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CHARACTERIZATION OF PROJECTIVE GENERAL LINEAR GROUPS

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Abstract. Let G be a finite group and $\pi_e(G)$ be the set of element orders of G. Let $k \in \pi_e(G)$ and s_k be the number of elements of order k in G. Set $\mathrm{nse}(G)$:=$\{s_k | k \in \pi_e(G)\}$. In this paper, it is proved if $|G| = |\mathrm{PGL}_2(q)|$, where q is odd prime power and $\mathrm{nse}(G) = \mathrm{nse}(\mathrm{PGL}_2(q))$, then $G \cong \mathrm{PGL}_2(q)$.

1. Introduction

If n is an integer, then we denote by $\pi(n)$ the set of all prime divisors of n. If G is a finite group, then $\pi(|G|)$ is denoted by $\pi(G)$. We denote by $\pi_e(G)$ the set of orders of its elements. It is clear that the set $\pi_e(G)$ is closed and partially ordered by divisibility, and hence it is uniquely determined by $\mu(G)$, the subset of its maximal elements. The prime graph $\Gamma(G)$ of a group G is defined as a graph with vertex set $\pi(G)$ in which two distinct primes p, $q \in \pi(G)$ are adjacent if G contains an element of order pq . Let $t(G)$ be the number of connected components of $\Gamma(G)$ and $\pi_1, \ldots, \pi_{t(G)}$ be the connected components of $\Gamma(G)$. If $2 \in \pi(G)$, then we always suppose that $2 \in \pi_1$. Then π_1 is called the even component of $\Gamma(G)$ and $\pi_2, \ldots, \pi_{t(G)}$ are called the odd components of $\Gamma(G)$.

Let p be a prime. A group G is called a C_{pp} if $p \in \pi(G)$ and p is an isolated vertex of the prime graph of G. In the other words, the centralizers of its elements of order p in G are p-groups.

Given a finite group G, we can express $|G|$ as a product of integers $m_1, m_2, \ldots, m_{t(G)}$, where $\pi(m_i) = \pi_i$ for each i. These numbers m_i are called the order components of G. In particular, if m_i is odd, then we call it an odd order component of G (see [14]). According to the classification theorem of finite simple groups and [7, 17, 19], we can list the order components of finite simple groups with disconnected prime graphs as in Tables 1-3 in [8].

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Set $s_i = s_i(G) := |\{g \in G \mid \text{the order of } g \text{ is } i\}|$ and $\mathrm{nse}(G) := \{s_i \mid i \in \pi_e(G)\}$. In fact, s_i is the number of elements of order i in G and $\mathrm{nse}(G)$ is the set of sizes of elements with the same order in G.

Throughout this paper we denote by ϕ the Euler's totient function. If G is a finite group, then we denote by P_q a Sylow $q-$ subgroup of G. All other notations are standard and we refer to [16], for example.

For the set $\mathrm{nse}(G)$, the most important problem is related to Thompson's problem. In 1987, J. G. Thompson posed a very interesting problem related to algebraic number fields as follows. For each finite group G and each integer $d \geq 1$, let $G(d) = \{x \in G \mid x^d = 1\}$. We say that the groups G_1 and G_2 are of the same type if $|G_1(d)| = |G_2(d)|$, for all $d = 1, 2, 3, \ldots$.

Thompson's problem. Suppose G_1 and G_2 are of the same order type. If G_1 is solvable, is G_2 necessarily solvable? (see [20, Problem 12.37])

It is easy to see that if G and H are of the same order type, then $\mathrm{nse}(G) = \mathrm{nse}(H)$ and $|G| = |H|$. W. J Shi in [21] made the above problem public in 1989. Unfortunately, no one can solve it or give a counterexample until now, and it remains open. The influence of $\mathrm{nse}(G)$ on the structure of finite groups was studied by some authors (see [3, 4, 5, 9]).

In [2], it is proved that $L_2(p)$, where p is prime, is characterizable by $\mathrm{nse}(G)$ and its order. In [1], it is proved that $\mathrm{PGL}_2(p)$, where $p > 3$ is prime number, is characterizable by $\mathrm{nse}(G)$ and $p \in \pi(G)$. In this paper it is proved that $\mathrm{PGL}_2(q)$, where $q > 3$ is odd prime power, is characterizable by $\mathrm{nse}(G)$ and its order. In fact the main theorem of our paper is as follows:

Main Theorem. Let G be a group such that $\mathrm{nse}(G) = \mathrm{nse}(\mathrm{PGL}_2(q))$, where $q > 3$ is odd prime power and $|G| = |\mathrm{PGL}_2(q)|$. Then $G \cong \mathrm{PGL}_2(q)$.

We note that there are finite groups G which are not characterizable even by $\mathrm{nse}(G)$ and $|G|$. For example see the Remark in [9].

2. **Preliminary Results**

We first quote some lemmas that are used in deducing the main theorem of this paper.

Lemma 2.1. [12] *Let G be a finite group and n be a positive integer dividing $|G|$. If $M_n(G) = \{g \in G \mid g^n = 1\}$, then $n \mid |M_n(G)|$.*

Lemma 2.2. [1] *Let G be a group such that nse$(G) =$ nse$(\mathrm{PGL}_2(p))$, where $p > 3$ is prime divisor of $|G|$ but p^2 does not divide $|G|$. Then $G \cong \mathrm{PGL}_2(p)$.*

Lemma 2.3. [18, Theorem 3] *Let G be a finite group. Then the number of elements whose orders are multiples of n is either zero, or a multiple of the greatest divisor of G that is prime to n.*

Lemma 2.4. [13] *Let G be a Frobenius group of even order with H and K its Frobenius kernel and Frobenius complement, respectively. Then $t(G) = 2$ and $T(G) = \{\pi(K), \pi(H)\}$.*

Lemma 2.5. *(*[10, Theorem 10:3:1]*,* [11, Theorem 18:6]*,* [15]*) Let G be a Frobenius group with kernel F and complement C. Then the following assertions are true.*
(a) F is nilpotent.
(b) $|F| \equiv 1 \pmod{|C|}$.
(c) Every subgroup of C of order $p\cdot q$, with p, q (not necessarily distinct) primes, is cyclic. In particular, every Sylow subgroup of C of odd order is cyclic and a Sylow 2-subgroup of C is either cyclic or a generalized quaternion group. If C is non-solvable, then C has a subgroup of index at most 2 isomorphic to $\mathrm{SL}_2(5) \times M$, where M has cyclic Sylow p-subgroups and $(|M|, 30) = 1$ in particular, $15,\ 20 \notin \pi_e(G)$. If C is solvable and $O(C) = 1$, then either C is a 2-group or C has a subgroup of index at most 2 isomorphic to $\mathrm{SL}_2(3)$.

A group G is a 2-Frobenius group if there exists a normal series $1 \trianglelefteq H \triangleleft K \trianglelefteq G$ such that K and G/H are Frobenius groups with kernels H and K/H, respectively.

Lemma 2.6. [13] *Let G be a 2-Frobenius group of even order which has a normal series $1 \trianglelefteq H \triangleleft K \trianglelefteq G$ such that K and G/H are Frobenius groups with kernels H and K/H, respectively. Then the following assertions hold.*
(a) $t(G) = 2$ and $T(G) = \{\pi_1(G) = \pi(H) \cup \pi(G/K),\ \pi_2(G) = \pi(K/H)\}$.
(b) G/K and K/H are cyclic, $|G/K|$ divides $|\mathrm{Aut}(K/H)|$, and $(|G/K|, |K/H|) = 1$.
(c) H is nilpotent and G is solvable.

Lemma 2.7. [17] *Let G be a finite group with $t(G) \geq 2$, then one of the following assertions is true:*
(a) G is a Frobenius or 2-Frobenius group;
(b) G has a normal series $1 \trianglelefteq H \triangleleft K \trianglelefteq G$ such that H and G/K are π_1-groups and K/H is simple, where H is a nilpotent group and $|G/K| \mid |\mathrm{Aut}(K/H)|$. Moreover, any odd order component of G is also an odd order component of K/H.

Lemma 2.8. *The set $nse(\mathrm{PGL}_2(q))$ consists of the numbers $1, q^2 - 1$ and q^2 together with all of the numbers of the form $\phi(r)q(q-1)/2$ and all of the numbers $\phi(t)q(q+1)/2$, where $r > 2$ is a divisor of $q + 1$ and $t > 2$ is a divisor of $q - 1$.*

Proof. The group $\mathrm{PGL}_2(q)$, where $q = p^n$, has one conjugacy class of size $q^2 - 1$, which is related to elements of order p. So $s_p(\mathrm{PGL}_2(q)) = (q^2 - 1)$. Also, this group has two conjugacy classes of sizes $q(q-1)/2$ and $q(q+1)/2$, which are related to elements of order 2. So $s_2(\mathrm{PGL}_2(q)) = q^2$.

Suppose that $2 < r \mid (q + 1)$. By [6, p. 464] we have $\mu(\mathrm{PGL}_2(q)) = \{q - 1, p, q + 1\}$. Then $r \in \pi_e(\mathrm{PGL}_2(q))$. To find $s_r(\mathrm{PGL}_2(q))$, let H be a cyclic subgroup of order r of $\mathrm{PGL}_2(q) = T$. We know $|T : C_T(H)|$ is the size of the conjugacy class of H. The group $\mathrm{PGL}_2(q)$ has $(q-1)/2$ conjugacy classes of order $q(q-1)$ and $(q-3)/2$ conjugacy classes of order $q(q+1)$. Since $r > 2$ divides $q+1$, we have $|T : C_T(H)| = q(q-1)$. Now we will show the number of conjugacy classes of such subgroups H is $\phi(r)/2$. Since $r > 2$ divides $q+1$, we have each element of order r lies in a unique, up to conjugation, subgroup R of order $q+1$ of $\mathrm{PGL}_2(q)$=T. Now, $N_T(R) = R \rtimes C_2$, is a dihedral group of order $2(q+1)$. So all elements of order r of $R \rtimes C_2$ lie in a unique subgroup of order r of R. Therefore there are

$\phi(r)$ elements of order r in $N_T(R)$. Now every element in R is conjugate to its inverse, so there are $\phi(r)/2$ classes of elements of order r in $N_T(R)$, hence there are $\phi(r)/2$ classes of elements of order r in $\mathrm{PGL}_2(q)$. Therefore, $s_r(\mathrm{PGL}_2(q)) = \phi(r)q(q-1)/2$.

Also if $t > 2$ divides $q-1$, then by $\mu(\mathrm{PGL}_2(q))$, $t \in \pi_e(\mathrm{PGL}_2(q))$ and $s_t(\mathrm{PGL}_2(q)) = \phi(t)q(q+1)/2$. □

Let s_n be the number of elements of order n. We note that $s_n = k\phi(n)$, where k is the number of cyclic subgroups of order n in G. Also we note that if $n > 2$, then $\phi(n)$ is even. If $n \mid |G|$, then by Lemma 2.1 and the above notation we have

$$\begin{cases} \phi(n) \mid s_n \\ \\ n \mid \sum_{d|n} s_d \end{cases} \qquad (*)$$

In the proof of the main theorem, we apply $(*)$ and the above comments.

3. Proof of the Main Theorem

Let $q = p^n > 3$, where p is an odd prime. Let G be a group such that $|G| = |\mathrm{PGL}_2(q)| = q(q^2-1)$ and $\mathrm{nse}(G) = \mathrm{nse}(\mathrm{PGL}_2(q))$. As q^2 is the only odd number in $\mathrm{nse}(G) = \mathrm{nse}(\mathrm{PGL}_2(q))$, by recalling that $s_2(G)$ is odd one deduces that $s_2(G) = q^2$. We now prove the main theorem of this paper. To continue we need to the following Lemmas.

Lemma 3.1. $s_p(G) = s_p(\mathrm{PGL}_2(q)) = (q^2-1)$.

Proof. By $(*)$ we know that $1 + s_p(G)$ is divisible by p, so $s_p(G) \equiv -1 \pmod p$. By $\mathrm{nes}(G)$, the only number m in $\mathrm{nse}(G)$ that $m \equiv -1 \pmod p$ is q^2-1, so we must have $s_p(G) = q^2-1$. □

Lemma 3.2. *$rp \notin \pi_e(G)$ for every $r \in \pi(G)$.*

Proof. First we show that if $r \in \pi(G)\backslash\{p\}$, then $s_r \neq q^2-1$. Suppose that $s_r = q^2-1$. By $(*)$ we know that $1+s_r(G)$ is divisible by r, so $r \mid q^2$, a contradiction. Thus by $\mathrm{nse}(G)$ if $r \in \pi(G)\backslash\{p\}$, then $q \mid s_r$. Now we show that if $rp \in \pi_e(G)$ where $r \in \pi(G)\backslash\{p\}$, then $q \mid s_{rp}$. Suppose $rp \in \pi_e(G)$, by $(*)$, $rp \mid 1+s_r+s_p+s_{rp}$. We know that $1+s_p$ is divisible by q, on the other hand, $q \mid s_r$. Therefore $q \mid s_{rp}$. Also if $p^i \in \pi_e(G)$ where $i \geq 2$, then since $\phi(p^i) \mid s_{p^i}$, $s_{p^i} \neq q^2-1$ and by $\mathrm{nse}(G)$, $q \mid s_{p^2}$.

By Lemma 2.3, the number of elements whose orders are multiples of p is either zero, or a multiple of the greatest divisor of G that is prime to p. By Lemma 3.1, $s_p(G) = q^2-1$ that is the greatest divisor of G prime to p, so the number of elements whose orders are multiples of p is $(q^2-1)\cdot l$, where $l \in \mathbb{N}$. If $l = 1$, then since $s_p(G) = q^2-1$, p is the only element of G whose order is multiples of p. Thus $rp \notin \pi_e(G)$ for every $r \in \pi(G)$. Let $l \neq 1$, and let there exists $r \in \pi(G)$ such that $rp \in \pi_e(G)$. Then the number of elements whose orders are multiples of p is $(q^2-1)k+q(q+1)t+q(q-1)s = (q^2-1)\cdot l$, where k, s and t are non-negative integers(we note that except p, $h \in \pi_e(G)$ may be such that $s_h = q^2-1$, so we write $(q^2-1)\cdot k$). Thus $q(q+1)t+q(q-1)s=(q^2-1)\cdot(l-k)$. So $q \mid (q^2-1)\cdot(l-k)$. Then $q \mid (l-k)$. Therefore $(q^2-1)\cdot l$ is greater than order of group G, a contradiction. □

Lemma 3.3. *$2r \in \pi_e(G)$ for every prime $r \neq 2$ distinct from p.*

Proof. Let $2r \notin \pi_e(G)$ for some prime divisor r of $|G|$, with $r \neq 2$ distinct from p. Then the group P_r acts fixed point free on the set of elements of order 2. Therefore $|P_r| \mid s_2 = q^2$, a contradiction. □

Lemma 3.4. *$OC(G) = OC(\mathrm{PGL}_2(q))$.*

Proof. By Lemma 3.2 and 3.3, we have $t(G)=2$ and $T(G)=\{(q^2-1),\ \{p\}\}$, which implies that $OC(G)=\{q^2-1,\ q\}=OC(\mathrm{PGL}_2(q))$. □

Lemma 3.5. *G is neither Frobenius nor 2-Frobenius.*

Proof. If q is prime, then by Lemma 2.2, $G \cong \mathrm{PGL}_2(q)$. Thus G is neither Frobenius nor 2-Frobenius. Let $q=p^n$, where p is prime and $n>1$. Assume $G=NH$ is a Frobenius group with Frobenius kernel N and Frobenius complement H. By Lemma 2.4, we have $T(G)=\{\pi(N),\ \pi(H)\}=\{\pi(q^2-1),\ \{p\}\}$. Since $|G|=q(q^2-1)$ and $|H| \mid (|N|-1)$ by Lemma 2.5(b), it follows that $|N|=q^2-1$ and $|H|=q$, which is impossible because $q \nmid (q^2-2)$ if $q \neq 2$.

Let G be a 2-Frobenius group. By Lemma 2.6, G has a normal series $1 \trianglelefteq H \triangleleft K \trianglelefteq G$ such that K and G/H are Frobenius groups with Frobenius kernels H and K/H, respectively. By Lemma 2.6(a), $\pi_2(G)=\pi(K/H)=\{p\}$ and by Lemma 2.6(b), K/H is cyclic. Thus K/H has an element of order p^n. By Lemma 3.2, we get a contradiction. □

Lemma 3.6. *G is isomorphic to $\mathrm{PGL}_2(q)$.*

Proof. By Lemma 3.5 and 2.7, G has a normal series $1 \trianglelefteq N \triangleleft G_1 \trianglelefteq G$ such that N is a nilpotent π_1-group, G/G_1 is a solvable π_1-group, and G_1/N is a simple C_{pp}-group. By the definition of the prime graph component, the odd order component q of G is of a certain odd component of G_1/N since G is a simple C_{pp}-group. In particular, $t(G_1/N) \geq 2$. Furthermore, $G_1/N \lesssim G/N \lesssim \mathrm{Aut}(G_1/N)$ by Lemma 2.7.

Now using the classification of finite simple groups and the results in Tables 1–3 in [8], we consider the following steps.

Step 1. We prove that G_1/N can not be an alternating group $A_{n'}$.

If $G_1/N \cong A_{n'}$, then since the odd order components of $A_{n'}$ are primes, say p' or $p'-2$, we conclude that $q=p'$ or $q=p'-2$. In both cases, q is a prime number. By Lemma 2.2, G is isomorphic to $\mathrm{PGL}_2(q)$, a contradiction.

Step 2. If $G_1/N \cong A_r(q')$, then we distinguish the following six cases.

2.1. Suppose $G_1/N \cong A_{p'-1}(q')$, where $(p',q') \neq (3,2),\ (3,4)$, p' is an odd prime and q' is a prime power. Then $q=\frac{q'^{p'}-1}{(q'-1)(p',q'-1)}$ and $q'^{\frac{p'(p'-1)}{2}}\Pi_{i=1}^{p'-1}(q'^{i}-1) \mid (q^2-1)$. Thus $q^2=\frac{(q'^{p'}-1)^2}{[(q'-1)(p',q'-1)]^2} \leq q'^{2p'}$

and $q'^{\frac{p'(p'-1)}{2}} < q'^{\frac{p'(p'-1)}{2}} \Pi_{i=1}^{p'-1}(q'^i-1) \leq q^2-1 < q^2$. Hence $q'^{\frac{p'(p'-1)}{2}} < q'^{2p'}$. Therefore $\frac{p'(p'-1)}{2} < 2p'$, which implies that $p' < 5$. Since p' is an odd prime, $p' = 3$. Now it follows that $q^2 = \frac{(q'^3-1)^2}{[(q'-1)(3,q'-1)]^2}$ and $q'^3(q'^2-1)(q'-1) \leq q^2-1$. Therefore $4q'^4 = q'^2(2q')^2 \geq q'^2(q'+2)^2 = (q'^2+2q')^2 > (q'^2+q'+1)^2 \geq \frac{(q'^3-1)^2}{[(q'-1)(3,q'-1)]^2} = q^2 > q'^3(q'^2-1)(q'-1)$. It follows that $4q' > (q'^2-1)(q'-1) = q'^3 - q'^2 - q' + 1 > q'^3 - q'^2 - q'$. Thus $4 > q'^2 - q' - 1$, which shows that $q'^2 - q - 5 < 0$. Hence $q' = 2$, a contradiction.

2.2. Suppose $G_1/N \cong A_{p'}(q')$, where $(q'-1) \mid (p'+1)$, p' is an odd prime and q' is a prime power. Then $q = \frac{q'^{p'}-1}{q'-1}$ and $q'^{\frac{p'(p'+1)}{2}}(q'^{p'+1}-1)\Pi_{i=2}^{p'-1}(q'^i-1) \mid (q^2-1)$. Thus $q^2 = \frac{(q'^{p'}-1)^2}{(q'-1)^2} \leq q'^{2p'}$ and $q'^{\frac{p'(p'+1)}{2}} < q'^{\frac{p'(p'+1)}{2}}(q'^{p'+1}-1)\Pi_{i=2}^{p'-1}(q'^i-1) \leq q^2-1 < q^2$. Hence $q'^{\frac{p'(p'+1)}{2}} < q'^{2p'}$. Therefore $\frac{p'(p'+1)}{2} < 2p'$, which implies that $p' < 3$, a contradiction.

2.3. $G_1/N \cong A_1(q')$, where $4 \mid (q'+1)$ and q' is a prime power. Then $q = q'$ or $\frac{q'-1}{2}$. Moreover, $\frac{q'(q'^2-1)}{2} \leq \frac{q(q^2-1)}{|N|}$ in both cases. If $q = q'$, then $\frac{q'(q'^2-1)}{2} \leq \frac{q(q^2-1)}{|N|}$, which implies that $|N| = 1$ or 2. If $|N| = 1$, then $A_1(q) \lesssim G \lesssim \mathrm{Aut}(A_1(q))$. It follows that $G \cong \mathrm{PGL}_2(q)$, where $4 \mid (q'+1)$ and q' is a prime power. If $|N| = 2$, then $G/N \cong A_1(q)$. Since $G/C_G(N) \lesssim \mathrm{Aut}(N) \cong C_1$, it follows that $G = C_G(N)$. Hence $N \leq Z(G)$, which implies that $2p \in \pi_e(G)$, a contradiction.

If $q = \frac{q'-1}{2}$, then $q' = 2q+1$. Since $\frac{q'(q'^2-1)}{2} \mid \frac{q(q^2-1)}{|N|}$, we have that $\frac{(2q+1)[(2q+1)^2-1]}{2} \leq \frac{q(q^2-1)}{|N|}$. It follows that $\frac{(2q+1)[(2q+1)^2-1]}{2} \leq q(q^2-1)$, which implies that $7q \leq -1$, a contradiction.

2.4. $G_1/N \cong A_1(q')$, where $4 \mid (q'-1)$ and q' is a prime power. Then $q = q'$ or $\frac{q'+1}{2}$. Moreover, $\frac{q'(q'^2-1)}{2} \leq \frac{q(q^2-1)}{|N|}$ in both cases. If $q = q'$, then $\frac{q'(q'^2-1)}{2} \leq \frac{q(q^2-1)}{|N|}$, which implies that $|N| = 1$ or 2. If $|N| = 1$, then $A_1(q) \lesssim G/N \lesssim \mathrm{Aut}(A_1(q))$. It follows that $G \cong \mathrm{PGL}_2(q)$, where $4 \mid (q'-1)$ and q' is a prime power. If $|N| = 2$, then $G/N \cong A_1(q)$. Since $G/C_G(N) \lesssim \mathrm{Aut}(N) \cong C_1$, it follows that $G = C_G(N)$. Hence $N \leq Z(G)$, which implies that $2p \in \pi_e(G)$, a contradiction.

If $q = \frac{q'+1}{2}$, then $q' = 2q-1$. Since $\frac{q'(q'^2-1)}{2} \mid \frac{q(q^2-1)}{|N|}$, we have that $\frac{(2q-1)[(2q-1)^2-1]}{2} \leq \frac{q(q^2-1)}{|N|}$. It follows that $\frac{(2q-1)[(2q-1)^2-1]}{2} \leq q(q^2-1)$, which implies that $q \leq 1$, a contradiction.

2.5. $G_1/N \cong A_1(q')$, where $4 \mid q'$ and q' is a prime power. Then $q = q'+1$ or $q'-1$, and $q'(q'^2-1) \mid \frac{q(q^2-1)}{|N|}$. If $q = q'+1$, then $q' = q-1$. It follows that $(q-1)[(q-1)^2-1] \mid \frac{q(q^2-1)}{|N|}$, which implies that $q+1 = (q-2)|N|t$, where t is a natural number. Hence $(q-2)(|N|t-1) = 3$, which shows that $q-2 = 1$ or $q-2 = 3$. Because $4 \mid q$, we get a contradiction.

If $q = q'-1$, then $q' = q+1$. Since $q'(q'^2-1) \mid \frac{q(q^2-1)}{|N|}$, we have that $(q+1)[(q+1)^2-1] \leq \frac{q(q^2-1)}{|N|}$. It follows that $q(q+1)(q+2) \leq q(q^2-1)$, which implies that $q+2 < q-1$, a contradiction.

2.6. $G_1/N \cong A_2(4)$. Then q must be equal to 3, 5 or 7. Since $q > 3$ so $q = 5$ or 7 by Lemma 2.2, $G \cong \mathrm{PGL}_2(5)$ or $\mathrm{PGL}_2(7)$, a contradiction.

Step 3. If $G_1/N \cong {}^2A_r(q')$, then we distinguish the following three cases.

3.1. Suppose $G_1/N \cong^2 A_{p'-1}(q')$, where p' is an odd prime and q' is a prime power. Then $q = \frac{q'^{p'}+1}{(q'+1)(p',q'+1)}$ and $q'^{\frac{p'(p'-1)}{2}}\Pi_{i=1}^{p'-1}(q'^i-(-1)^i) \mid (q^2-1)$. Thus $q^2 = \frac{(q'^{p'}+1)^2}{[(q'+1)(p',q'+1)]^2} \leq q'^{2(p'+1)}$ and $q'^{\frac{p'(p'-1)}{2}} < q'^{\frac{p'(p'-1)}{2}}\Pi_{i=1}^{p'-1}(q'^i-(-1)^i) \leq q^2-1 < q^2$. Hence $q'^{\frac{p'(p'-1)}{2}} < q'^{2(p'+1)}$. Therefore $\frac{p'(p'-1)}{2} < 2(p'+1)$, which implies that $p' < 7$. Since p' is an odd prime, we have that $p' = 3$ or 5. If $p' = 3$, then $q^2 = \frac{(q'^3+1)^2}{[(q'+1)(3,q'+1)]^2} \leq (q'^2-q'+1)^2 < q'^4$ and $q'^4 < q'^3(q'+1) < q'^3(q'+1)(q'^2-1) \mid (q^2-1) < q^2$. Thus $q'^4 < q^2 < q'^4$, a contradiction. If $p' = 5$, then $q^2 = \frac{(q'^5+1)^2}{[(q'+1)(5,q'+1)]^2} \leq (q'^4-q'^3+q'^2-q'+1)^2 < q'^8$ and $q'^{10} < q'^{10}\Pi_{i=1}^{4}(q'^i-(-1)^i) \leq (q^2-1) < q^2$. Thus $q'^{10} < q^2 < q'^{10}$, a contradiction.

3.2. Suppose $G_1/N \cong^2 A_{p'}(q')$, where $(q'+1) \mid (p'+1)$ such that p' is an odd prime, q' is a prime power and $(p', q') \neq (3,3), (5,2)$. Then $q = \frac{q'^{p'}+1}{q'+1}$ and $q'^{\frac{p'(p'+1)}{2}}(q'^{p'+1}-1)\Pi_{i=2}^{p'-1}(q'^i-(-1)^i) \mid (q^2-1)$. Thus $q^2 = \frac{(q'^{p'}+1)^2}{(q'+1)^2} \leq q'^{2(p'+1)}$ and $q'^{\frac{p'(p'+1)}{2}} < q'^{\frac{p'(p'+1)}{2}}\Pi_{i=1}^{p'-1}(q'^i-(-1)^i) \leq q^2-1 < q^2$. Hence $q'^{\frac{p'(p'+1)}{2}} < q'^{2(p'+1)}$. Therefore $\frac{p'(p'+1)}{2} < 2(p'+1)$, which implies that $p' < 4$. Since p' is an odd prime, we have that $p' = 3$. Therefore, $q'^6(q'^4-1)(q'^2-1) \mid (q^2-1) < q^2 = \frac{(q'^3+1)^2}{(q'+1)^2} = (q'^2-q'+1)^2 < q'^4$, a contradiction.

3.3. Suppose $G_1/N \cong^2 A_3(2)$, $^2A_3(3)$ or $^2A_5(2)$.

If $G_1/N \cong^2 A_3(2)$, then $q = 5$ by Lemma 2.2, $G \cong \mathrm{PGL}_2(5)$, a contradiction.

If $G_1/N \cong^2 A_3(3)$, then $q = 5$ or 7 by Lemma 2.2, $G \cong \mathrm{PGL}_2(5)$ or $\mathrm{PGL}_2(7)$, a contradiction.

If $G_1/N \cong {}^2A_5(2)$, then $q = 7$ or 11 by Lemma 2.2, $G \cong \mathrm{PGL}_2(7)$ or $\mathrm{PGL}_2(11)$, a contradiction.

Step 4. If $G_1/N \cong B_r(q')$, then we consider the following two cases.

4.1. Suppose $G_1/N \cong B_r(q')$, $r = 2^t \geq 4$ and q' is an odd prime power. Then $q = \frac{q'^r+1}{2}$ and $q'^{r^2}(q'^r-1)\Pi_{i=1}^{r-1}(q'^{2i}-1) \mid (q^2-1)$. Thus $q^2 = \frac{(q'^r+1)^2}{4} \leq q'^{2(r+1)}$ and $q'^{r^2} < q'^{r^2}(q'^r-1)\Pi_{i=1}^{r-1}(q'^{2i}-1) \leq q^2-1 < q^2$. Hence $q'^{r^2} < q'^{2(r+1)}$. Therefore $r^2 < 2(r+1)$, which implies that $r < 3$, a contradiction.

4.2. Suppose $G_1/N \cong B_{p'}(3)$, where p' is an odd prime. Then $q = \frac{3^{p'}-1}{2}$ and $3^{p'^2}(3^{p'}+1)\Pi_{i=1}^{p'-1}(3^{2i}-1) \mid (q^2-1)$. Thus $q^2 = \frac{(3^{p'}-1)^2}{4} \leq 3^{2p'}$ and $3^{p'^2}(3^{p'}+1)\Pi_{i=1}^{p'-1}(3^{2i}-1) \mid (q^2-1) < q^2$. Hence $3^{p'^2} < 3^{2p'}$. Therefore $p'^2 < 2p'$, which implies that $p' < 2$, a contradiction.

Step 5. If $G_1/N \cong C_r(q')$, then we consider the following two cases.

5.1. Suppose $G_1/N \cong C_r(q')$, $r = 2^t \geq 2$ and q' is an odd prime power. Then $q = \frac{q'^r+1}{(2,q'-1)}$ and $q'^{r^2}(q'^r-1)\Pi_{i=1}^{r-1}(q'^{2i}-1) \mid (q^2-1)$. Thus $q^2 = \frac{(q'^r+1)^2}{(2,q'-1)^2} \leq q'^{2(r+1)}$ and $q'^{r^2} < q'^{r^2}(q'^r-1)\Pi_{i=1}^{r-1}(q'^{2i}-1) \leq q^2-1 < q^2$. Hence $q'^{r^2} < q'^{2(r+1)}$. Therefore $r^2 < 2(r+1)$, which implies that $r < 3$. Since $r = 2^t \geq 2$, we have $r = 2$. Now $q = \frac{q'^2+1}{(2,q'-1)}$ and $q'^4(q'^2-1)^2 < (q'^2+1)^2$. Therefore $q'^2 < 3$, which is a contradiction since q' is a prime power.

5.2. Suppose $G_1/N \cong C_{p'}(q')$, where p' is an odd prime and $q' = 2$ or 3. Then $q = \frac{q'^{p'}-1}{(2,q'-1)}$ and $q'^{p'^2}(q'^{p'}+1)\Pi_{i=1}^{p'-1}(q'^{2i}-1) \mid (q^2-1)$. Thus $q^2 = \frac{(q'^{p'}-1)^2}{(2,q'-1)^2} \leq q'^{2p'}$ and $q'^{p'^2} < q'^{p'^2}(q'^{p'}+1)\Pi_{i=1}^{p'-1}(q'^{2i}-1) \mid (q^2-1)$. Hence $q'^{p'^2} < q'^{2p'}$. Therefore $p'^2 < 2p'$, which implies that $p' < 2$, a contradiction.

Step 6. If $G_1/N \cong D_r(q')$, then we consider the following two cases.

6.1. Suppose $G_1/N \cong D_{p'}(q')$, where $p' \geq 5$ is an odd prime and $q' = 2$, 3 or 5.

Then $q = \frac{q'^{p'}-1}{q'-1}$ and $q'^{p'(p'-1)}\Pi_{i=1}^{p'-1}(q'^{2i}-1) \mid (q^2-1)$. Thus $q^2 = \frac{(q'^{p'}-1)^2}{(q'-1)^2} \leq q'^{2p'}$ and $q'^{p'(p'-1)} < q'^{p'(p'-1)}\Pi_{i=1}^{p'-1}(q'^{2i}-1) \leq (q^2-1) < q^2$. Hence $q'^{p'(p'-1)} < q'^{2p'}$. Therefore $p'(p'-1) < 2p'$, which implies that $p' < 3$, a contradiction.

6.2. Suppose $G_1/N \cong D_{p'+1}(q')$, where p' is an odd prime and $q' = 2$ or 3.

Then $q = \frac{q'^{p'}-1}{(2,q'-1)}$ and $\frac{1}{(2,q'-1)}q'^{p'(p'+1)}(q'^{p'}+1)q'^{p'+1}-1)\Pi_{i=1}^{p'-1}(q'^{2i}-1) \mid (q^2-1)$. Thus $q^2 = \frac{(q'^{p'}-1)^2}{(2,q'-1)^2} \leq q'^{2p'}$ and $q'^{p'(p'+1)} < \frac{1}{(2,q'-1)}q'^{p'(p'+1)}(q'^{p'}+1)(q'^{p'+1}-1)\Pi_{i=1}^{p'-1}(q'^{2i}-1) \leq q^2-1 < q^2$. Hence $q'^{p'(p'+1)} < q'^{2p'}$. Therefore $p'(p'+1) < 2p'$, which implies that $p' < 1$, a contradiction.

Step 7. If $G_1/N \cong {}^2B_2(q')$, where $q' = 2^{2t+1} > 2$, then we distinguish the following three cases.

7.1. Suppose $q = q'-1$. Then $q' = q+1$. Since $q'^2(q'-\sqrt{2q'}+1)(q'+\sqrt{2q'}+1) \mid (q^2-1)$, it follows that $(q+1)^2[(q+1)^2+1] \leq q^2-1 < q^2$, a contradiction.

7.2. Suppose $q = q'-\sqrt{2q'}+1$. Since $q'^2(q'-1)(q'+\sqrt{2q'}+1) \mid (q^2-1)$ and $q' > 2$, it follows that $q'^2(q'-\sqrt{2q'}+1)(q'+\sqrt{2q'}+1) \leq q'^2(q'-1)(q'+\sqrt{2q'}+1) \leq q^2-1 < q^2 = (q'-\sqrt{2q'}+1)^2$. Therefore $q'^2(q'+\sqrt{2q'}+1) < q'-\sqrt{2q'}+1 < q'+\sqrt{2q'}+1$, which shows that $q'^2 < 1$, a contradiction.

7.3. Suppose $q = q'+\sqrt{2q'}+1$. Since $q'^2(q'-1)(q'-\sqrt{2q'}+1) \mid (q^2-1)$, it follows that $q'^2(q'-\sqrt{2q'}+1)^2 \leq q'^2(q'-1)(q'-\sqrt{2q'}+1) \leq q^2-1 < q^2 = (q'+\sqrt{2q'}+1)^2$. Therefore $q'(q'-\sqrt{2q'}) < q'(q'-\sqrt{2q'}+1) < q'+\sqrt{2q'}+1 < 2q'+\sqrt{2q'}$, which shows that $q'(q'-\sqrt{2q'}) < 2q'+\sqrt{2q'}$. Thus $\sqrt{q'}(q'-\sqrt{2q'}) < 2\sqrt{q'}+\sqrt{2} < 3\sqrt{q'}$. Hence $q'-\sqrt{2q'} < 3$. It follows that $4-\sqrt{7} < q' < 4+\sqrt{7}$, which shows that $1 < q' < 7$. This is a contradiction since $q' = 2^{2t+1} \geq 8$.

Step 8. If $G_1/N \cong {}^2D_r(q')$, where q' is a prime power, then we distinguish the following six cases.

8.1. Suppose that $G_1/N \cong {}^2D_r(q')$, where $r = 2^t \geq 4$. Then $q = \frac{q'^r+1}{(2,q'+1)}$ and $q'^{r(r-1)}\Pi_{i=1}^{r-1}(q'^{2i}-1) \mid (q^2-1)$. Thus $q^2 = \frac{(q'^r+1)^2}{(2,q'+1)^2} \leq q'^{2(r+1)}$ and $q'^{r(r-1)} < q'^{r(r-1)}\Pi_{i=1}^{r-1}(q'^{2i}-1) \leq q^2-1 < q^2$. Hence $q'^{r(r-1)} < q'^{2(r+1)}$. Therefore $r(r-1) < 2(r+1)$, which implies that $0 < r < 4$, a contradiction.

8.2. Suppose that $G_1/N \cong {}^2D_r(2)$, where $r = 2^t+1 \geq 5$. Then $q = 2^{r-1}+1$ and $2^{r(r-1)}(2^r+1)(2^{r-1}-1)\Pi_{i=1}^{r-2}(2^{2i}-1) \mid (q^2-1)$. Thus $q^2 = (2^{r-1}+1)^2 \leq 2^{2r}$ and $2^{r(r-1)} < 2^{r(r-1)}(2^r+1)(2^{r-1}-1)\Pi_{i=1}^{r-2}(2^{2i}-1) \leq q^2-1 < q^2$. Hence $2^{r(r-1)} < 2^{2r}$, which implies that $r < 3$, a contradiction.

8.3. Suppose that $G_1/N \cong^2 D_{p'}(3)$, where $5 \leq p' \neq 2^t + 1$, p' is an odd prime.

Then $q = \frac{3^{p'}+1}{4}$ and $3^{p'(p'-1)}\Pi_{i=1}^{p'-1}(3^{2i}-1) \mid (q^2-1)$. Thus $q^2 = \frac{(3^{p'}+1)^2}{16} \leq 3^{2(p'+1)}$ and $3^{p'(p'-1)} < 3^{p'(p'-1)}\Pi_{i=1}^{p'-2}(3^{2i}-1) \leq q^2-1 < q^2$. Hence $3^{p'(p'-1)} < 3^{2(p'+1)}$, which implies that $p'(p'-1) < 2(p'+1)$. Thus $0 < p' < 4$, a contradiction.

8.4. Suppose that $G_1/N \cong^2 D_r(3)$, where $r = 2^t + 1 \neq p'$ such that $t \geq 2$ and p' is an odd prime. Then $q = \frac{3^{r-1}+1}{2}$ and $\frac{1}{2} \cdot 3^{r(r-1)}(3^r+1)(3^{r-1}-1)\Pi_{i=1}^{r-2}(3^{2i}-1) \mid (q^2-1)$. Thus $q^2 = \frac{(3^{r-1}+1)^2}{4} \leq 3^{2r}$ and $3^{r(r-1)} < \frac{1}{2} \cdot 3^{r(r-1)}(3^r+1)(3^{r-1}-1)\Pi_{i=1}^{r-2}(3^{2i}-1) \leq q^2-1 < q^2$. Hence $3^{r(r-1)} < 3^{2r}$, which implies that $r(r-1) < 2r$. Thus $r < 3$, a contradiction.

8.5. Suppose that $G_1/N \cong^2 D_{p'}(3)$, where $p' = 2^t + 1$, $t \geq 2$ and p' is an odd prime. Then $q = \frac{3^{p'}+1}{4}$ or $\frac{3^{p'-1}+1}{2}$. If $q = \frac{3^{p'}+1}{4}$, then $3^{p'(p'-1)}(3^{p'-1}-1)(3^{p'-1}+1)\Pi_{i=1}^{p'-2}(3^{2i}-1) \mid (q^2-1)$. Thus $q^2 = \frac{(3^{p'}+1)^2}{16} \leq 3^{2(p'+1)}$ and $3^{p'(p'-1)} < 3^{p'(p'-1)}(3^{p'-1}-1)(3^{p'-1}+1)\Pi_{i=1}^{p'-2}(3^{2i}-1) \leq q^2-1 < q^2$. Hence $3^{p'(p'-1)} < 3^{2(p'+1)}$, which implies that $p'(p'-1) < 2(p'+1)$. Thus $0 < p' < 4$, a contradiction. If $q = \frac{3^{p'-1}+1}{2}$, then $\frac{1}{2}3^{p'(p'-1)}(3^{p'-1}-1)(3^{p'}+1)\Pi_{i=1}^{p'-2}(3^{2i}-1) \mid (q^2-1)$. Thus $q^2 = \frac{(3^{p'-1}+1)^2}{4} \leq 3^{2p'}$ and $3^{p'(p'-1)} < \frac{1}{2}3^{p'(p'-1)}(3^{p'-1}-1)(3^{p'}+1)\Pi_{i=1}^{p'-2}(3^{2i}-1) \leq q^2-1 < q^2$. Hence $3^{p'(p'-1)} < 3^{2p'}$, which implies that $p'(p'-1) < 2p'$. Thus $p' < 3$, a contradiction.

Step 9. If $G_1/N \cong G_2(q')$, where q' is a prime power, then we distinguish the following three cases.

9.1. Suppose $G_1/N \cong G_2(q')$, where $2 < q' \equiv 1 \pmod 3$. Then $q = q'^2 - q' + 1$ and $q'^6(q'^3-1)(q'^2-1)(q'+1) \mid (q^2-1)$. Thus $q^2 = (q'^2-q'+1)^2 \leq q'^4$ and $q'^6 < q'^6\,(q'^3-1)(q'^2-1)(q'+1) \leq q^2-1 < q^2$. Hence $q'^6 < q'^4$, which implies that $q' < 1$, a contradiction.

9.2. Suppose $G_1/N \cong G_2(q')$, where $2 < q' \equiv -1 \pmod 3$. Then $q = q'^2 + q' + 1$ and $q'^6(q'^3+1)(q'^2-1)(q'-1) \mid (q^2-1)$. Thus $q^2 = (q'^2+q'+1)^2 \leq (q'^3-1)^2 \leq q'^6$ and $q'^6(q'-1) < q'^6\,(q'^3+1)(q'^2-1)(q'-1) \leq q^2-1 < q^2$. Hence $q'^6(q'-1) < q'^6$, which implies that $q' < 2$, a contradiction.

9.3. Suppose $G_1/N \cong G_2(q')$, where $3 \mid q'$. Then $q = q'^2 + q' + 1$ or $q'^2 - q' + 1$. If $q = q'^2 + q' + 1$, then $q'^6(q'^2-1)^2(q'^2-q'+1) \mid (q^2-1)$. Thus $q^2 = (q'^2+q'+1)^2 \leq (q'^3-1)^2 \leq q'^6$ and $q'^6(q'^2-1) < q'^6\,(q'^2-1)^2(q'^2-q'+1) \leq q^2-1 < q^2$. Hence $q'^6(q'^2-1) < q'^6$, which implies that $q' < 2$, a contradiction. If $q = q'^2 - q' + 1$, then $q'^6(q'^2-1)^2(q'^2+q'+1) \mid (q^2-1)$. Thus $q^2 = (q'^2-q'+1)^2 \leq q'^4$ and $q'^6 < q'^6\,(q'^2-1)^2(q'^2+q'+1) \leq q^2-1 < q^2$. Hence $q'^6 < q'^4$, which implies that $q' < 1$, a contradiction.

Step 10. If $G_1/N \cong E_7(2)$, $E_7(3)$, ${}^2E_6(2)$ or ${}^2F_4(2)'$.

If $G_1/N \cong E_7(2)$, then $|G_1/N| = |E_7(2)| = 2^{63} \cdot 3^{11} \cdot 5^2 \cdot 7^3 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 31 \cdot 43 \cdot 73 \cdot 127$ and $q = 73$ or 127. By Lemma 2.2, $G \cong \mathrm{PGL}_2(73)$ or $\mathrm{PGL}_2(127)$, a contradiction.

If $G_1/N \cong E_7(3)$, then $|G_1/N| = |E_7(3)| = 2^{23} \cdot 3^{63} \cdot 5^2 \cdot 7^3 \cdot 11^2 \cdot 13^3 \cdot 17 \cdot 19 \cdot 37 \cdot 41 \cdot 61 \cdot 73 \cdot 547 \cdot 757 \cdot 1093$ and $q = 757$ or 1093. By Lemma 2.2, $G \cong \mathrm{PGL}_2(757)$ or $\mathrm{PGL}_2(1093)$, a contradiction.

If $G_1/N \cong {}^2E_6(2)$, then $|G_1/N| = |{}^2E_6(2)| = 2^{36}\cdot 3^9\cdot 5^2\cdot 7^2\cdot 11\cdot 13\cdot 17\cdot 19$ and $q = 13$, 17 or 19. By Lemma 2.2, $G \cong$PGL$_2(13)$, PGL$_2(17)$ or PGL$_2(19)$, a contradiction.

If $G_1/N \cong {}^2F_4(2)'$, then $|G_1/N| = |{}^2F_4(2)'| = 2^{11}\cdot 3^3\cdot 5^2\cdot 13$ and $q = 13$. By Lemma 2.2, $G \cong$PGL$_2(13)$, a contradiction.

Step 11. If $G_1/N \cong {}^3D_4(q')$, where q' is a prime power, then $q = q'^4 - q'^2 + 1$ and $q'^{12}(q'^6-1)(q'^2-1)(q'^4+q'^2+1) \mid (q^2-1)$. Thus $q^2 = (q'^4 - q'^2+1)^2 < q'^8$ and $q'^{12} < q'^{12}(q'^6-1)(q'^2-1)(q'^4+q'^2+1) \leq q^2-1 < q^2$. Hence $q'^{12} < q'^8$, which implies that $q' < 1$, a contradiction.

Step 12. If $G_1/N \cong F_4(q')$, where q' is a prime power, then we distinguish the following two cases.

12.1. Suppose $G_1/N \cong F_4(q')$, where q' is a an odd prime power. Then $q = q'^4 - q'^2 + 1$ and $q'^{24}(q'^8-1)(q'^6-1)^2(q'^4-1) \mid (q^2-1)$. Thus $q^2 = (q'^4-q'^2+1)^2 \leq q'^8$ and $q'^{24} < q'^{24}(q'^8-1)(q'^6-1)^2(q'^4-1) \leq q^2-1 < q^2$. Hence $q'^{24} < q'^8$, which implies that $q' < 1$, a contradiction.

12.2. Suppose $G_1/N \cong F_4(q')$, where $2 \mid q'$ and $q' > 2$. Then $q = q'^4+1$ or $q'^4 - q'^2+1$. If $q = q'^4+1$, then $q'^{24}(q'^6-1)^2(q'^4-1)^2(q'^4-q'^2+1) \mid (q^2-1)$. Thus $q^2 = (q'^4+1)^2 < q'^{10}$ and $q'^{24} < q'^{24}(q'^6-1)^2(q'^4-1)^2(q'^4-q'^2+1) \leq q^2-1 < q^2$. Hence $q'^{24} < q'^{10}$, which implies that $q' < 1$, a contradiction. If $q = q'^4 - q'^2+1$, then $q'^{24}(q'^6-1)^2(q'^4-1)^2(q'^4+1) \mid (q^2-1)$. Thus $q^2 = (q'^4-q'^2+1)^2 < q'^8$ and $q'^{24} < q'^{24}(q'^6-1)^2(q'^4-1)^2(q'^4+1) \leq q^2-1 < q^2$. Hence $q'^{24} < q'^8$, which implies that $q' < 1$, a contradiction.

Step 13. If $G_1/N \cong {}^2F_4(q')$, where $q' = 2^{2t+1} > 2$, then $q = q'^2 \pm \sqrt{2q'^3} + q' \pm \sqrt{2q'} + 1$ and $q'^{12}(q'^4-1)(q'^3+1)(q'^2+1)(q'-1)(q'^2 \pm \sqrt{2q'^3} + q' \pm \sqrt{2q'}+1) \mid (q^2-1)$. Thus $q^2 = (q'^2 \pm \sqrt{2q'^3} + q' \pm \sqrt{2q'}+1)^2 \leq q'^{10}$ and $q'^{12} < q'^{12}(q'^4-1)(q'^3+1)(q'^2+1)(q'-1)(q'^2 \pm \sqrt{2q'^3} + q' \pm \sqrt{2q'}+1) \leq q^2-1 < q^2$. Hence $q'^{12} < q'^{10}$, which implies that $q' < 1$, a contradiction.

Step 14. If $G_1/N \cong {}^2G_2(q')$, where $q' = 3^{2t+1} > 3$, then $q = q' \pm \sqrt{3q'} + 1$ and $q'^3(q'^2-1)(q' \pm \sqrt{3q'}+1) \mid (q^2-1)$. Thus $q^2 = (q' \pm \sqrt{3q'}+1)^2 \leq [(q'+1)^2 - 3q']^2 = (q'^2 - q'+1)^2 < q'^4$ and $q'^3(q'^2-1) < q'^3(q'^2-1)(q' \pm \sqrt{3q'}+1) \leq q^2-1 < q^2$. Hence $q'^3(q'^2-1) < q'^4$, which implies that $q' < 2$, a contradiction.

Step 15. If $G_1/N \cong E_6(q')$ $(q' \geq 2)$ or ${}^2E_6(q')$ $(q' > 2)$, where q' is a prime power. Then $q = \frac{q'^6 \pm q'^3+1}{(3,q'\pm1)}$ and $q'^{36}(q'^{12}-1)(q'^8-1)(q'^6-1)(q'^5\pm1)(q'^3\pm1)(q'^2-1) \mid (q^2-1)$. Thus $q^2 = \frac{(q'^6\pm q'^3+1)^2}{(3,q'\pm1)^2} \leq (q'^9-1)^2 \leq q'^{18}$ and $q'^{36} < q'^{36}(q'^{12}-1)(q'^8-1)(q'^6-1)(q'^5\pm1)(q'^3\pm1)(q'^2-1) \leq q^2-1 < q^2$. Hence $q'^{36} < q'^{18}$, which implies that $q' < 1$, a contradiction.

Step 16. If G_1/N is a sporadic simple group, then $q = 5$, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 59, 67 or 71. By Lemma 2.2, G is isomorphic to PGL$_2(5)$, PGL$_2(7)$, PGL$_2(11)$, PGL$_2(13)$,

$\mathrm{PGL}_2(17)$, $\mathrm{PGL}_2(19)$, $\mathrm{PGL}_2(23)$, $\mathrm{PGL}_2(29)$, $\mathrm{PGL}_2(31)$, $\mathrm{PGL}_2(37)$, $\mathrm{PGL}_2(41)$, $\mathrm{PGL}_2(43)$, $\mathrm{PGL}_2(47)$, $\mathrm{PGL}_2(59)$, $\mathrm{PGL}_2(67)$, $\mathrm{PGL}_2(71)$, a contradiction.

Step 17. If $G_1/N \cong E_8(q')$, where q' is a prime power. Thus $q \in \{q'^8 + q'^7 - q'^5 - q'^4 - q'^3 + q' + 1, q'^8 - q'^7 + q'^5 - q'^4 + q'^3 - q' + 1, q'^8 - q'^6 + q'^4 - q'^2 + 1, q'^8 - q'^4 + 1\}$. Since $q'^8 + q'^7 - q'^5 - q'^4 - q'^3 + q' + 1 < (q' - 1)(q'^8 + q'^7 + q'^6 + q'^5 + q'^4 + q'^3 + q'^2 + q' + 1) = q'^9 - 1 < q'^9$, it follows that $q < q'^9$ in all cases. Since $q'^{120} \mid |E_8(q')| \mid |G|$ and $|G| = q(q^2 - 1)$, we get a contradiction.

We have thus examined all possibilities of G_1/N. Now we have just seen if $G_1/N \cong A_1(q')$, where $4 \mid (q' - 1)$ and q' is a prime power or $G_1/N \cong A_1(q')$, where $4 \mid (q' + 1)$ and q' is a prime power, then $q = q'$ and $G \cong \mathrm{PGL}_2(q)$. In the other cases we get a contradiction. Since q is odd prime power, $4 \mid (q - 1)$ or $4 \mid (q + 1)$. Therefore, we have proved that $G \cong \mathrm{PGL}_2(q)$. □

This completes the proof of the main theorem.

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