International Journal of Group Theory
ISSN (print): 2251-7650, ISSN (on-line): 2251-7669
Vol. 5 No. 1 (2016), pp. 11-16.

www.theoryofgroups.ir
www.ui.ac.ir

FINITE SIMPLE GROUPS WHICH ARE THE PRODUCTS OF SYMMETRIC OR ALTERNATING GROUPS WITH $L_3(4)$

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Communicated by Alireza Abdollahi

Abstract. In this paper, we determine the simple groups $G = AB$, where B is isomorphic to $L_3(4)$ and A isomorphic to an alternating or a symmetric group on $n \geq 5$, letters.

1. Introduction

In this paper, all groups considered are finite. By definition, a group G is called **factorizable** if $G = AB$ for some proper subgroups A and B of G, otherwise it is called **non-factorizable**. The subgroups A and B are called the **factors of the factorization**, and we call the factorization **exact** if $A \cap B = 1$. If both A and B are maximal, the factorization is called a **maximal factorization**. Group factorizations play an important role in permutation group theory, for if G is a transitive permutation group on Ω, then a subgroup A of G is transitive if and only if $G = AG_\alpha$, where G_α is the stabilizer in G of $\alpha \in \Omega$. We should remark that there are groups which are not factorizable. Tarski group (an infinite group all of whose proper subgroups are finite) in infinite groups and $PSL(2, 61)$ in finite groups are two examples that are non-factorizable.

In this paper we will mention some research work done concerning the factorization of a group G with specified factors. All exact factorizations of almost simple groups for which the factors are maximal subgroups have been determined in [7]. In [9] all finite groups $G = AB$ are found where A is a non-abelian simple group and B is isomorphic to $\mathbb{A}_5$. Similarly all finite groups $G = AB$, where $A \cong \mathbb{S}_6$ and B is a simple group, were found in [3]. In [8] all maximal factorizations of finite simple groups and their automorphism groups are found. Recently all factorization of $G = AB$ where the

MSC(2010): Primary: 20D40; Secondary: 20D08.

Keywords: Factorization, product of groups, symmetric groups.

Received: 29 July 2013, Accepted: 27 May 2014.

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socle of G is a sporadic simple group have been determined [5]. Motivated by [2] and the results mentioned above, we determine the simple groups G with a factorization $G = AB$, where B is isomorphic to $L_3(4)$ and A is isomorphic to an alternating or a symmetric group on $n \geq 5$, letters. Notation for the names of the simple groups are taken from [1]. First we present some results which are useful when dealing with factorizable groups.

Lemma 1.1. [2, Lemma 1] *Let A and B be subgroups of a group G. Then the following statements are equivalent:*
(a) $G = AB$.
(b) A acts transitively on the coset space $\Omega(G : B)$ of right cosets of B in G.
(c) B acts transitively on the coset space $\Omega(G : A)$ of right cosets of A in G.
(d) $(\pi_A, \pi_B) = 1$, where π_A and π_B are permutation characters of G on $\Omega(G : A)$ and $\Omega(G : B)$, respectively.

Lemma 1.2. [2, Lemma 2] *Let G be a permutation group on a set Ω of size n. Suppose the action of G on Ω is k-homogeneous for $1 \leq k \leq n$. If a subgroup H of G acts on Ω k-homogeneously, then $G = G_{(\Delta)}H$, where Δ is a k-subset of Ω and $G_{(\Delta)}$ denotes its global stabilizer.*

By using Lemma 1.2, we have $\mathbb{A}_{21} = L_3(4)\mathbb{A}_{19}$ (this follows easily from the fact that $L_3(4)$ has 2-transitive action on points of $P_2(4)$).

The following lemma is an immediate consequence of [2, Lemmas 6, 7 and 8].

Lemma 1.3. *For any natural number $m \geq 2$ and a prime power $q \geq 3$, we have the following inequalities:*
(i) $|L_m(q)| > \frac{1}{6}(m+2)!q^{\frac{m^2-3}{2}}$.
(ii) $|S_{2m}(q)| \geq \frac{1}{12}(2m+2)!q^{\frac{m(2m+1)}{2}}$.
(iii) $|U_m(q)| > \frac{1}{6}(m+2)!q^{\frac{m^2-5}{2}}$.

Now we describe some useful facts about $L_3(4)$. Using GAP we can find the indices of the proper subgroups of $L_3(4)$ which are: $21, 56, 105, 120, 126, 210, 252, 280, 315, 336, 420, 560, 630, 840, 960, 1120, 1260, 1680, 2016, 2240, 2520, 2880, 3360, 4032, 5040, 6720, 10080$. Now we will have following lemma.

Lemma 1.4. *If $\mathbb{A}_m = AB$ is a factorization of $\mathbb{A}_m$ with A a non-abelian simple group and $B \cong L_3(4)$, then one of the following occurs:*
(i) $\mathbb{A}_m = \mathbb{A}_{m-1}L_3(4)$ where $m = 21, 56, 105, 120, 126, 210, 252, 280, 315, 336, 420, 560, 630, 840, 960, 1120, 1260, 1680, 2016, 2240, 2520, 2880, 3360, 4032, 5040, 6720, 10080$.
(ii) $\mathbb{A}_{21} = L_3(4)\mathbb{A}_{19}$.

Proof. Let $\mathbb{A}_m = AB$, where A is a simple group and $B \cong L_3(4)$. By [8, Theorem D], we have the following two cases:

Case (i): (1) $\mathbb{A}_{m-k} \trianglelefteq A \trianglelefteq \mathbb{S}_{m-k} \times \mathbb{S}_k$ for some k with $1 \leq k \leq 5$, and B is k-homogeneous on m letters.

If $k = 3, 4, 5$, then by [4, Theorems 9.4A and 9.4B] we obtain that B is 2-transitive and so it is primitive. Therefore $m = 21, 56, 120$ or 280 that results from the fact that these values are the indices of the maximal subgroups of $B \cong L_3(4)$. It is easy to see that if $m = 21, 56, 120$ or 280 and $k = 3, 4, 5$ the decomposition is impossible. The simplicity of A and $k = 2$ conclude (ii). If $k = 1$, then we have a transitive action on m letters, so (i) occurs.

(2) $\mathbb{A}_{m-k} \trianglelefteq B \trianglelefteq \mathbb{S}_{m-k} \times \mathbb{S}_k$ for some k with $1 \leq k \leq 5$ and A is k-homogeneous on m letters.

Using simplicity of B we obtain $\mathbb{A}_{m-k} = 1$, the trivial group. Thus $m - k = 1$ and $2 \leq m \leq 6$. It is easy to see that $m \geq 7$. This is a contradiction.

Case (ii): $m = 6, 8, 10$. This case is impossible because we are assuming $\mathbb{A}_m$ contains a copy of $L_3(4)$. □

The proof of the next lemma is similar to the proof of Lemma 1.4, because again it is concerned with a factorization of $\mathbb{A}_m$ and [8, Theorem D] can be applied.

Lemma 1.5. *Let $\mathbb{A}_m = AB$, where A is isomorphic to a symmetric group $\mathbb{S}_n$ and $B \cong L_3(4)$. Then $m = 21$ and $n = 19$ and we have the factorization $\mathbb{A}_{21} = \mathbb{S}_{19} L_3(4)$.*

2. Main results

This section is devoted to our main result. Our results are concerned with the factorization of finite non-abelian simple groups. Every finite non-abelian simple group G is isomorphic to one of the following: alternating group $\mathbb{A}_m$, $m \geq 5$, a sporadic group or a group of Lie type. Therefore to see if G has an appropriate factorization we have to go thorough all members of the above list. Later we will discuss the case of the alternating subgroup, and now we will continue with the rest of the simple groups.

Lemma 2.1. *Let G be a sporadic finite simple group. Then it is impossible to write $G = AB$ where $B \cong L_3(4)$ and A isomorphic to an alternating group or a symmetric group on more than 5 letters.*

Proof. It follows from [5]. □

Simple groups of Lie type are divided into large families called classical groups and the exceptional groups of Lie type. Now we study classical groups which are the projective special linear, symplectic, unitary, and orthogonal groups.

Theorem 2.2. *The decomposition $L_m(q) = AB$ where $A \cong \mathbb{A}_n$ or $\mathbb{S}_n$, $n \geq 5$, and $B \cong L_3(4)$ is impossible unless one has $L_3(4) = B \supseteq A$ with $A \cong \mathbb{A}_5$ or $\mathbb{A}_6$.*

Proof. By [6], the minimum degree of a projective modular representation of $A \cong \mathbb{A}_n$ or $\mathbb{S}_n$ is $n - 2$ and therefore $m \geq n - 2$ which implies $n \leq m + 2$. We break the proof into three cases:

(i) Let $q = 2$. We know that the highest power of 2 that divides the order of $\mathbb{S}_n$ i.e. $|\mathbb{S}_n|_2$ satisfies $|\mathbb{S}_n|_2 \leq 2^n$. Thus the highest power of 2 that divides the order of AB is at most $2^{(n+6)}$. Therefore we have $\frac{m(m-1)}{2} < (n+6)$ that implies $\frac{(m-4)(m+3)}{2} < n$. If $m > 5$, we obtain $n > m+2$, which is a contradiction. For $m \leq 5$ we have $|L_5(2)| = 2^{10} \cdot 3^2 \cdot 5 \cdot 7 \cdot 31$. Since $31 \bigm| |L_5(2)|$ and 31 is a prime number, then $31 \bigm| |A|$ or $31 \bigm| |B|$. But $31 \nmid |B|$ so $31 \bigm| |\mathbb{A}_n|$ or $|\mathbb{S}_n|$, which implies $n \geq 31$. This is a contradiction.

(ii) $q \geq 3$ and $m \geq 5$. From $|L_m(q)| \leq |A||B| \leq 2^6 \cdot 3^2 \cdot 5 \cdot 7 \cdot n!$, $n \leq m+2$, and Lemma 1.3, we obtain

$$q^{\frac{m^2-3}{2}} < 2^7 \cdot 3^3 \cdot 5 \cdot 7. \tag{2.1}$$

Since we assume $q \geq 3$ and $m \geq 5$, we obtain $3^{11} < 2^7 \cdot 3^3 \cdot 5 \cdot 7$, which is a contradiction.

(iii) $q \geq 3$, $m < 5$. In this case from inequality $n \leq m+2$ and $n \geq 5$, we deduce $n = 5$ or $n = 6$. Note that $L_3(4)$ has subgroups that are isomorphism to $\mathbb{A}_5$ or $\mathbb{A}_6$. Therefore if $A \cong \mathbb{A}_5$ or $A \cong \mathbb{A}_6$, we have $A \subseteq B \cong L_3(4)$. □

Theorem 2.3. *The decomposition $S_{2m}(q) = AB$ where $A \cong \mathbb{A}_n$ or $\mathbb{S}_n$, $n \geq 5$ and $B \cong L_3(4)$ is impossible.*

Proof. By [6] we have $2m \geq n-2$. Hence by inequality $|S_{2m}(q)| \leq |A||B| \leq 2^6 \cdot 3^2 \cdot 5 \cdot 7 \cdot n!$ and Lemma 1.3, we deduce

$$q^{\frac{m(2m+1)}{2}} \leq 2^8 \cdot 3^3 \cdot 5 \cdot 7. \tag{2.2}$$

Now We break the proof into three cases:

(i) $m \geq 4$. In this case by inequality (2.2), and $q \geq 2$, we obtain $2^{18} < 2^8 \cdot 3^3 \cdot 5 \cdot 7$, which is contradiction.

(ii) $m = 3$. By inequality (2.2), we have that $q \leq 3$. If $q = 2$, then by structure of the maximal subgroups of $S_6(2)$ ([1, p.80]), it is easy to show that $S_6(2)$ does not contain $L_3(4)$. If $q = 3$, then $|S_6(3)| = 2^9 \cdot 3^9 \cdot 5 \cdot 7 \cdot 13$. Hence $13 \bigm| |A|$ or $13 \bigm| |B|$. Since $13 \nmid |B|$, we have $13 \bigm| |A|$. Therefore $n \geq 13$, a contradiction.

(iii) $m = 2$. By inequality (2.2), we obtain $q \leq 11$. For $q = 2, 3, 4, 5, 9$, we have $7 \nmid |S_4(q)|$, but $7 \bigm| |L_3(4)$. This is a contradiction.

If $q = 7$, we have $7^4 \bigm| |S_4(4)|$, which implies $n \geq 7$. This is a contradiction. For $q = 8$ and $q = 11$, we similarly obtain a contradiction. □

Theorem 2.4. *Decomposition of the unitary groups or the orthogonal groups as the product of $\mathbb{A}_n$ or $\mathbb{S}_n$, $n \geq 5$, with the group $L_3(4)$ is impossible.*

Proof. Using Lemma 1.3, the proof is similar to Theorem 2.2 and Theorem 2.3. □

We continue discussion with exceptional groups of Lie type. An exceptional group of Lie type is one of the groups:

Table 1.

1	$C_2(q)$	$q \geq 3$
2	$F_4(q); E_6(q);^2 E_6(q);^3 D_4(q); E_7(q); E_8(q)$	q *is prime power*
3	$^2B_2(2^{2n+1});^2 G_2(3^{2n+1});^2 F_4(2^{2n+1})$	$n \geq 1$

According to [7] factorization of exceptional groups of Lie type are given in Theorem B from which it follows that none of this groups have desired factorization.

Now we can prove the following main result of this paper:

Theorem 2.5. *Let G be a finite non-abelian simple group such that $G = AB$, where $A \cong \mathbb{A}_n$ or $\mathbb{S}_n$, $n \geq 5$, and $B \cong L_3(4)$. Then the following possibilities occur:*
(i) $\mathbb{A}_m \cong \mathbb{A}_{m-1}L_3(4)$ *where* $m = 21, 56, 105, 120, 126, 210, 252, 280, 315, 336, 420, 560, 630, 840, 960$
$1120, 1260, 1680, 2016, 2240, 2520, 2880, 3360, 4032, 5040, 6720, 10080.$
(ii) $\mathbb{A}_{21} = L_3(4)\mathbb{A}_{19}$.
(iii) $\mathbb{A}_{21} = L_3(4)\mathbb{S}_{19}$.

Acknowledgments

The authors would like to thank the referees for some helpful corrections. We would like to express our thanks to Shahrekord university for financial support and the authors partially supported by the Center of Excellence for Mathematics, University of shahrekord, Iran.

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