www.theoryofgroups.ir

International Journal of Group Theory
ISSN (print): 2251-7650, ISSN (on-line): 2251-7669
Vol. 4 No. 4 (2015), pp. 43-48.

www.ui.ac.ir

MODULES OVER GROUP RINGS OF GROUPS WITH RESTRICTIONS ON THE SYSTEM OF ALL PROPER SUBGROUPS

OLGA YU. DASHKOVA*

Communicated by Victor Bovdi

Abstract. We consider the class $\mathfrak{M}$ of $\mathbf{R}$–modules where $\mathbf{R}$ is an associative ring. Let A be a module over a group ring $\mathbf{R}G$, G be a group and let $\mathfrak{L}(G)$ be the set of all proper subgroups of G. We suppose that if $H \in \mathfrak{L}(G)$ then $A/C_A(H)$ belongs to $\mathfrak{M}$. We investigate an $\mathbf{R}G$–module A such that $G \neq G'$, $C_G(A) = 1$. We study the cases: 1) $\mathfrak{M}$ is the class of all artinian $\mathbf{R}$–modules, $\mathbf{R}$ is either the ring of integers or the ring of p–adic integers; 2) $\mathfrak{M}$ is the class of all finite $\mathbf{R}$–modules, $\mathbf{R}$ is an associative ring; 3) $\mathfrak{M}$ is the class of all finite $\mathbf{R}$–modules, $\mathbf{R}= F$ is a finite field.

1. **Introduction**

Let A be a vector space over a field F. Subgroups of the group $GL(F, A)$ of all automorphisms of A are called linear groups. If A has a finite dimension over F then $GL(F, A)$ can be considered as the group of non-singular $(n \times n)$-matrices, where $n = dim_F A$. Finite dimensional linear groups have played an important role in various fields of mathematics, physics and natural sciences, and have been studied many times. When A is infinite dimensional over F, the situation is totally different. Infinite dimensional linear groups have been investigated little. Study of this class of groups requires additional restrictions.

The definition of the central dimension of an infinite dimensional linear group was introduced in [5]. Let H be a subgroup of $GL(F, A)$. H acts on the quotient space $A/C_A(H)$ in a natural way. The authors define $centdim_F H$ to be $dim_F(A/C_A(H))$. The subgroup H is said to have the finite central dimension if $centdim_F H$ is finite and H has infinite central dimension otherwise.

MSC(2010): Primary: 20F19; Secondary: 20H25.
Keywords: group ring, linear group, module.
Received: 3 May 2014, Accepted: 26 May 2014.

Let $G \leq GL(F, A)$. The system $\mathbf{L_{id}(G)}$ of all subgroups of G of infinite central dimension was considered in [5]. In order to investigate infinite dimensional linear groups that are close to finite dimensional, it is natural to consider the case where the system $\mathbf{L_{id}(G)}$ is "very small". Locally soluble infinite dimensional linear groups in which $\mathbf{L_{id}(G)}$ satisfies the minimal condition as an ordered set have been studied in [5]. Soluble infinite dimensional linear groups in which $\mathbf{L_{id}(G)}$ satisfies the maximal condition as an ordered set have been investigated in [10].

If $G \leq GL(F, A)$ then A can be considered as an FG-module. The natural generalization of this case is the consideration of an $\mathbf{R}$G-module A, where $\mathbf{R}$ is a ring whose structure is near to a field. At this point the generalization of the notion of the central dimension of a subgroup of a linear group is the notion of the cocentralizer of a subgroup. This notion was introduced in [14].

Let A be an $\mathbf{R}G$-module where $\mathbf{R}$ is a ring, G is a group. If $H \leq G$ then the quotient module $A/C_A(H)$ considered as an $\mathbf{R}$-module is called the cocentralizer of the subgroup H in the module A.

Investigation of modules over group rings is an important direction of algebra. Modules over group rings of finite groups have been studied by many authors. Recently this class of modules was investigated in [7]. Study of modules over group rings of infinite groups requires some additional restrictions as in the case of infinite dimensional linear groups. Modules over group rings of infinite groups have been considered recently in [13]. Artinian and noetherian modules over group rings are a broad class of modules over group rings. We remind that a module is called artinian if the partially ordered set of all its submodules satisfies the minimal condition. A module is called noetherian if the partially ordered set of all its submodules satisfies the maximal condition.

Another question arises here: investigation of modules over group rings which are not artinian or noetherian but which are similar to these modules in some sense. Let $\mathfrak{M}$ be the class of $\mathbf{R}$–modules where $\mathbf{R}$ is an associative ring and let A be a module over a group ring $\mathbf{R}G$ where G is a group. Let $\mathfrak{L}(G)$ be the set of subgroups of G such that if $H \in \mathfrak{L}(G)$ then $A/C_A(H)$ belongs to $\mathfrak{M}$. B.A.F. Wehrfritz has considered groups G of automorphisms of a module A over a ring $\mathbf{R}$ if $\mathfrak{M}$ is one of the classes of noetherian, artinian or finite $\mathbf{R}$–modules and $\mathfrak{L}(G)$ contains all finitely generated subgroups of G [16]–[20].

In [5] the authors has considered an $\mathbf{R}G$–module A such that $\mathbf{R}= F$ is a field of prime characteristic, $C_G(A) = 1$, G is an almost locally soluble group, all proper subgroups of G belong to $\mathfrak{L}(G)$ but G does not belong to $\mathfrak{L}(G)$. In [3] the authors has considered the case where A is an $\mathbf{R}G$–module, $\mathbf{R}$ is the ring $\mathbb{Z}_{p^\infty}$ of p–adic integers, $C_G(A) = 1$, G is an infinite soluble group and $\mathfrak{M}$ is the class of all artinian $\mathbf{R}$–modules. As well the case has been considered where A is an $\mathbf{R}G$–module, $\mathbf{R}$ is the ring $\mathbb{Z}$ of integers, $C_G(A) = 1$, G is an infinite soluble group and $\mathfrak{M}$ is the class of all artinian $\mathbf{R}$–modules [1]. In [12] the authors have investigated the case where A is an $\mathbf{R}G$–module, $\mathbf{R}$ is the ring $\mathbb{Z}$ of integers, $C_G(A) = 1$, G is a locally generalized radical group and $\mathfrak{M}$ is the class of all artinian-by-(finite rank) $\mathbf{R}$–modules. In all these cases G is isomorphic to a quasi–cyclic q–group for some prime q.

We study $\mathbf{R}G$–module A such that G is an infinite group, $G \neq G'$, $C_G(A) = 1$, and $\mathfrak{M}$ is one of the classes of artinian $\mathbf{R}$–modules or finite $\mathbf{R}$–modules.

The main results of this paper are Theorems 2.4, 2.5, 3.1. It should be noted that Theorem 3.1 was announced in [2]. Theorems 2.4, 2.5 were announced in [4].

2. **Modules over group rings**

Later we consider $\mathbf{R}G$–module A such that $C_G(A) = 1$, $\mathbf{R}$ is an associative ring.

Lemma 2.1. *Let A be an $\mathbf{R}G$–module, K, L be subgroups of G. If $A/C_A(K)$ and $A/C_A(L)$ are artinian $\mathbf{R}$–modules then $A/C_A(\langle K, L\rangle)$ is an artinian $\mathbf{R}$–module also.*

Proof. Since $A/C_A(K)$ and $A/C_A(L)$ are artinian $\mathbf{R}$–modules then $A/(C_A(K) \cap C_A(L))$ is artinian also. Therefore $A/C_A(\langle K, L\rangle)$ is an artinian $\mathbf{R}$–module. □

Lemma 2.2. *Let A be an $\mathbf{R}G$–module where G is an infinite group, $G \neq G'$. Suppose that $A/C_A(G)$ is not an artinian $\mathbf{R}$–module and $A/C_A(H)$ is an artinian $\mathbf{R}$–module for every proper subgroup H of G. Then G has not proper subgroups of finite index and G/G' is isomorphic to a quasi–cyclic q–group for some prime q.*

Proof. We prove that G is infinite generated. Otherwise let $\{x_1, x_2, \dots, x_m\}$ be a minimal system of generatings of G. If $m = 1$ then G is an infinite cyclic group. Therefore G is generated by two proper subgroups. By Lemma 2.1 $A/C_A(G)$ is an artinian $\mathbf{R}$–module. Contradiction. If $k > 1$ then G is generated by proper subgroups $\langle x_1, x_2, \dots, x_{m-1}\rangle$ and $\langle x_m\rangle$. We have a contradiction also. It follows that G is an infinite generated group. Now we prove that G has not proper subgroups of finite index. Otherwise if N is a proper subgroup of G of finite index then we can choose a finitely generated subgroup M such that $G = MN$ where M and N are proper subgroups of G. By Lemma 2.1 $A/C_A(G)$ is an artinian $\mathbf{R}$–module. Contradiction.

Let D be the derived subgroup of G. As G has not proper subgroups of finite index then G/D is infinite. By Lemma 2.1 G/D can not be generated by two proper subgroups. Suppose that G/D is a nonperiodic group. Let T/D be the periodic part of G/D. Then G/T can be generated by two proper subgroups. Contradiction with Lemma 2.1. Therefore G/D is periodic. Hence G/D is isomorphic to a quasi–cyclic q–group for some prime q (p.152 [15]). □

Lemmas 2.1 and 2.2 are valid if the artinian condition is replaced by the finiteness condition.

Lemma 2.3. *Let A be an artinian $\mathbf{R}$–module where $\mathbf{R}= \mathbb{Z}_{p^\infty}$ is the ring of p–adic integers. Then the additive group of A is Chernikov and its divisible part is a p–group.*

Proof. Let P be the maximal ideal of $\mathbf{R}$. Then the additive group of $\mathbf{R}/P$ has the order p and the additive group of $\mathbf{R}/P^k$ is a cyclic group of the order p^k. Let $\mathbf{R}/P^k = \langle a_k\rangle$, $k = 1, 2, \dots,$ $\pi_k^{k+1} : \mathbf{R}/P^k \longrightarrow \mathbf{R}/P^{k+1}$ where

$$\pi_k^{k+1}(a_k) = pa_{k+1}$$

and let $\pi_k^m : \mathbf{R}/P^k \longrightarrow \mathbf{R}/P^m$ where $m > k$ and

$$\pi_k^m(a_k) = p^{m-k}a_m.$$

We consider the injective limit of the set of $\mathbf{R}/P^k$, $k = 1, 2, \dots, n, \dots$. From the choice of a_1 it follows that $pa_1 = 0$. Therefore this injective limit is isomorphic to the quasi–cyclic p–group C_{p^∞}. It follows that the additive group of a Prüfer $\mathbf{R}$–module is isomorphic to the quasi–cyclic p–group C_{p^∞} (ch. 5 [11]). By Theorem 7.13 [11] an artinian $\mathbf{R}$–module is decomposed in the direct sum $A = a_1\mathbf{R} \oplus a_2\mathbf{R} \oplus \dots \oplus a_n\mathbf{R} \oplus C_1 \oplus \dots \oplus C_k$ where C_i is a Prüfer P_i–module, $P_i \in Spec(\mathbf{R}), i = 1, \dots, k, Ann_{\mathbf{R}}(a_j) = P_j^{m_j}, P_j \in Spec(\mathbf{R}), j = 1, \dots, n$. Every ideal of $\mathbf{R}$ has a finite index in $\mathbf{R}$ (ch. 6 [15]). Therefore $a_j\mathbf{R}$ is a finite $\mathbf{R}$–module for each $j = 1, \dots, n$. It follows that the additive group of an artinian $\mathbb{Z}_{p^\infty}$–module A is Chernikov and its divisible part is a p–group. □

Theorem 2.4. *Let A be an $\mathbf{R}G$–module where G is an infinite group, $G \neq G'$, $\mathbf{R}$ is either the ring $\mathbb{Z}$ of integers or the ring $\mathbb{Z}_{p^\infty}$ of p–adic integers. If $A/C_A(G)$ is not an artinian $\mathbf{R}$–module and $A/C_A(H)$ is an artinian $\mathbf{R}$–module for every proper subgroup H then G is isomorphic to a quasi–cyclic q–group C_{q^∞} for some prime q.*

Proof. Let D be the derived subgroup of G. By Lemma 2.2 G/D is isomorphic to a quasi–cyclic q–group for some prime q. Let H/D be any finite subgroup of G/D. Since H is a proper subgroup of G then $A/C_A(H)$ is an artinian $\mathbf{R}$–module. If $\mathbf{R}$ is the ring $\mathbb{Z}$ of integers then $A/C_A(H)$ is an abelian group with the minimal condition on subgroups. Therefore $A/C_A(H)$ is a Chernikov group. If $\mathbf{R}$ is the ring $\mathbb{Z}_{p^\infty}$ of p–adic integers then $A/C_A(H)$ is a Chernikov group by Lemma 2.3. It follows that $A/C_A(H)$ is the union of finite characteristic subgroups $A_n/C_A(H)$, $n = 1, 2, \dots,$ and for each $n = 1, 2, \dots,$ we have that $G/C_G(A_n/C_A(H))$ is finite. By Lemma 2.2 G has not proper subgroups of finite index. Then $G = C_G(A_n/C_A(H))$ for each $n = 1, 2, \dots$. It follows that $[G, A_n] \leq C_A(H)$ for each $n = 1, 2, \dots$. Therefore $[G, A] \leq C_A(H)$. From the choice of H it follows that $[G, A] \leq C_A(G)$ and so G acts trivially in every factor of the series $0 \leq C_A(G) \leq A$. By Kaluzhnin Theorem (p. 144 [8]) G is abelian. It follows that G is isomorphic to a quasi–cyclic q–group C_{q^∞} for some prime q. □

Theorem 2.5. *Let A be an $\mathbf{R}G$–module where G is an infinite group, $G \neq G'$, $\mathbf{R}$ is an associative ring. If $A/C_A(G)$ is an infinite $\mathbf{R}$–module and $A/C_A(H)$ is a finite $\mathbf{R}$–module for every proper subgroup H of G then G is isomorphic to a quasi–cyclic q–group C_{q^∞} for some prime q.*

Proof. Let D be the derived subgroup of G. By Lemma 2.2 G/D is isomorphic to a quasi–cyclic q–group for some prime q. Let H/D be any finite subgroup of G/D. Since H is a proper subgroup of G then $A/C_A(H)$ is finite. Therefore $G/C_G(A/C_A(H))$ is finite. Since by Lemma 2.2 G has not proper subgroups of finite index then $G = C_G(A/C_A(H))$. It follows that $[G, A] \leq C_A(H)$. From the choice of H it follows that $[G, A] \leq C_A(G)$. So G acts trivially in every factor of the series $0 \leq C_A(G) \leq A$. By Kaluzhnin Theorem (p. 144 [8]) G is an abelian group. Therefore G is isomorphic to a quasi–cyclic q–group C_{q^∞} for some prime q. □

In [12] the authors have constructed the example of $\mathbb{Z}G$–module with the prescribed properties.

3. Linear groups

Now we shall prove that in contrast to a module over group ring a linear group with all proper subgroups of finite central dimensions can not have infinite central dimension.

Theorem 3.1. *Let $G \leq GL(F, A)$, $G \neq G'$, F be a finite field. If every proper subgroup of G has a finite central dimension and $|G| \neq q^k$, where q is prime then G has a finite central dimension.*

Proof. At first we consider the case of infinite group G. Assume that $centdim_F(G)$ is infinite. Let D be the derived subgroup of G. By Lemma 2.2 G/D is isomorphic to a quasi–cyclic q–group C_{q^∞} for some prime q. It follows that G/D is the union of finite characteristic subgroups. Let H be a proper subgroup of G such that $D \leq H$. The central dimension of H is finite. Since F is a finite field then $A/C_A(H)$ is finite. Therefore $G/C_G(A/C_A(H))$ is finite. By Lemma 2.2 G has not proper subgroups of finite index. It follows that $G = C_G(A/C_A(H))$. Then $[G, A] \leq C_A(H)$. From the choice of H it follows that $[G, A] \leq C_A(G)$ and so G acts trivially in every factor of the series $0 \leq C_A(G) \leq A$. By Kaluzhnin Theorem (p. 144 [8]) G is abelian. Suppose that G is nonperiodic. Let T be the periodic part of G. Then G/T can be generated by two proper subgroups. Contradiction with Lemma 1.1. Therefore G is periodic. Hence G/D is isomorphic to a quasi–cyclic q–group for some prime q (p.152 [15]). Since G is an infinite finitary abelian Chernikov q–subgroup of $GL(F, A)$ then by Lemma 5.1 [5] $q \neq p$ where p is the characteristic of F. On the other hand G acts trivially in every factor of the series $0 \leq C_A(G) \leq A$. Every factor of this series is an elementary abelian p–group. By Proposition 1.C.3 [9] and the results of §43 [6] we obtain that G is the bounded abelian p–group. Contradiction. Therefore $centdim_F(G)$ is finite.

Now we assume that G is finite. Since $|G| \neq q^k$ for some prime q then we can choose the system of generatings $\{g_1, g_2 \ldots, g_m\}$ of G such that $m > 1$ and for any $l = 1, \ldots, m$, the set $\{g_1, g_2 \ldots, g_m\} \setminus \{g_l\}$ is not a system of generatings of G. Therefore $\langle g_l \rangle$ is a proper subgroup of G for any $l = 1, \ldots, m$. Since $centdim_F(\langle g_l \rangle)$ is finite for any $l = 1, \ldots, m$, then $A/\cap_{i=1,\ldots,m} C_A(g_i)$ is a finite dimensional quotient space. As $C_A(G) = \cap_{i=1,\ldots,m} C_A(g_i)$ then $centdim_F(G)$ is finite. □

Corollary 3.2. *Let $G \leq GL(F, A)$, $G \neq G'$, F be a finite field of prime characteristic p. If every proper subgroup of G has a finite central dimension and $|G| \neq q^k$, where q is prime then G has the normal elementary abelian p–subgroup H such that G/H is isomorphic to some subgroup of $GL_n(F)$.*

Proof. By Theorem 3.1 $centdim_F(G)$ is finite. Then G contains the normal elementary abelian p–subgroup H such that G/H is isomorphic to some subgroup of $GL_n(F)$ [5]. □

Example.

Let $A = B \times C$ be a vector space over a field F, $charF = p$, $dim_F B = n$, $C = Dr_{l \in \mathbb{N}} C_l$, $C_l = \langle c_l \rangle, l \in \mathbb{N}$, $G = W \times H$, $G \leq GL(F, A)$, $W \leq GL(F, B)$, $H = Dr_{k \in \mathbb{N}} H_k$, $H_k = \langle h_k \rangle, k \in \mathbb{N}$, $|h_k| = p$.

Let $c_1 h_i = c_1 + c_{i+1}$, $c_l h_i = c_l, l = 2, 3, \ldots$, $i \in \mathbb{N}$, $bh = b$ for each $b \in B, h \in H$, $cw = c$ for each $c \in C, w \in W$.

Since $C \leq C_A(W)$ then $centdim_F W \leq n$. As $C_A(H) = B \times (Dr_{j\geq 2}\langle c_j \rangle)$ then $centdim_F H = 1$. Then by Lemma 2.1 $centdim_F G$ is finite.

References

[1] O.Yu. Dashkova, Locally soluble AFA–groups, *Ukr. Math. J.*, **65** no. 4 (2013) 459–469 (in Russian).

[2] O.Yu. Dashkova, On linear groups with restrictions on the system of all proper subgroups, *Reports of the Academy of Sciences of Ukraine*, no.12 (2013) 7–10.

[3] O.Yu. Dashkova, On modules over group rings of locally soluble groups for a ring of p–adic integers, *Algebra Discrete Math.*, no. 1 (2009) 32–43.

[4] O.Yu. Dashkova, On modules over group rings of groups with restrictions on the system of all proper subgroups, *arXiv:1305.0744v2* (2013), 6 p.

[5] M.R. Dixon, M.J. Evans, L.A. Kurdachenko, Linear groups with the minimal condition on subgroups of infinite central dimension, *J. Algebra*, **277** no. 1 (2004) 172–186.

[6] L. Fuchs, Infinite Abelian Groups, **1** Academic Press, New York, 1973.

[7] P. M. Gudivok, V. P. Rudko, V. A. Bovdi, *Crystallographic groups*, Uzhgorod: Uzhgorodskii Natsionalnii Universitet, 2006, 174 pp. (in Ukrainian).

[8] M.I. Kargapolov, Yu.I. Merzlyakov, Bases of the Theory of Groups, Moscow, Nauka, 1975 (in Russian).

[9] O.H. Kegel, B.A.F. Wehrfritz, Locally Finite Groups, North–Holland Mathematical Library, **3** North-Holland, Amsterdam, London, 1973.

[10] L.A. Kurdachenko, I.Ya. Subbotin, Linear groups with the maximal condition on subgroups of infinite central dimension, *Publ. Mat.*, **50** no. 1 (2006) 103–131.

[11] L.A. Kurdachenko, I.Ya. Subbotin, N.N. Semko, Insight into Modules over Dedekind Domains, National Academy of Sciences of Ukraine, Institute of Mathematics, Kiev, 2008.

[12] L.A. Kurdachenko, I.Ya. Subbotin, V.A. Chepurdya, On the structure of some modules over generalized soluble groups, *arXiv:1302.2115* (2013), 9 p.

[13] L.A. Kurdachenko, J. Otal, I.Ya. Subbotin, *Artinian Modules over Group Rings*, Birkhäuser, Boston, Berlin, 2007.

[14] L.A. Kurdachenko, On groups with minimax classes of conjugate elements, *Infinite groups and adjoining algebraic structures*, Academy of Sciences of Ukraine, Institute of Mathematics, Kiev (1993) 160–177 (in Russian).

[15] A.G. Kurosh, The Theory of Groups, Moscow, Nauka, 1967 (in Russian).

[16] B.A.F. Wehrfritz, Artinian–finitary groups are locally normal-finitary, *J. Algebra*, **287** no. 2 (2005) 417–431.

[17] B.A.F. Wehrfritz, Artinian–finitary groups over commutative rings and non-commutative rings, *J. Lond. Math. Soc. (2)*, **70** no. 2 (2004) 325–340.

[18] B.A.F. Wehrfritz, Artinian–finitary groups over commutative rings, *Illinois J. Math.*, **47** no. 1–2 (2003) 551–565.

[19] B.A.F. Wehrfritz, Finite–finitary groups of automorphisms, *J. Algebra Appl.*, **1** no. 4 (2002) 375–389.

[20] B.A.F. Wehrfritz, On generalized finitary groups, *J. Algebra*, **247** no. 2 (2002) 707–727.

Olga Yu. Dashkova

Department of Mathematics, The Branch of Moscow State University in Sevastopol, Sevastopol, Ukraine

`Email: odashkova@yandex.ru`