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A NOTE ON FIXED POINTS OF AUTOMORPHISMS OF INFINITE GROUPS

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Abstract. Motivated by a celebrated theorem of Schur, we show that if Γ is a normal subgroup of the full automorphism group $Aut(G)$ of a group G such that $Inn(G)$ is contained in Γ and $Aut(G)/\Gamma$ has no uncountable abelian subgroups of prime exponent, then $[G,\Gamma]$ is finite, provided that the subgroup consisting of all elements of G fixed by Γ has finite index. Some applications of this result are also given.

1. Introduction

A famous result by I. Schur [7] states that if the centre $Z(G)$ of a group G has finite index, then the commutator subgroup G' of G is finite. If $Inn(G)$ is the group of all inner automorphisms of G, then $G' = [G, Inn(G)]$ and $Z(G)$ coincides with the centralizer $C_G\big(Inn(G)\big)$ (i.e. the set of all points of G fixed under the action of $Inn(G)$). This remark shows that Schur's theorem can be restated in terms of automorphisms, and suggests investigating situations when $Inn(G)$ is replaced by another suitable (normal) subgroup Γ of the full automorphism group $Aut(G)$ of G. For the extreme choice $\Gamma = Aut(G)$, it was proved by P. Hegarty [4] that a result corresponding to Schur's theorem holds, i.e. if the *absolute centre* $C_G\big(Aut(G)\big)$ of a group G has finite index, then the *autocommutator subgroup* $[G, Aut(G)]$ is finite. Some further results of this type on the absolute centre and the autocommutator subgroup of a group can be found in a recent paper by H. Dietrich and P. Moravec [2].

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Here we prove the following theorem, which contains Hegarty's result but is proved independently of it and holds for an arbitrary normal subgroup Γ of $Aut(G)$ containing $Inn(G)$.

Theorem 1.1. *Let G be a group, and let Γ be a normal subgroup of $Aut(G)$ such that the factor group $Aut(G)/\Gamma$ has no uncountable abelian subgroups of prime exponent. If the index $|G : C_G(\Gamma)|$ is finite and $C_G(\Gamma)$ is contained in the centre of G, then the subgroup $[G, \Gamma]$ is finite.*

Observe that in the above statement the assumption that the group $Aut(G)/\Gamma$ does not contain uncountable abelian subgroups of prime exponent cannot be removed. To see this, let p be a prime number, $A = \langle a \rangle$ a cyclic group of order p^2 and B a countably infinite abelian group of exponent p. Consider the abelian p-group $G = A \times B$ and its socle $S = \langle a^p \rangle \times B$. The group Γ, consisting of all automorphisms of G acting trivially on S and G/S, is a normal subgroup of $Aut(G)$ and it is isomorphic to the homomorphism group $Hom(G/S, S)$. Moreover, $C_G(\Gamma) = [G, \Gamma] = S$ is an infinite subgroup of finite index of G.

If the full automorphism group $Aut(G)$ of a group G satisfies a given finiteness condition, it often happens that $Aut(G)$ must be very small; for instance, it is known that an infinite Černikov group cannot be the full automorphism group of any group (see [6]). As an application of our main statement, we prove the following results, which show that the sets of fixed points under the action of certain automorphism groups cannot be too large.

Corollary 1.2. *Let G be a group admitting infinitely many automorphisms, and let Γ be a normal subgroup of $Aut(G)$ containing $Inn(G)$. If $Aut(G)/\Gamma$ is a Černikov group, then the index $|G : C_G(\Gamma)|$ is infinite.*

Corollary 1.3. *Let G be a periodic group of infinite exponent, and let Γ be a normal subgroup of $Aut(G)$ containing $Inn(G)$. If the factor group $Aut(G)/\Gamma$ is countable, then the index $|G : C_G(\Gamma)|$ is infinite.*

Most of our notation is standard and can be found in [5].

2. Proofs

Let Γ be a group of automorphisms of a group G, and consider in G the subgroups

$$C_G(\Gamma) = \{g \in G \mid g^\gamma = g \text{ for all } \gamma \in \Gamma\}$$

and

$$[G, \Gamma] = \langle [g, \gamma] \mid g \in G, \gamma \in \Gamma \rangle.$$

The interaction between Γ, $C_G(\Gamma)$ and $[G, \Gamma]$ was first studied by R. Baer [1], who obtained in particular the following result.

Lemma 2.1. *Let G be a group, and let Γ be a group of automorphisms of G. If the index $|G : C_G(\Gamma)|$ and the subgroup $[G, \Gamma]$ are finite, then Γ is likewise finite.*

This investigation was later taken up by R. F. Turner Smith [8], who proved in particular that if Γ is any group of automorphisms of a group G such that the index $|G : C_G(\Gamma)|$ is finite, then the subgroup $[G, \Gamma]$ is central-by-finite.

The following result gives some further interesting information of this type, and it is needed in our proof.

Lemma 2.2. *Let G be a group, and let Γ be a group of automorphisms of G such that the index $|G : C_G(\Gamma)| = n$ is finite and $C_G(\Gamma)$ is contained in the centre of G. Then the subgroup $[G, \Gamma]$ has finite exponent dividing n. Moreover Γ is an abelian-by-finite group of finite exponent.*

Proof. Put $|G : C_G(\Gamma)| = n$. As the subgroup $C_G(\Gamma)$ lies in the centre of G, the transfer homomorphism of G into $C_G(\Gamma)$ is the map

$$\tau : g \in G \longmapsto g^n \in C_G(\Gamma).$$

Let g and γ be elements of G and Γ, respectively. Then $(g^\gamma)^n = (g^n)^\gamma = g^n$, and hence

$$[g, \gamma]^\tau = g^{-\tau} g^{\gamma\tau} = g^{-n} g^n = 1.$$

Therefore $[G, \Gamma]$ is contained in the kernel of τ, so that $[G, \Gamma]^n = \{1\}$ and $[G, \Gamma]$ has finite exponent dividing n.

Let Γ_0 be the subgroup of Γ consisting of all automorphisms acting trivially on $G/C_G(\Gamma)$. Clearly, Γ_0 is abelian and Γ/Γ_0 is finite. If g is any element of G and γ belongs to Γ_0, then $[g, \gamma]$ lies in $C_G(\Gamma)$, and so $[g, \gamma^n] = [g, \gamma]^n = 1$. Therefore γ^n is the identity map, and hence Γ_0 has finite exponent (dividing n). It follows that also Γ has finite exponent. □

Proof of Theorem 1.1 – Assume for a contradiction that $[G, \Gamma]$ is infinite so that also the group Γ is infinite by the result of Baer quoted in the introduction. Put $C = C_G(\Gamma)$, and let Γ_0 be the subgroup of Γ consisting of all automorphisms acting trivially on G/C. Then $[G, \Gamma_0]$ is contained in C, and in particular it is abelian. As Γ/Γ_0 is finite, it follows that Γ_0 is infinite and so a second application of Baer's theorem yields that $[G, \Gamma_0]$ is likewise infinite. Since $[G, \Gamma_0]$ has finite exponent by Lemma 2.2, we have that the p-component P of C has infinite rank for some prime number p. Let D be the largest divisible subgroup of P. Then C splits over D, so that $C = D \times H$ and hence $P = D \times R$, where $R = H \cap P$ is a reduced subgroup.

Suppose that D has infinite rank, and let E be a finitely generated subgroup of G such that $G = EC$. As G is central-by-finite, E satisfies the maximal condition on subgroups. In particular, $E \cap C$ is generated by finitely many elements and so D can be decomposed into a direct product $D = D_1 \times D_2$, where D_1 has finite rank and $E \cap C$ is contained in $D_1 H$. Then

$$(ED_1H) \cap D_2 = (ED_1H) \cap C \cap D_2 = D_1H \cap D_2 = \{1\},$$

and hence

$$G = EC = (ED_1H)D_2 = (ED_1H) \times D_2.$$

It follows that each automorphism of D_2 can be extended to an automorphism of G, so that in particular $Aut(G)$ contains an uncountable subgroup Λ of exponent 2 and $\Gamma \cap \Lambda = \{1\}$ because all elements of Γ acts trivially on C. This contradiction shows that the subgroup D must have finite rank, and hence the reduced abelian p-group R has infinite rank.

Therefore there exists a collection $(X_n)_{n\in\mathbb{N}}$ of cyclic non-trivial subgroups of R such that

$$R = X_1 \times \cdots \times X_n \times R_n$$

for all positive integers n, and suitable subgroups R_n. Moreover, since $X_1 \times \cdots \times X_n$ is a finite direct factor of P, and P is pure in C, for each n there exists a subgroup C_n of C such that

$$C = X_1 \times \cdots \times X_n \times C_n.$$

For each non-negative integer n, put

$$U_{2n+1} = X_1^p \times X_2 \times X_3^p \times \cdots \times X_{2n} \times X_{2n+1}^p \times C_{2n+1},$$

and let

$$U = \bigcap_{n\in\mathbb{N}_0} U_{2n+1}.$$

Then C/U is an infinite abelian group of exponent p, and the subgroup $U[p]$ of U, consisting of all elements u such that $u^p = 1$, is likewise infinite. Consider a countably infinite subgroup V of $U[p]$. As the contravariant functor $Hom(-, V)$ is left exact, corresponding to the central extension

$$C/U \overset{\mu}{\rightarrowtail} G/U \overset{\varepsilon}{\twoheadrightarrow} G/C$$

there is an exact sequence

$$Hom(G/C, V) \overset{\varepsilon^*}{\rightarrowtail} Hom(G/U, V) \overset{\mu^*}{\longrightarrow} Hom(C/U, V),$$

where μ^* maps every homomorphism f to the homomorphism $f \circ \mu$. Then the kernel of μ^* is isomorphic to $Hom(G/C, V)$, and in particular it is at most countable. On the other hand, as G/C and G' are finite, the group G/U admits an infinite abelian homomorphic image of exponent p, and so $Hom(G/U, V)$ is uncountable. It is well-known that $Hom(G/U, V)$ is naturally isomorphic to a subgroup Λ of $Aut(G)$, and the factor group $\Lambda\Gamma/\Gamma$ is an uncountable abelian group of exponent p, because also the image of μ^* is uncountable. This contradiction completes the proof of the theorem. □

Proof of Corollary 1.2 – Assume for a contradiction that the index $|G : C_G(\Gamma)|$ is finite. As Γ contains all inner automorphisms of G, the centralizer $C_G(\Gamma)$ lies in the centre of G, and hence it follows from Theorem 1.1 that the subgroup $[G, \Gamma]$ is finite. Then Γ is finite by Lemma 2.1, and so $Aut(G)$ is a Černikov group. On the other hand, the full automorphism group of an arbitrary group cannot be infinite and Černikov (see [6]), so that $Aut(G)$ must be finite, and this contradiction proves the statement. □

Proof of Corollary 1.3 – Assume for a contradiction that the index $|G : C_G(\Gamma)|$ is finite. As the centralizer $C_G(\Gamma)$ is contained in the centre of G, the commutator subgroup of G is finite by Schur's theorem. Moreover, it follows from Theorem 1.1 that the subgroup $[G, \Gamma]$ is finite. Then also Γ is finite by Lemma 2.1, and so $Aut(G)$ is a countable group. On the other hand, it is known that any periodic finite-by-abelian group admitting only countably many automorphisms must have finite exponent (see [3]). This contradiction completes the proof. □

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