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# SYLOW LIKE THEOREMS FOR $V(\mathbb{Z}G)$

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Abstract. The main part of this article is a survey on torsion subgroups of the unit group of an integral group ring. It contains the major parts of my talk given at the conference "Groups, Group Rings and Related Topics" at UAEU in Al Ain October 2013. In the second part special emphasis is layed on $p$ - subgroups and on the open question whether there is a Sylow like theorem in the normalized unit group of an integral group ring. For specific classes of finite groups we prove that $p$ - subgroups of the normalized unit group of its integral group rings $V(\mathbb{Z}G)$ are isomorphic to subgroups of $G$. In particular for $p = 2$ this is shown when $G$ has abelian Sylow 2 - subgroups. This extends results known for soluble groups to classes of groups which are not contained in the class of soluble groups.

## 1. The subgroup isomorphism problem

Throughout $G$ denotes a finite group. The integral group ring of $G$ is denoted by $\mathbb{Z}G$ and $V(\mathbb{Z}G)$ is the subgroup of the unit group $U(\mathbb{Z}G)$ consisting of all units with augmentation one.

The question up to which extend torsion subgroups of $V(\mathbb{Z}G)$ are determined by $G$ has been studied since G. Higman's thesis [21] which completely settles the question when $G$ is abelian or hamiltonian. In this case the torsion subgroups with augmentation one are precisely the subgroups of $G$. This result was certainly a main ingredient for the following conjectures of H. Zassenhaus [39, Chapter 5].

**ZC–1** Let $u$ be a unit of finite order of $V(\mathbb{Z}G)$. Then $u$ is conjugate within $\mathbb{Q}G$ to an element of $G$ [45].

**ZC–2** Let $H$ be a subgroup of $V(\mathbb{Z}G)$ with the same order as $G$. Then $H$ is conjugate within $\mathbb{Q}G$ to $G$.

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**ZC–3** Let $H$ be a finite subgroup of $V(\mathbb{Z}G)$. Then $H$ is conjugate within $\mathbb{Q}G$ to a subgroup of $G$.

Note that ZC–3 implies ZC–1 as well as ZC – 2. Finite subgroups of $V(\mathbb{Z}G)$ of the same order as $G$ consist of linearly independent elements over $\mathbb{Z}$ and span $\mathbb{Z}G$. Thus such torsion subgroups are called a **group basis**. Note also that ZC – 2 contains a statement about automorphisms of $\mathbb{Z}G$. If ZC – 2 holds for $V(\mathbb{Z}G)$ then an augmentation preserving automorphism of $\mathbb{Z}G$ may be written as the composition of an automorphism of $\mathbb{Z}G$ induced from a group automorphism of $G$ followed by an automorphism of $\mathbb{Z}G$ given by the conjugation with an element of $\mathbb{Q}G$ which normalizes $\mathbb{Z}G$. The Noether - Skolem theorem shows that such rational conjugations are precisely the central automorphisms of $\mathbb{Z}G$, i.e. the automorphisms which fix the centre of $\mathbb{Z}G$ elementwise.

Nowadays we know that the conjectures ZC – 2 and ZC – 3 do not hold in general. The first counterexample to ZC - 2 (and therefore also a counterexample to ZC – 3) was constructed by K. W. Roggenkamp and L. L. Scott [30], [37], [41], [33]. But for many important classes of groups, e.g. for nilpotent groups [44], they are valid. We say that the conjecture ZC - i (i = 1, 2 or 3) holds for a group $G$, if it is valid for $V(\mathbb{Z}G)$. The first Zassenhaus conjecture ZC - 1 is still an open question and may be valid for all finite groups. The best result concerning metabelian groups is that ZC – 1 holds provided $G$ is cyclic - by - abelian [9]. For a recent survey on further known results concerning ZC – 1 see [28].

Knowing that torsion subgroups are in general not rationally conjugate to subgroups of a group base it is natural to pose the following question.

**Subgroup Question.** Are torsion subgroups of $V(\mathbb{Z}G)$ isomorphic to a subgroup of $G$?

This question contains as subcase the well known isomorphism problem **IP**, i.e., the question whether an isomorphism of $\mathbb{Z}G$ and $\mathbb{Z}H$ as rings implies that $G$ and $H$ are isomorphic. Equivalent to IP is to ask whether all group bases of $V(\mathbb{Z}G)$ are isomorphic. Note that not each torsion subgroup of $V(\mathbb{Z}G)$ is contained in a group basis. Even for the integral group ring of the smallest nonabelian group, the symmetric group $S_3$, there are involutions in $V(\mathbb{Z}S_3)$ which are not elements of a group basis, for a detailed description see e.g. [26].

1998 M. Hertweck showed that there are two nonisomorphic groups of order $2^{21} \cdot 97^{28}$ [14]. Nevertheless for many groups IP has a positive answer. It follows from a theorem of Roggenkamp and Scott, cf. [34], [40], [36] and [20], that for each finite group $G$ there exists a finite abelian extension $E = A \cdot G$ such that IP has a positive answer for $E$. However concerning torsion subgroups which are not contained in a group basis we know much less. At the IMC satellite conference 2006 in Granada Z.Marciniak posed the question whether $G$ has a Kleinian fourgroup as subgroup when $V(\mathbb{Z}G)$ has such a subgroup. Marciniak's question leads naturally to the following problem.

**Subgroup Isomorphism Problem SIP.** Classify all finite groups $H$ with the following property. Suppose that $H$ occurs as subgroup of $V(\mathbb{Z}G)$, where $G$ is finite. Then $H$ is isomorphic to a subgroup of $G$.

Till 2006 the only groups for which SIP has been established were the cyclic groups of prime power order [10, Corollary 4.1]. An affirmative answer to Marciniak's question has been given in [25] and that SIP holds for $C_p \times C_p$, where $p$ is an odd prime, has been shown in [17]. Note that the arguments for $C_p \times C_p$, with $p$ odd and $C_2 \times C_2$ are different. The latter uses the Brauer - Suzuki theorem.

The open main cases of SIP are the following.

- **Question A.** Are finite abelian subgroups of $V(\mathbb{Z}G)$ isomorphic to a subgroup of $G$?
- **Question C.** [39, Research Problem 8] In particular, are finite cyclic subgroups of $V(\mathbb{Z}G)$ isomorphic to a subgroup of $G$?
- **Question P.** Are finite $p$ - subgroups of $V(\mathbb{Z}G)$ isomorphic to a subgroup of $G$?

Even for groups of order six SIP is an open problem. If one restricts himself to a specific class of groups (e.g. to the case that $G$ soluble) then much more is known. This show the results of A. Weiss [44], mentioned above in the context of the Zassenhaus conjecture ZC − 3, or that one of M. Hertweck [18] that a cyclic subgroup of $V(\mathbb{Z}G)$ is isomorphic to subgroup of $G$ provided $G$ is soluble. The main obstruction for general results is that very little is known with respect to integral group rings of finite nonabelian simple groups.

## 2. The Gruenberg - Kegel graph of $\mathbb{Z}G$

If $U$ is a group (not necessarily finite) then its prime graph $\Pi(U)$ (also called its Gruenberg - Kegel graph) is the graph whose vertices are the primes dividing the order of a torsion element of $U$. Two vertices $p, q$ are connected if $U$ has an element of order $p \cdot q$.

If $R$ is a ring then its Gruenberg - Kegel graph $\Gamma(R)$ is the prime graph of its unit group. In the case when $R$ is an integral group ring it makes sense also to consider the prime graph of its normalized unit group. $\Pi(V(\mathbb{Z}G))$ will provide more information because in $U(\mathbb{Z}G)$ the prime 2 is connected with each other prime. On the other hand it seems to be more difficult to determine $\Pi(V(\mathbb{Z}G))$.

**Prime graph question PQ.** Let $G$ be a finite group. Is

$$\Pi(V(\mathbb{Z}G)) = \Pi(G)?$$

Concerning $\Gamma(\mathbb{Z}G)$ the question is whether it coincides with $\Pi(G \times C_2)$. The subgroup isomorphism problem has a positive answer for $C_{pq}$ iff PQ holds. So the prime graph question is a special case of the subgroup isomorphism problem. Clearly a positive answer to ZC − 1 implies a positive answer to PQ.

Especially with respect to the prime graph question a lot of activities have been made within the last years. V. Bovdi, A. Konovalov, E. Jespers, S. Siciliano, M. Salim established also with the aid of computer algebra (mainly with [5]) a positive answer for about half of the sporadic simple groups and for simple groups of small order, e.g. alternating groups $A_n$ for $n \leq 10$, see [8], [38]. The basic tool is a character-theoretical method, nowadays called the **HeLP** - method, which originates from the paper of **L**uthar and **P**assi in which they show that ZC – 1 holds for the alternating group $A_5$ [31]. A first algorithmic description of it is given in [7]. **He**rtweck extended this method to Brauer characters, cf. [16, §4].

For a certain time it was not clear whether the PQ - test of simple groups with the HeLP - method provides just interesting examples for PQ and possibly candidates for a counterexample of ZC – 1. The following result shows that a complete answer for a general finite group may be reduced to almost simple groups [1]. Thus the computer aided test of simple groups (especially of sporadic simple groups) and their automorphism groups is an essential step towards a general solution.

**Theorem 2.1.** [27, Theorem 2.1] Let $G$ be a finite group. Supppose that PQ has a positive answer for all almost simple images of $G$ then PQ holds for $G$.

The HeLP - method does not decide whether PQ holds for a given group $G$. Recently additional arguments have been developed by A. Bächle and L. Margolis [3]. With this they could complete the proof of part b) of the following result. For part a) the HeLP - method is sufficient [27, §3].

**Theorem 2.2** [27], [3] Let $G$ be a finite group whose almost simple images involve only simple groups whose order is divisible by three primes. Then

a) $\Pi(G \times C_2) = \Gamma(\mathbb{Z}G)$.
b) PQ has a positive answer for $G$.

For a more detailed overview of known results on PQ we refer to [28], [38], [8].

## 3. Sylow like results

The previous sections certainly support the question whether a Sylow like theorem may hold in $V(\mathbb{Z}G)$.

We say that in $V(\mathbb{Z}G)$ a **Sylow like theorem** holds provided for each prime $p$ each finite $p$ - subgroup of $V(\mathbb{Z}G)$ is conjugate within $\mathbb{Q}G$ to a Sylow $p$-subgroup of $G$.

We use Sylow like because the conjugation takes place in $\mathbb{Q}G$ and not in $\mathbb{Z}G$. If $G$ is a $p$-group then the results by Roggenkamp and Scott [35] and Weiss [43] show that even Sylow's theorem holds in $V(\mathbb{Z}_pG)$, where $\mathbb{Z}_p$ stands for the $p$-adic integers. But in general conjugation in $\mathbb{Q}G$ cannot be

[1]We call a finite group $X$ almost simple if $\mathrm{Inn}S \leq X \leq \mathrm{Aut}S$ for some nonabelian simple group S.

replaced by conjugation in $\mathbb{Z}_pG$. In $\mathbb{Z}S_3$ there is an involution which is not conjugate within $V(\mathbb{Z}_2S_3)$ to a trivial unit [2] [26, p.104]. On the other hand in $\mathrm{GL}(n,\mathbb{Z})$ Sylow's theorem is valid [1]. Thus one could consider also conjugation in a larger ring, may be in a larger group ring.

Similarly, if for a fixed prime $p$ the $p$-subgroups of $V(\mathbb{Z}G)$ have the property stated above then we say that in $V(\mathbb{Z}G)$ a Sylow like theorem is valid for $p$.

If the $p$-subgroups of $V(\mathbb{Z}G)$ are isomorphic to a subgroup of $G$ then we say that for $p$ a **weak Sylow like theorem** holds in $V(\mathbb{Z}G)$.

Clearly if ZC–3 holds for $\mathbb{Z}G$ then a Sylow like theorem holds in $V(\mathbb{Z}G)$. Thus a Sylow like theorem holds in $V(\mathbb{Z}G)$ when $G$ is nilpotent by [44].

Note that Question P has an affirmative answer if a weak Sylow like theorem holds on $V(\mathbb{Z}G)$ for each group $G$. The methods developed by A. Weiss show even more. The following result by M. Dokuchaev and S. O. Juriaans covers the supersoluble groups.

**Theorem 3.1.** *[11, Theorem 2.9] Let $G$ be a finite group with a nilpotent normal subgroup $N$ such that $G/N$ is nilpotent. Then the Sylow like theorem holds in $V(\mathbb{Z}G)$.*

With respect to group bases even more is known.

**Theorem 3.2.** *[29] Let $G$ be a finite soluble group and let $H$ be a group basis of $\mathbb{Z}G$. Let $p$ be a prime. Then each $p$-subgroup of $H$ is conjugate within $\mathbb{Q}G$ to a subgroup of a Sylow $p$-subgroup of $G$.*

The following result indicates that a Sylow like theorem in nonabelian simple groups of small Lie rank may be true.

**Theorem 3.3.** *[19] Let $G = \mathrm{PSL}(2,q)$ then finite $2$ - subgroups of $V(\mathbb{Z}G)$ are isomorphic to subgroups of $G$.*

By [10] this is clear when $q$ is even because $G = \mathrm{PSL}(2,2^f)$ has elementary abelian subgroups. So the case $q$ odd is the main part of Theorem 3.3, cf. [19, Theorem 2.1]. Note if $q$ is odd the Sylow 2 - subgroups of $\mathrm{PSL}(2,q)$ are dihedral or elementary abelian of order 4.

Next we study the behaviour under extensions.

**Proposition 3.4.** *[11] Let $G$ be a finite group and let $p$ be a prime. Suppose that $N \lhd G$ is a $p'$-group. Then*

- *$p$-subgroups of $V(\mathbb{Z}G)$ are isomorphic to $p$-subgroups of $V(\mathbb{Z}G/N)$.*
- *$p$-subgroups of $V(\mathbb{Z}G)$ are conjugate within $\mathbb{Q}G$ to subgroups of $G$ if, and only if, $p$-subgroups of $V(\mathbb{Z}G/N)$ are conjugate within $\mathbb{Q}G/N$ to subgroups of $G/N$.*

[2] Elements of $G$ are called the trivial units of $\mathbb{Z}G$.

**Proposition 3.5.** *Let $N \lhd G$ and suppose that $G/N$ is a $p'$-group.*

a) *The order of a $p$ - subgroup $U$ of $V(\mathbb{Z}G)$ divides $|N|$.*
b) *Suppose that $N \cong N_1 \times N_2$ with $N_i \lhd G$. Let $\kappa_i$ be the reductions of $G$ to $G/N_i$ and denote by $\hat{\kappa}_i$ its extensions to $V(\mathbb{Z}G) \longrightarrow V(\mathbb{Z}G/N_i)$. Let $K_i$ be the kernel of $\hat{\kappa}_i$. Assume further that torsion units in $K_1$ have only nonzero partial augmentations [3] in $N_1$ (i.e. if $C$ is a conjugacy class of $G$ then $\varepsilon_C(u) \neq 0$ only if $C \subset N_1$). If $U$ is a $p$ - subgroup of $V(\mathbb{Z}G)$ then $U$ is isomorphic to a subgroup of $V_1 \times V_2$, where $V_i$ denotes the projection of $U$ onto $V(\mathbb{Z}G/N_i)$.*

**Proof.** a) Because the order of a finite subgroup of $V(\mathbb{Z}G)$ divides the order of $G$ and $G/N$ is a $p'$ - group it is clear that $|U|$ divides $|N|$.

b) Note that the exact sequence of groups

$$1 \longrightarrow N_i \longrightarrow G \xrightarrow{\kappa_i} G/N_i \longrightarrow 1$$

induces a ring homomorphism $\lambda : V(\mathbb{Z}G) \longrightarrow V(\mathbb{Z}G/N_i)$ and thus a group homomorphism $\hat{\kappa}_i : V(\mathbb{Z}G) \longrightarrow V(\mathbb{Z}G/N_i)$.

We show that $U \cap K_1 \cap K_2 = 1$. Let $u \in U$ and suppose that $u \in K_1 \cap K_2$ has order $p$. Elements of order $p$ in $N_1 \times N_2$ are of the form $(a,b), (1,b)$ or $(a,1)$ with $o(a) = o(b) = p$. Let $C_1$ be the set of conjugacy classes with representatives of type $(a,1)$ and $C_2, C_3$ rsp. that one with representatives of type $(a,b), (1,b)$ rsp.

Put $\gamma_i(u) = \sum_{C \in C_i} \varepsilon_C(u)$.

By [16, Theorem 2.3] and [32, Theorem 2.7] we know that $\varepsilon_C(u) = 0$ provided the order of a representative of the conjugacy class $C$ is different from $p$. Because $u$ is normalized we know that $\sum_{i=1}^{3} \gamma_i = 1$. $\hat{\kappa}_1(u) = 1$ shows that $\gamma_1(u) = 1$ and $\gamma_2(u) + \gamma_3(u) = 0$. Similarly $\hat{\kappa}_2(u) = 1$ yields that $\gamma_3(u) = 1$ and $\gamma_1(u) + \gamma_2(u) = 0$. It follows that $\gamma_1(u) = \gamma_3(u) = 1$ and $\gamma_2(u) = -1$. But by assumption for units in $K_1$ we have that $\gamma_2(u) = \gamma_3(u) = 0$. Thus there exists no unit of order $p$ in $K_1 \cap K_2$ and it follows $U \cap K_1 \cap K_2 = 1$.

Consider the map $u \mapsto (\hat{\kappa}_1(u), \hat{\kappa}_2(u)) \in V_1 \times V_2$. Because $U \cap K_1 \cap K_2 = 1$ this map is injective. Thus $U \leq V_1 \times V_2$. □

**Theorem 3.6.** *[15, Theorem 4.2] Suppose that $N$ is a normal $p$ - subgroup of $G$. Let $\hat{\kappa} : V(\mathbb{Z}G) \longrightarrow V(\mathbb{Z}G/N)$ be the induced map from the reduction $\kappa : G \longrightarrow G/N$. Suppose that $V \leq V(\mathbb{Z}G)$ is a torsion subgroup in the kernel of $\hat{\kappa}$. Then $V$ is conjugate within $\mathbb{Q}G$ to subgroup of $N$.*

[3] If $C$ is a conjugacy class of $G$ and $u = \sum_{g \in G} z_g g \in \mathbb{Z}G$ then $\varepsilon_C = \sum_{g \in C} z_g$ is called the partial augmentation of $u$ with respect to $C$.

**Remark.** It would be extremely important if a similar result as Theorem 3.6 could be established in the case when $N$ is a perfect minimal normal subgroup of $G$.

**Corollary 3.7.** *Let the notations as in Proposition 3.5. Assume that $N_1$ is a p-group. If $U$ is a $p$ - subgroup of $V(\mathbb{Z}G)$ then $U$ is isomorphic to a subgroup of $V_1 \times V_2$, where $V_i$ denotes the projection of $U$ onto $V(\mathbb{Z}G/N_i)$.*

**Proof.** By Theorem 3.6 a unit $u \in K_1$ has non-trivial partial augmentations only in $N_1$. So we may apply Proposition 3.5 and the corollary follows. □

**Corollary 3.8.** *Assume that the finite group $G$ is the direct product of $G_1$ and $G_2$. Assume that $G_1$ is a p-group and that each finite p-subgroup of $V(\mathbb{Z}G_2)$ is isomorphic to a subgroup of $G_2$. Then for $p$ a weak Sylow like theorem holds in $V(\mathbb{Z}G)$.*

## 4. **Special Sylow subgroups**

If one assumes that Sylow $p$-subgroups of $G$ have a special structure then with respect to Questions A, C and P much more is known. This is especially the case when $G$ has abelian Sylow subgroups.

The case of cyclic Sylow subgroups is covered by the following.

**Theorem 4.1.**
a) *The weak Sylow like theorem holds for a prime $p$ in $V(\mathbb{Z}G)$ provided $G$ has cyclic Sylow $p$-subgroups.*
b) *The weak Sylow like theorem holds for 2 in $V(\mathbb{Z}G)$ provided $G$ has generalized quaternion Sylow 2-subgroups.*
c) *The weak Sylow like theorem holds for an odd prime $p$ in $V(\mathbb{Z}G)$ provided Sylow $p$-subgroups of $G$ are isomorphic to $C_p \times C_p$. If $p = 2$ it holds provided $G$ has elementary abelian Sylow 2-subgroups.*

**Proof.** a) and b) are for $p$ odd an immediate consequence of [17] and for $p = 2$ one of [25].

c) By [10, Corollary 4.1] each $p$-element of $V(\mathbb{Z}G)$ has order $p$. Because the order of a finite subgroup of $V(\mathbb{Z}G)$ has to divide the order of $G$ the result follows.

If $G$ has elementary abelian Sylow 2-subgroups then all 2-elements of $V(\mathbb{Z}G)$ are involutions. Thus all 2-subgroups of $V(\mathbb{Z}G)$ are abelian. Because the order of a torsion subgroup of $V(\mathbb{Z}G)$ divides $|G|$ it follows that each 2-subgroup is isomorphic to a subgroup of a Sylow 2-subgroup of $G$. □

**Corollary 4.2.** *Suppose that ZC –1 holds for $V(\mathbb{Z}G)$. Assume that $G$ has a nilpotent normal Hall subgroup $N$ and that all Sylow subgroups of $G/N$ are cyclic or generalized quaternion. Then a weak Sylow like result holds for $V(\mathbb{Z}G)$.*

The structure theorems for Frobenius groups show that the previous corollary may be applied to Frobenius groups. But for Frobenius groups more is known.

**Theorem 4.3.** *[12, Theorem 6.1] Let $G$ be a Frobenius group. Then for odd primes $p$ the Sylow like theorem holds for $V(\mathbb{Z}G)$. For $p = 2$ the same holds provided $G$ does not have $S_5$ as image.*

**Theorem 4.4.** *[6, Theorem 3], [12, Theorem 6.1]. Let $G$ be a finite Frobenius group. Then each torsion unit of $V(\mathbb{Z}G)$ of prime power order is conjugate within $\mathbb{Q}G$ to an element of $G$.*

V. Bovdi and M. Hertweck completed the proof of Theorem 4.4 by showing that ZC − 1 holds for the covering group $\hat{S}_5$ of the symmetric group $S_5$ which has a unique conjugacy class of involutions.[4]

This was the missing piece in earlier work on integral group rings of Frobenius groups done in [11,12,23]. I expect that with Theorem 4.4 the assumption on $p = 2$ in Theorem 4.3 can be removed.

The following should be seen in this direction.

**Proposition 4.5.** *Suppose that $Q$, the quaternion group of order 8, is a Sylow 2-subgroup of $G$ and that 2-elements of $V(\mathbb{Z}G)$ are conjugate rationally to elements of $G$. Then for $p = 2$ the Sylow like theorem holds in $V(\mathbb{Z}G)$.*

**Proof.** By Theorem 4.1 finite 2-subgroup of $V(\mathbb{Z}G)$ are isomorphic to subgroups of $Q$. Thus torsion 2-subgroups of order $\leq 4$ are cyclic and so by assumption conjugate to a subgroup of $G$. It remains to show that subgroups of $V(\mathbb{Z}G)$ isomorphic to $Q$ are rationally conjugate. Let $Q_1$ be such a subgroup.

Let $\sigma : Q_1 \longrightarrow Q$ be an isomorphism. Then we may modify $\sigma$ by $\tau \in \mathrm{Aut}Q$ such that $\tau \circ \sigma$ maps each element of $Q_1$ to an element of $Q$ which is within $\mathbb{Q}G$ conjugate to an element of $Q$. Let $W$ be a Wedderburn component of $G$ and $\pi : G \longrightarrow W$ be the projection. Then by the previous it follows that $\pi(Q_1)$ and $\pi(Q)$ have the same characters and therefore they are conjugate within $W$. So $Q_1$ and $Q$ are conjugate in $G$ and hence by Noether - Deuring also in $\mathbb{Q}G$.

**Proposition 4.6.** *[11, Proposition 2.11] Assume that $G$ is a finite soluble group. Suppose that $G$ has abelian Sylow $p$-subgroups. Then a $p$-subgroup of $V(\mathbb{Z}G)$ is rationally conjugate to a subgroup of $G$.*

This proposition may easily be generalized to $p$-constrained groups [2, Theorem 3.2].

Nevertheless these are results which still circulate, as all other results before in this section, around the class of soluble groups. But for abelian Sylow 2-subgroups there are also results which hold for each group with such Sylow subgroups.

For small abelian Sylow 2 - subgroups it is known that even a Sylow-like theorem holds.

**Proposition 4.7.** *Let $G$ be a group whose Sylow 2-subgroups are abelian of order $\leq 8$. Then each 2-subgroup of $V(\mathbb{Z}G)$ is rationally conjugate to a subgroup of $G$.*

**Proof.** If $G$ has cyclic Sylow 2-subgroups or Sylow 2-subgroups isomorphic to $C_4 \times C_2$ then $G$ is soluble and the result follows from Proposition 4.6. If $S \in Syl_2(G)$ is elementary abelian of order 4 or 8 then [2, Proposition 3.4] completes the proof. □

[4] $\hat{S}_5$ is the group with number 89 in the GAP Small Groups Library [13]

We remark that it is unknown whether a Sylow like theorem holds in $V(\mathbb{Z}G)$ when $G$ has elementary abelian Sylow 2-subgroups of order 16. In the case when $G$ is soluble it is shown in [11, Theorem 5.3] that a Sylow-like theorem holds in $V(\mathbb{Z}G)$ provided Sylow $p$-subgroups of $G$ have order dividing $p^3$.

**Proposition 4.8.** *Let $G$ be a finite group with abelian Sylow 2-subgroups. Then for $p = 2$ in $V(\mathbb{Z}G)$ a weak Sylow like theorem holds.*

**Proof.** Let $N = O_{2'}(G)$. Then it follows from Proposition 3.4 that 2-subgroups of $V(\mathbb{Z}G)$ are isomorphic to those of $V(\mathbb{Z}G/N)$. Thus we may assume that $G$ has no normal subgroup of odd order.

Now it follows from the structure theorem of finite groups with abelian Sylow 2-subgroups that $G$ has a normal subgroup $N$ which is the direct product of non-abelian simple groups $S_i$ with abelian Sylow 2- subgroups and an abelian 2-group $P$. Moreover $P$ is a normal subgroup of $G$ as well as the join $M$ of all $S_i$ and $G/N$ has odd order.

If $P$ is trivial the Sylow 2-subgroups of the factors $S_i$ are elementary abelian and so the Sylow 2-subgroup of $G$ as well. By [10] 2 - elements of $V(\mathbb{Z}G)$ are involutions and thus each finite 2 - subgroup $H$ of $V(\mathbb{Z}G)$ is elementary abelian. Clearly $H$ divides the order of a Sylow 2-subgroup $S$ of $G$ because it divides the order of $G$ [21], [4]. Consequently $H$ is isomorphic to a subgroup of $S$.

If all factors $S_i$ are trivial then $G$ is soluble and the result follows from [11].

Thus we can assume that $G$ has nontrivial normal subgroups $M$ and $P$ and we may apply Proposition 3.5 with $P = N_1$. Thus a finite 2-subgroup $H$ of $V(\mathbb{Z}G)$ is isomorphic to a subgroup of $P \times V$, where $V$ is a finite 2-subgroup of $V(\mathbb{Z}G/P)$. As before we see that $V$ is isomorphic to subgroup of a Sylow 2-subgroup $S$ of $G/P$. Because Sylow 2-subgroups of $G$ are isomorphic to a direct product of $P$ and $S$ the proposition follows. $\square$

**Remark.** For a similar result for groups with abelian Sylow $p$-subgroups with $p$ odd one needs more knowledge on simple groups with such Sylow $p$-subgroups. It is e.g. unknown whether in $V(\mathbb{Z}PSL(2,p^f))$ a weak Sylow like theorem for $p$ is valid if $f \geq 3$. Note that all Sylow subgroups of $PSL(2,p^f)$ are abelian if $p^f \equiv 3$ or 5 mod 8. Of course together with Theorem 4.1 we get

**Corollary 4.9.** *Suppose that all Sylow subgroups of odd order are cyclic or of order $\leq p^2$ and Sylow 2-subgroups are abelian or generalized quaternion. Then a weak Sylow like theorem holds in $V(\mathbb{Z}G)$.*

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