsigmondy's theorem, we conclude that $r'_s \notin \pi(S)$, where $s \geq 3$, since $\pi(41^s - 1) \nsubseteq \pi(G)$, where $s \geq 3$. Hence S only can be isomorphic to $L_2(41)$. If $S \cong L_2(41)$, then $3 \sim 7$ in $\Gamma(S)$, whereas $3, 7 \in \rho(3, G)$, which is a contradiction.

Let $p = 2$ and $S \cong L_m(2^\alpha)$. We note that $31 \notin \pi(G)$ and $e(31, 2) = 5$. Hence $m \leq 4$. Since $41 \in \pi(S)$ and $e(41, 2) = 20$, we conclude that $20 \mid t\alpha$ for $1 \leq t \leq 4$ and by Zsigmondy's theorem, $t\alpha = 20$. Since $t \leq 4$, we get that $\alpha \geq 5$ and $5 \mid \alpha$. Therefore $31 \in \pi(S) \setminus \pi(G)$, a contradiction.

We have a similar argument for other groups of Lie type over $\mathrm{GF}(2^\alpha)$ and we omit the proof for convenience. □

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References

[1] A. Babai and B. Khosravi, Recognition by prime graph of ${}^2D_{2^m+1}(3)$, *Sib. Math. J.*, **52** no. 5 (2011) 788–795.

[2] A. Babai, B. Khosravi and N. Hasani, Quasirecognition by prime graph of ${}^2D_p(3)$ where $p = 2^n + 1 \geq 5$ is a prime, *Bull. Malays. Math. Sci. Soc. (2)*, **32** no. 3 (2009) 343–350.

[3] J. H. Conway, R. T. Curtis, S. P. Norton, R. A. Parker and R. A. Wilson, *Atlas of finite groups*, Oxford University Press, Oxford, 1985.

[4] M. F. Ghasemabadi, A. Iranmanesh and N. Ahanjideh, Characterizations of the simple group ${}^2D_n(3)$ by prime graph and spectrum, *Monatsh Math.*, **168** no. (3-4) (2012) 347–361.

[5] B. Khosravi, Z. Akhlaghi and M. Khatami, Quasirecognition by prime graph of simple group $D_n(3)$, *Publ. Math. Debrecen*, **78** no. 2 (2011) 469–484.

[6] B. Khosravi and H. Moradi, Quasirecognition by prime graph of finite simple groups $L_n(2)$ and $U_n(2)$, *Acta Math. Hungar.*, **132** no. (1-2) (2011) 140-153.

[7] B. Khosravi and H. Moradi, Quasirecognition by prime graph of some orthogonal groups over the binary field, *J. Algebra Appl.*, **11** no. 3 (2012) 15 pp..

[8] Z. Momen and B. Khosravi, On r-recognition by prime graph of $B_p(3)$ where p is an odd prime, *Monatsh Math.*, **166** no. 2 (2012) 239-253.

[9] A. V. Vasil'ev, On connection between the structure of a finite group and the properties of its prime graph, *Sib. Math. J.*, **46** no. 3 (2005) 396–404.

[10] A. V. Vasil'ev and I. B. Gorshkov, On recognition of finite groups with connected prime graph, *Sib. Math. J.*, **50** no. 2 (2009) 233–238.

[11] A. V. Vasil'ev and E. P. Vdovin, An adjacency criterion for the prime graph of a finite simple group, *Algebra Logic*, **44** no. 6 (2005) 381–406.

[12] A. V. Vasil'ev and E. P. Vdovin, Cocliques of maximal size in the prime graph of a finite simple group, *Algebra Logic*, **50** no. 4 (2011) 291-322. Arxiv: 0905.1164v1

[13] K. Zsigmondy, Zur theorie der potenzreste, *Monatsh. Math. Phys.,* **3** (1892) 265–284.

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