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SYMMETRY CLASSES OF POLYNOMIALS ASSOCIATED WITH THE DIRECT PRODUCT OF PERMUTATION GROUPS

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Abstract. Let G_i be a subgroup of S_{m_i}, $1 \le i \le k$. Suppose χ_i is an irreducible complex character of G_i. We consider $G_1 \times \cdots \times G_k$ as subgroup of S_m, where $m = m_1 + \cdots + m_k$. In this paper, we give a formula for the dimension of $H_d(G_1 \times \cdots \times G_k, \chi_1 \times \cdots \times \chi_k)$ and investigate the existence of an o-basis of this type of classes.

1. Introduction

The relative symmetric polynomials as a generalization of symmetric polynomials are introduced in [7]. In [1, 10, 11], the authors studied the space of relative symmetric polynomials (symmetry class of polynomials) with respect to the irreducible characters of certain groups. In this paper we study the symmetry class of polynomials with respect to the direct product of permutation groups. First we give a review of this notion (for more details, see [7]).

Let $H_d[x_1, \ldots, x_m]$ be the complex space of homogeneous polynomials of degree d with independent commuting variables $x_1, \ldots, x_m$. Let $\Gamma^+_{m,d}$ be the set of all m-tuples of non-negative integers $\alpha = (\alpha_1, \ldots, \alpha_m)$, such that $\sum_{i=1}^m \alpha_i = d$. For any $\alpha \in \Gamma^+_{m,d}$, let X^α be the monomial $x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_m^{\alpha_m}$. Then the set $\{X^\alpha | \ \alpha \in \Gamma^+_{m,d}\}$ is a basis of $H_d[x_1, \ldots, x_m]$. An inner product on $H_d[x_1, \ldots, x_m]$ is defined by

$$\langle X^\alpha, X^\beta \rangle = \delta_{\alpha,\beta}.$$

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Suppose G is a subgroup of the symmetric group S_m. Then G acts on $H_d[x_1,\dots,x_m]$ by

$$q^{\sigma}(x_1,\dots,x_m)=q(x_{\sigma^{-1}(1)},\dots,x_{\sigma^{-1}(m)}),$$

and this action is extended linearly to the group algebra $\mathbb{C}G$. Let χ be an irreducible complex character of G. Consider the idempotent

$$T(G,\chi)=\frac{\chi(1)}{|G|}\sum_{\sigma\in G}\chi(\sigma)\sigma,$$

in the group algebra $\mathbb{C}G$. The image of $H_d[x_1,\dots,x_m]$ under the map $T(G,\chi)$ is called the *symmetry class of polynomials of degree d with respect to G and χ*, and it is denoted by $H_d(G,\chi)$. For any $q\in H_d[x_1,\dots,x_m]$,

$$q^*=T(G,\chi)(q)$$

is called a *symmetrized polynomial with respect to G and χ*. For $\alpha\in\Gamma^+_{m,d}$, we denote the symmetrized monomial $(X^{\alpha})^*$ by $X^{\alpha,*}$. So

$$H_d(G,\chi)=\langle X^{\alpha,*}|\ \alpha\in\Gamma^+_{m,d}\rangle.$$

The group G also acts on $\Gamma^+_{m,d}$ by

$$\alpha\sigma=(\alpha_{\sigma(1)},\dots,\alpha_{\sigma(m)}).$$

Let Δ be a set of representatives of orbits of $\Gamma^+_{m,d}$ under the action G.
For any $\alpha\in\Gamma^+_{m,d}$, we have

$$\|X^{\alpha,*}\|^2 = \chi(1)\frac{[\chi,1]_{G_\alpha}}{|G:G_\alpha|}, \tag{1.1}$$

where G_α is the stabilizer subgroup of α under the action of G and $[\ ,\ ]_G$ is the inner product of characters (see [5]). Hence, $X^{\alpha,*}\neq 0$ if and only if $[\chi,1]_{G_\alpha}\neq 0$.

Let Ω be the set of all $\alpha\in\Gamma^+_{m,d}$ with $[\chi,1]_{G_\alpha}\neq 0$ and suppose $\bar{\Delta}=\Delta\cap\Omega$. We have

$$\dim H_d(G,\chi) = \chi(1)\sum_{\alpha\in\bar{\Delta}}[\chi,1]_{G_\alpha}. \tag{1.2}$$

An orthogonal basis of $H_d(G,\chi)$ of the form $\{X^{\alpha,*}|\ \alpha\in S\}$, where S is a subset of $\Gamma^+_{m,d}$ is called an *o-basis* of $H_d(G,\chi)$. If χ is linear, then the set $\{X^{\alpha,*}|\ \alpha\in\bar{\Delta}\}$ is an o-basis of $H_d(G,\chi)$. If χ is not linear, then $H_d(G,\chi)$ may have no o-basis.
In this paper, we consider $G=G_1\times\cdots\times G_k$, where G_i is permutation group and we find a formula for the dimension of $H_d(G,\chi)$. Then we investigate the existence of an o-basis for this symmetry class. A similar result has been obtained for symmetry classes of tensors in [2, 3, 4, 6, 8, 9, 12].

2. Main Results

A partition of m is a non-increasing finite sequence of positive integers, whose sum is m. If π is a partition of m we denote $\pi\vdash m$. Let $\pi=(\pi_1,\dots,\pi_k)\vdash m$. For any $1\le i\le k$, suppose

$$\Lambda_i=\{t:\sum_{j=1}^{i-1}\pi_j<t\le\sum_{j=1}^{i}\pi_j\}.$$

The corresponding Young subgroup is defined by

$$S_\pi = S_{\Lambda_1} \times \cdots \times S_{\Lambda_k},$$

where S_{Λ_i} is the symmetric group on Λ_i.

Now, let G_i be a subgroup of S_{π_i}, $1 \leq i \leq k$. Let $m = \sum_{i=1}^{k} \pi_i$. Rearranging the order, we may assume $\pi = (\pi_1, \ldots, \pi_k) \vdash m$. Let $G = G_1 \times \cdots \times G_k$. Then $G \leq S_\pi$. In particular, we can consider G as a subgroup of symmetric group S_m. Let χ be an irreducible character of G. Then $\chi = \prod_{i=1}^{k} \chi_i$, where χ_i is an irreducible character of G_i, $1 \leq i \leq k$.

Let $\Gamma^+_{\Lambda_i,\rho_i}$ be the set of all π_i-tuples of non-negative integers $\alpha^i = (\alpha_{r_1}, \ldots, \alpha_{r_{\pi_i}})$, such that $\sum_{j=1}^{\pi_i} \alpha_{r_j} = \rho_i$, where $r_k = \sum_{j=1}^{i-1} \pi_j + k$, $\; 1 \leq k \leq \pi_i$. Let $X_i = \{x_{r_1}, \ldots, x_{r_{\pi_i}}\}$, and suppose $H_{\rho_i}[X_i]$ is the complex space of homogeneous polynomials of degree ρ_i with independent variables in X_i. For any $\alpha^i \in \Gamma^+_{\Lambda_i,\rho_i}$, let $X_i^{\alpha^i}$ be the monomial

$$x_{r_1}^{\alpha_{r_1}} \cdots x_{r_{\pi_i}}^{\alpha_{r_{\pi_i}}}.$$

Then the set $\{X_i^{\alpha^i} \mid \alpha^i \in \Gamma^+_{\Lambda_i,\rho_i}\}$ is a basis of $H_{\rho_i}[X_i]$. Naturally, we define an inner product on $H_{\rho_i}[X_i]$ by $\langle X_i^{\alpha^i}, X_i^{\beta^i} \rangle = \delta_{\alpha^i,\beta^i}$. We define $H_{\rho_i}(G_i, \chi_i) = T(G_i, \chi_i)(H_{\rho_i}[X_i])$ and $X_i^{\alpha^i,*} = T(G_i, \chi_i)(X_i^{\alpha^i})$. So we have $H_{\rho_i}(G_i, \chi_i) = \langle X_i^{\alpha^i,*} | \alpha^i \in \Gamma^+_{\Lambda_i,\rho_i} \rangle$.

Let $\alpha = (\alpha_1, \ldots, \alpha_m) \in \Gamma^+_{m,d}$. Then $(\alpha_{|\Lambda_1}, \ldots, \alpha_{|\Lambda_k}) \in \prod_{i=1}^{k} \Gamma^+_{\Lambda_i,\rho_i}$, where $\rho_i = \sum_{j \in \Lambda_i} \alpha_j$. There is a bijection between $\Gamma^+_{m,d}$ and the set

$$\bigcup_{(\rho_1,\ldots,\rho_k) \in \Gamma^+_{k,d}} \prod_{i=1}^{k} \Gamma^+_{\Lambda_i,\rho_i}.$$

We denote $\alpha_{|\Lambda_i}$ by α^i. If $g = g_1 \cdots g_k$, where $g_i \in G_i, 1 \leq i \leq k$, then $\alpha g = (\alpha^1 g_1, \ldots, \alpha^k g_k)$, so we have

$$\Delta = \bigcup_{(\rho_1,\ldots,\rho_k) \in \Gamma^+_{k,d}} \prod_{i=1}^{k} \Delta_{\Lambda_i,\rho_i},$$

where $\Delta_{\Lambda_i,\rho_i}$ is a set of representatives of orbits of $\Gamma^+_{\Lambda_i,\rho_i}$ under the action G_i.

Now, let $\alpha = (\alpha^1, \ldots, \alpha^k) \in \prod_{i=1}^{k} \Gamma^+_{\Lambda_i,\rho_i}$ for some $(\rho_1, \ldots, \rho_k) \in \Gamma^+_{k,d}$. Then $G_\alpha = \prod_{i=1}^{k} (G_i)_{\alpha^i}$, where $(G_i)_{\alpha^i}$ is the stabilizer subgroup of $\alpha^i \in \Gamma^+_{\Lambda_i,\rho_i}$ in G_i. Hence we have $[\chi, 1]_{G_\alpha} = \prod_{i=1}^{k} [\chi_i, 1]_{(G_i)_{\alpha^i}}$. Thus $[\chi, 1]_{G_\alpha} \neq 0$ if and only if $[\chi_i, 1]_{(G_i)_{\alpha^i}} \neq 0$, for all i, $1 \leq i \leq k$. We denote by $\Omega_{\Lambda_i,\rho_i}$, the set of all $\alpha^i \in \Gamma^+_{\Lambda_i,\rho_i}$ with $[\chi_i, 1]_{(G_i)_{\alpha^i}} \neq 0$ and define $\bar{\Delta}_{\Lambda_i,\rho_i} = \Omega_{\Lambda_i,\rho_i} \cap \Delta_{\Lambda_i,\rho_i}$. Summarizing previous statements, we have the following theorem.

Theorem 2.1. *With respect to the above mentioned notations, the following equality holds.*

$$\bar{\Delta} = \bigcup_{(\rho_1,\ldots,\rho_k)\in\Gamma^+_{k,d}} \prod_{i=1}^{k} \bar{\Delta}_{\Lambda_i,\rho_i}.$$

In the following theorem, we give a formula for the dimension of $H_d(G,\chi)$.

Theorem 2.2. *We have*

$$\dim H_d(G,\chi) \;=\; \sum_{(\rho_1,\ldots,\rho_k)\in\Gamma^+_{k,d}} \prod_{i=1}^{k} \dim H_{\rho_i}(G_i,\chi_i).$$

Proof. Applying (1.2) and Theorem 2.1, we have

$$\begin{aligned}
\dim H_d(G,\chi) &= \chi(1)\sum_{\alpha\in\bar{\Delta}}[\chi,1]_{G_\alpha} \\
&= \prod_{i=1}^{k}\chi_i(1) \sum_{\rho\in\ \Gamma^+_{k,d}} \sum_{\alpha\in\prod_{i=1}^{k}\bar{\Delta}_{\Lambda_i,\rho_i}} \prod_{i=1}^{k}[\chi_i,1]_{(G_i)_{\alpha^i}} \\
&= \sum_{\rho\in\ \Gamma^+_{k,d}} \prod_{i=1}^{k}\chi_i(1) \sum_{\alpha^i\in\ \bar{\Delta}_{\Lambda_i,\rho_i}} [\chi_i,1]_{(G_i)_{\alpha^i}} \\
&= \sum_{\rho\in\ \Gamma^+_{k,d}} \prod_{i=1}^{k}\dim H_{\rho_i}(G_i,\chi_i),
\end{aligned}$$

where $\alpha=(\alpha^1,\ldots,\alpha^k)\in\prod_{i=1}^{k}\Gamma^+_{\Lambda_i,\rho_i}$ and $\rho=(\rho_1,\ldots,\rho_k)\in\Gamma^+_{k,d}$. □

Corollary 2.3. *If $G=G_1\times G_2$ and $\chi=\chi_1\times\chi_2$ where $\chi_i\in Irr(G_i)$, $i=1,2$, then*

$$\dim H_d(G,\chi)=\sum_{i=0}^{d}\dim H_i(G_1,\chi_1)\dim H_{d-i}(G_2,\chi_2).$$

Proof. Since $\Gamma^+_{2,d}=\{(i,d-i),\ 0\le i\le d\}$, the result is immediate by Theorem 2.2. □

Let $D^+_{k,d}$ be the set of all $(\rho_1,\ldots,\rho_k)\in\Gamma^+_{k,d}$ such that $H_{\rho_i}(G_i,\chi_i)\neq 0$, for any $1\le i\le k$. We define the subset $H_{\rho_1}(G_1,\chi_1)\cdots H_{\rho_k}(G_k,\chi_k)$ of $H_d(G,\chi)$ by

$$\Big\{\sum_{\text{finite}} f_1\cdots f_k \,\Big|\; f_i\in H_{\rho_i}(G_i,\chi_i),\ 1\le i\le k\Big\}.$$

It is easy to see that $H_{\rho_1}(G_1,\chi_1)\cdots H_{\rho_k}(G_k,\chi_k)$ is a vector space on $\mathbb{C}$ and

$$\dim\ H_{\rho_1}(G_1,\chi_1)\cdots H_{\rho_k}(G_k,\chi_k)=\prod_{i=1}^{k}\dim\ H_{\rho_i}(G_i,\chi_i).$$

Theorem 2.4. *Let $G=G_1\times\cdots\times G_k$ and $\chi=\prod_{i=1}^{k}\chi_i\in Irr(G)$. Then*

$$H_d(G,\chi)=\bigoplus_{(\rho_1,\ldots,\rho_k)\in D^+_{k,d}} H_{\rho_1}(G_1,\chi_1)\cdots H_{\rho_k}(G_k,\chi_k)$$

Proof. Let $\alpha = (\alpha^1, \dots, \alpha^k) \in \prod_{i=1}^k \Gamma^+_{\Lambda_i,\rho_i}$, for some $\rho = (\rho_1, \dots, \rho_k) \in D^+_{k,d}$. Then $X^{\alpha,*} = X_1^{\alpha^1,*} \cdots X_k^{\alpha^k,*}$. Hence

$$\langle X^{\alpha,*} | \ \alpha \in \prod_{i=1}^k \Gamma^+_{\Lambda_i,\rho_i} \rangle = H_{\rho_1}(G_1, \chi_1) \cdots H_{\rho_k}(G_k, \chi_k).$$

If $t = (t_1, \dots, t_k) \in D^+_{k,d}$ with $\rho \neq t$ and suppose $\beta = (\beta^1, \dots, \beta^k) \in \prod_{i=1}^k \Gamma^+_{\Lambda_i,t_i}$, then $\langle X^{\alpha,*}, \ X^{\beta,*} \rangle = 0$, so we obtain

$$\begin{aligned} H_d(G, \chi) &= \langle X^{\alpha,*} | \ \alpha \in \Gamma^+_{m,d} \rangle \\ &= \bigoplus_{(\rho_1,\dots,\rho_k) \in D^+_{k,d}} \langle X^{\alpha,*} | \ \alpha \in \prod_{i=1}^k \Gamma^+_{\Lambda_i,\rho_i} \rangle \\ &= \bigoplus_{(\rho_1,\dots,\rho_k) \in D^+_{k,d}} H_{\rho_1}(G_1, \chi_1) \cdots H_{\rho_k}(G_k, \chi_k). \end{aligned}$$

□

Let $G = S_\pi$ and $\chi = 1_{S_\pi}$. Let $\Lambda^i_{\rho_i}$ be the space of symmetric homogeneous polynomials of degree ρ_i with independent variables in X_i. Using Theorem 2.4, we have

$$H_d(G, \chi) = \bigoplus_{(\rho_1,\dots,\rho_k) \in D^+_{k,d}} \Lambda^1_{\rho_1} \cdots \Lambda^k_{\rho_k}.$$

If $\alpha = (\alpha^1, \dots, \alpha^k), \ \beta = (\beta^1, \dots, \beta^k) \in \prod_{i=1}^k \Gamma^+_{\Lambda_i,\rho_i}$ for some $(\rho_1, \dots, \rho_k) \in \Gamma^+_{k,d}$, then $\alpha = \beta$ if and only if $\alpha^i = \beta^i$ for any $1 \leq i \leq k$. Thus

$$\langle X^\alpha, X^\beta \rangle = \delta_{\alpha,\beta} = \prod_{i=1}^k \delta_{\alpha^i,\beta^i} = \prod_{i=1}^k \langle X_i^{\alpha^i}, X_i^{\beta^i} \rangle.$$

Therefore

$$\begin{aligned} \langle X^{\alpha,*}, X^{\beta,*} \rangle &= \frac{\chi(1)^2}{|G|^2} \sum_{\sigma,\theta \in G} \chi(\sigma)\overline{\chi(\theta)} \langle X^{\alpha\sigma}, X^{\beta\theta} \rangle \\ &= \prod_{i=1}^k \frac{\chi_i(1)^2}{|G_i|^2} \sum_{\sigma_i,\theta_i \in G_i} \chi_i(\sigma_i)\overline{\chi_i(\theta_i)} \langle X_i^{\alpha^i\sigma_i}, X^{\beta^i\theta_i} \rangle \\ &= \prod_{i=1}^k \langle X_i^{\alpha^i,*}, X_i^{\beta^i,*} \rangle. \qquad (2.1) \end{aligned}$$

In the following theorem we give a necessary and sufficient condition for existence of an o-basis of $H_d(G, \chi)$.

Theorem 2.5. *Symmetry class $H_d(G, \chi)$ has an o-basis if and only if for any $\rho \in D^+_{k,d}$ and for any $1 \leq i \leq k$, $H_{\rho_i}(G_i, \chi_i)$ has an o-basis.*

Proof. Let $\rho = (\rho_1, \ldots, \rho_k) \in D^+_{k,d}$. Suppose $\{X_i^{\alpha^i,*} |\ \alpha^i \in S_{\rho_i} \subseteq \Gamma^+_{\Lambda_i,\rho_i}\}$ is an o-basis of $H_{\rho_i}(G_i, \chi_i)$, $1 \le i \le k$. Let $S_\rho = \{(\alpha^1, \ldots, \alpha^k) | \alpha^i \in S_{\rho_i}\}$. Then, by (2.1), $\{X^{\alpha,*} |\ \alpha \in S_\rho\}$ is an o-basis of $H_{\rho_1}(G_1, \chi_1) \cdots H_{\rho_k}(G_k, \chi_k)$. Now if we set $S = \bigcup_{\rho \in D^+_{k,d}} S_\rho$, then $\{X^{\alpha,*} |\ \alpha \in S\}$ is an o-basis of $H_d(G, \chi)$, by Theorem 2.4.

Conversely, if $H_d(G, \chi)$ has an o-basis, then by Theorem 2.4, for any $(\rho_1, \ldots, \rho_k) \in D^+_{k,d}$,

$$\langle X^{\alpha,*} |\ \alpha \in \prod_{i=1}^{k} \Gamma^+_{\Lambda_i,\rho_i} \rangle = H_{\rho_1}(G_1, \chi_1) \cdots H_{\rho_k}(G_k, \chi_k),$$

has an o-basis, say $\{X^{\alpha,*} |\ \alpha \in S_\rho\}$, where $S_\rho \subseteq \prod_{i=1}^{k} \Gamma^+_{\Lambda_i,\rho_i}$. Then there are subsets $S_{i,\rho_i} \subseteq \Gamma^+_{\Lambda_i,\rho_i}$, $1 \le i \le k$, such that $S_\rho = \prod_{i=1}^{k} S_{i,\rho_i}$. We show that $E_{i,\rho_i} = \{X_i^{\alpha^i,*} \mid \alpha^i \in S_{i,\rho_i}\}$ is an o-basis of $H_{\rho_i}(G_i, \chi_i)$ for any $1 \le i \le k$.

Let $q_i \in H_{\rho_i}[X_i]$, $1 \le i \le k$. Then $q = q_1 \cdots q_k \in H_d[x_1, \ldots, x_m]$ and we have

$$q^* = T(G_1, \chi_1)(q_1) \cdots T(G_k, \chi_k)(q_k).$$

Since $q^* \in \langle X^{\alpha,*} |\ \alpha \in \prod_{i=1}^{k} \Gamma^+_{\Lambda_i,\rho_i} \rangle$, we can suppose $q^* = \sum_{\alpha \in S_\rho} c_\alpha X^{\alpha,*}$, where c_α is complex number. Let $c_{\alpha^1} = \cdots = c_{\alpha^k} = c_\alpha^{1/k}$. Then

$$\begin{aligned} q^* &= \sum_{\alpha \in S_\rho} c_\alpha X^{\alpha,*} \\ &= \sum_{\alpha^1 \in S_{1,\rho_1}} c_{\alpha^1} X_1^{\alpha^1,*} \cdots \sum_{\alpha^k \in S_{k,\rho_k}} c_{\alpha^k} X_k^{\alpha^k,*}. \end{aligned}$$

Hence there are complex numbers $\lambda_1, \ldots, \lambda_k$ such that $T(G_i, \chi_i)(q_i) = \lambda_i \sum_{\alpha^i \in S_{i,\rho_i}} c_{\alpha^i} X_i^{\alpha^i,*}$, $1 \le i \le k$. Since

$$\begin{aligned} |S_\rho| &= \dim H_{\rho_1}(G_1, \chi_1) \cdots H_{\rho_k}(G_k, \chi_k) \\ &= \prod_{i=1}^{k} \dim H_{\rho_i}(G_i, \chi_i) \\ &\le |S_{1,\rho_1}| \times \cdots \times |S_{k,\rho_k}| \\ &= |S_\rho|, \end{aligned}$$

so, for any i, $1 \le i \le k$, the set E_{i,ρ_i} is a basis of $H_{\rho_i}(G_i, \chi_i)$. It remains to show the orthogonality. Let $\alpha^i \in S_{i,\rho_i}$. Then there is $\alpha = (\alpha^1, \ldots, \alpha^k) \in \prod_{i=1}^{k} \Gamma^+_{\Lambda_i,\rho_i}$ such that $\alpha_{|\Lambda_i} = \alpha^i$. Suppose $\beta^i \in S_{i,\rho_i}$ such that $\alpha^i \ne \beta^i$. Let $\beta = (\alpha^1, \ldots, \beta^i, \ldots, \alpha^k)$. Then $\alpha,\ \beta \in S_\rho$ and $\alpha \ne \beta$. Using (2.1), we have

$$\begin{aligned} 0 &= \langle X^{\alpha,*}, X^{\beta,*} \rangle \\ &= \big(\prod_{j \ne i} \|X_j^{\alpha^j,*}\|^2\big) \langle X_i^{\alpha^i,*}, X_i^{\beta^i,*} \rangle. \end{aligned}$$

Since $\alpha^i \in S_{i,\rho_i}$, $1 \le i \le k$, we have $\|X_i^{\alpha^i,*}\| \ne 0$. Then $\langle X_i^{\alpha^i,*}, X_i^{\beta^i,*} \rangle = 0$. This completes the proof. $\square$

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