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NORMAL EDGE-TRANSITIVE AND $\frac{1}{2}-$ARC$-$TRANSITIVE CAYLEY GRAPHS ON NON-ABELIAN GROUPS OF ORDER $2pq$, $p > q$ ARE ODD PRIMES

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Abstract. Darafsheh and Assari in [Normal edge-transitive Cayley graphs on non-abelian groups of order 4p, where p is a prime number, Sci. China Math. **56** (1) (2013) 213−219.] classified the connected normal edge transitive and $\frac{1}{2}-$arc-transitive Cayley graph of groups of order $4p$. In this paper we continue this work by classifying the connected Cayley graph of groups of order $2pq$, $p > q$ are primes. As a consequence it is proved that $Cay(G, S)$ is a $\frac{1}{2}-$arc-transitive Cayley graph of order $2pq$, $p > q$ if and only if $|S|$ is an even integer greater than 2, $S = T \cup T^{-1}$ and $T \subseteq \{cb^ja^i \mid 0 \leq i \leq p-1\}$, $1 \leq j \leq q-1$, such that T and T^{-1} are orbits of $Aut(G, S)$ and

$$G \cong \langle a, b, c \mid a^p = b^q = c^2 = e, ac = ca, bc = cb, b^{-1}ab = a^r\rangle, \text{ or}$$
$$G \cong \langle a, b, c \mid a^p = b^q = c^2 = e, cac = a^{-1}, bc = cb, b^{-1}ab = a^r\rangle,$$

where $r^q \equiv 1 \ (mod p)$.

1. Introduction

All groups considered here are finite. For notations and definitions not defined here we refer the reader to [1]. Let $\Gamma = (V, E)$ be a simple graph, where $V = V(\Gamma)$ is the set of vertices and $E = E(\Gamma)$ is the set of edges of Γ. The group of automorphisms $Aut(\Gamma)$ is acting obviously on the set of vertices, edges and arcs of Γ. If $Aut(\Gamma)$ acts transitively on vertices, edges or arcs of Γ, then Γ is called vertex−, edge− or arc−transitive, respectively. If Γ is vertex− and edge−transitive but not arc−transitive, then Γ is called $1/2-$arc−transitive.

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Let G be a finite group and S be a subset of G such that $S = S^{-1}$ and $S \subseteq G \setminus \{1\}$. The Cayley graph $\Gamma = Cay(G, S)$ is defined by $V(\Gamma) = G$ and $E(\Gamma) = \{\{g, sg\} | g \in G, s \in S\}$. For every $g \in G$, the mapping $\rho_g : G \to G$ given by $\rho_g(x) = xg$ is an automorphism of Γ. Thus, $R(G) = \{\rho_g \mid g \in G\}$ is a subgroup of $Aut(\Gamma)$ isomorphic to G. Define $Aut(G, S) = \{\alpha \in Aut(G) \mid \alpha(S) = S\}$.

Following Xu [10], the Cayley graph $\Gamma = Cay(G, S)$ is called normal, if $R(G) \trianglelefteq Aut(\Gamma)$. The graph Γ is said to be normal edge transitive, if $N_{Aut(\Gamma)}(R(G))$ is transitive on the set of edges of Γ. Wang et al. [9], obtained all disconnected normal Cayley graphs on finite groups. Thus for studying the problem of normality in Cayley graphs, it suffices to consider the connected Cayley graphs. The following theorem is crucial throughout this paper:

Theorem 1.1. *Let $\Gamma = Cay(G, S)$ and $A = Aut(\Gamma)$, then the following hold:*

(1) [4] *$N_A(R(G)) = R(G) \rtimes Aut(G, S)$. The group $R(G)$ is normal in A if and only if $A = R(G) \rtimes Aut(G, S)$;*

(2) [4] *Γ is normal if and only if $A_1 = Aut(G, S)$;*

(3) [7] *Let $\Gamma = Cay(G, S)$ be a connected Cayley graph on S. Then Γ is normal edge-transitive if and only if $Aut(G, S)$ is either transitive on S, or has two orbits in S in the form of T and T^{-1}, where T is a non-empty subset of S such that $S = T \bigcup T^{-1}$;*

(4) [2, Corollary 2.3] *Let $\Gamma = Cay(G, S)$ and H be the subset of all involutions of the group G. If $< H > \neq G$ and Γ is connected normal edge−transitive, then its valency is even;*

(5) [4] *If $\Gamma = Cay(G, S)$ is a connected Cayley graph on S then Γ is normal arc−transitive if and only if $Aut(G, S)$ acts transitively on S;*

(6) [2, Corollary 2.5] *If G is a Cayley graph of an abelian group, then G is not a normal $\frac{1}{2}$−arc−transitive Cayley graph.*

Throughout this paper our notation is standard and taken from [1, 4]. We encourage the interested readers to consult papers [3, 5, 6] for more information on this topic. Our work is a continuation of recent paper of Darafsheh and Assari [2]. Our aim is to classify all normal edge-transitive and $\frac{1}{2}$−arc−transitive Cayley graphs on non-abelian groups of order $2pq$, when p and q are odd distinct primes. We encourage to the interested readers to consult [11] for more information on groups of order $2pq$, $p > q$ are primes.

2. Main Results

It is well−known that a Cayley graph $\Gamma = Cay(G, S)$ is connected if and only if G is generated by S. In this section, the connected Cayley graphs of groups of orders $2pq$, p and q are distinct primes, are investigated. Since the Cayley graph $Cay(G, S)$ is not normal arc−transitive, when G is abelian, it is enough to consider non-abelian groups of orders $2pq$. In [8], Talebi proved that if $G = D_{2n}$, the dihedral group of order $2n$, and $\Gamma = Cay(G, S)$ is connected normal edge−transitive then $Aut(D_{2n}, S)$ is transitive on S. Using this result, he proved that the Cayley graphs of dihedral groups are not normal $\frac{1}{2}$−arc−transitive. So, it is enough to investigate the groups G_3, G_4, G_5 and G_6. All Cayley graphs considered here are assumed to be undirected.

As a consequence of our result, it is proved that $\Gamma = Cay(G,S)$ is a $\frac{1}{2}-$arc$-$transitive Cayley graph of order $2pq$, $p > q$, if and only if $|S|$ is an even integer greater than 2, $S = T \cup T^{-1}$ and $T \subseteq \{cb^ja^i \mid 0 \leq i \leq p-1\}$, $1 \leq j \leq q-1$, such that T and T^{-1} are orbits of $Aut(G,S)$ and

$$G = \langle a,b,c \mid a^p = b^q = c^2 = e, ac = ca, bc = cb, b^{-1}ab = a^r\rangle, \text{ or}$$
$$G = \langle a,b,c \mid a^p = b^q = c^2 = e, cac = a^{-1}, bc = cb, b^{-1}ab = a^r\rangle,$$

where $r^q \equiv 1 \ (mod\ p)$.

Theorem 2.1. *(See [11]) A group of order $2pq$, p and q with $p > q$ are distinct odd primes, is isomorphic to one of the following groups:*

(1) $G_1 = \langle a\rangle$,
(2) $G_2 = \langle a,b \mid a^{pq} = b^2 = 1, b^{-1}ab = a^{-1}\rangle$,
(3) $G_3 = \langle a,b,c \mid a^p = b^q = c^2 = 1, ab = ba, cac^{-1} = a^{-1}, bc = cb\rangle$,
(4) $G_4 = \langle a,b,c \mid a^p = b^q = c^2 = 1, ab = ba, ac = ca, cbc = b^{-1}\rangle$,
(5) $G_5 = \langle a,b,c \mid a^p = b^q = c^2 = 1, ac = ca, bc = cb, b^{-1}ab = a^r\rangle$,
(6) $G_6 = \langle a,b,c \mid a^p = b^q = c^2 = 1, cac = a^{-1}, bc = cb, b^{-1}ab = a^r\rangle$,

where r is an element of order q in U_p.

Lemma 2.2. *The automorphism groups of G_3, G_4, G_5 and G_6 can be computed as follows:*

(1) $Aut(G_3) \cong (Z_p \times U_p) \rtimes U_q$.
(2) $Aut(G_4) \cong (Z_q \times U_q) \rtimes U_p$.
(3) $Aut(G_5) \cong Z_p \rtimes U_p$.
(4) $Aut(G_6) \cong Z_p \rtimes U_p$.

Proof. From the presentations of G_3, G_4, G_5 and G_6, Theorem 2.1, we record their element orders in Table 1. In this table, $1 \leq i \leq p-1$, $1 \leq j \leq q-1$ and $0 \leq k \leq p-1$. Our main proof will consider four cases as follows:

(1) If $\sigma \in Aut(G_3)$ then σ is an order preserving function. This implies that $\sigma(a) = a^i, 1 \leq i \leq p-1$, $\sigma(b) = b^j, 1 \leq j \leq q-1$, and $\sigma(c) = ca^k$, where $0 \leq k \leq p-1$. Thus, $Aut(G_3) = \{\sigma_{i,j,k} \mid \sigma_{i,j,k}(a) = a^i, \sigma_{i,j,k}(b) = b^j \ \&\ \sigma_{i,j,k}(c) = ca^k \ ;\ 1 \leq i \leq p-1,\ 1 \leq j \leq q-1,\ 0 \leq k \leq p-1\}$. On the other hand, $\sigma_{i,j,k}\sigma_{i',j',k'} = \sigma_{ii',jj',k+k'i}$. It is clear that $e_{Aut(G_3)} = \sigma_{1,1,0}$ and $\sigma^{-1}_{i,j,k} = \sigma_{i^{-1},j^{-1},-i^{-1}k}$, where i^{-1} and j^{-1} are computed in U_p and U_q, respectively. Define $A = \{\sigma_{i,1,k} \mid 1 \leq i \leq p-1,\ 0 \leq k \leq p-1\}$ and$B = \{\sigma_{1,j,0}, 1 \leq j \leq q-1\}$. Then by an easy calculation, one can see that $Aut(G_3) = A \rtimes B \cong (Z_p \times U_p) \rtimes U_q$, which completes this case.
(2) A similar argument as (1) shows that $Aut(G_4) = \{\sigma_{i,j,k} \mid \sigma_{i,j,k}(a) = a^i, \sigma_{i,j,k}(b) = b^j \ \&\ \sigma_{i,j,k}(c) = cb^k \ ;\ 1 \leq i \leq p-1,\ 1 \leq j \leq q-1,\ 0 \leq k \leq q-1\}$. We now define $A = \{\sigma_{1,j,k} \mid 1 \leq j \leq q-1,\ 0 \leq k \leq q-1\}$ and $B = \{\sigma_{i,1,0} \mid 1 \leq i \leq p-1\}$. Then $A \trianglelefteq Aut(G_4)$, $B \leq Aut(G_4)$, $A \cap B = 1$ and $G = AB$. Therefore, $Aut(G_4) = A \rtimes B \cong (Z_q \times U_q) \rtimes U_p$, which completes this part.

(3) Suppose $\sigma \in Aut(G_5)$. Since σ is an order preserving function, $\sigma(a) = a^i, 1 \leq i \leq p-1$, $\sigma(b) = ba^j, 0 \leq j \leq p-1$, and $\sigma(c) = c$. Thus, $Aut(G_5) = \{\sigma_{i,j} \mid \sigma_{i,j}(a) = a^i, \sigma_{i,j}(b) = ba^j \ \& \ \sigma_{i,j}(c) = c \ ; \ 1 \leq i \leq p-1, \ 0 \leq j \leq p-1\}$. On the other hand, $\sigma_{i,j}\sigma_{i',j'}(b) = \sigma_{i,j}(ba^{j'}) = \sigma_{i,j}(b)\sigma_{i,j}(a^{j'}) = ba^j a^{j'i} = ba^{j+j'i}$ and $\sigma_{i,j}\sigma_{i',j'}(a) = \sigma_{i,j}(a^{i'}) = a^{ii'}$, where $ii' \equiv 1 \ (mod \ p)$. So, $\sigma_{i,j}\sigma_{i',j'} = \sigma_{ii',j+j'i}$. Since $\sigma_{1,0}(a) = a, \sigma_{1,0}(b) = b$ and $\sigma_{1,0}(c) = c$, $\sigma_{1,0} = id$. This shows that $\sigma_{i,j}^{-1} = \sigma_{i^{-1},-i^{-1}j}$, where i^{-1} is computed in U_p. Set $A = \{\sigma_{1,j} \mid 0 \leq j \leq p-1\}$. It is clear that A is normal in $Aut(G_5)$. Put $B = \{\sigma_{i,0} \mid 1 \leq i \leq p-1\}$. Then obviously B is a subgroup of $Aut(G_5)$ and $Aut(G_5) = A \rtimes B \cong Z_p \rtimes U_p$, as desired.

(4) By a similar argument as above, $Aut(G_6) = \{\sigma_{i,j} \mid \sigma_{i,j}(a) = a^i, \sigma_{i,j}(b) = ba^j \ \& \ \sigma_{i,j}(c) = ca^j \ ; \ 1 \leq i \leq p-1, \ 0 \leq j \leq p-1\}$. Again, we define $A = \{\sigma_{1,j} \mid 0 \leq j \leq p-1\}$ and $B = \{\sigma_{i,0} \mid 1 \leq i \leq p-1\}$. Then $Aut(G_6) \cong Z_p \rtimes U_p$.

This completes the proof. □

We now apply Lemma 2.2 to compute the orbits of $Aut(G_i)$ under natural action on G_i, $3 \leq i \leq 6$. Suppose n_i, $3 \leq i \leq 6$, denote the number of orbits of $Aut(G_i)$ on G_i under natural group action. Then by a tedious calculation, one can see that $n_3 = n_4 = 6$, $n_5 = 2q+2$ and $n_6 = 2q+1$. Moreover, we assume that Ω_i^j, $3 \leq j \leq 6$ and $1 \leq i \leq n_j$, denote the i^{th} orbit of $Aut(G_j)$ on G_j. Our calculations are recorded in Table 2.

Example 2.3. *Define* $S = \{cb^ja^l, cb^ja^{l'}, (cb^ja^l)^{-1}, (cb^ja^{l'})^{-1}\}, l \neq l'$ *and* $\Gamma = Cay(G_3, S)$. *Obviously,* $(cb^ja^l)^{-1} = cb^{q-j}a^l$, $(cb^ja^{l'})^{-1} = cb^{q-j}a^{l'}$ *and* S *is a generating set for* G_3. *Hence,* $Cay(G_3, S)$ *is connected. We now consider the automorphisms* $\sigma_{-1,1,l+l'}, \sigma_{-1,q-1,l+l'}$ *and* $\sigma_{1,q-1,0}$ *that introduced in the proof of Lemma 2.2(1). By a simple calculation, one can see that*

$$\begin{aligned}
\sigma_{-1,1,l+l'}(b^jca^l) &= \sigma_{-1,q-1,l+l'}(b^{q-j}ca^l) = \sigma_{1,q-1,0}(b^{q-j}ca^{l'}) = b^jca^{l'},\\
\sigma_{-1,1,l+l'}(b^jca^{l'}) &= \sigma_{-1,q-1,l+l'}(b^{q-j}ca^{l'}) = \sigma_{1,q-1,0}(b^{q-j}ca^l) = b^jca^l,\\
\sigma_{-1,1,l+l'}(b^{q-j}ca^l) &= \sigma_{-1,q-1,l+l'}(b^jca^l) = \sigma_{1,q-1,0}(b^jca^{l'}) = b^{q-j}ca^{l'},\\
\sigma_{-1,1,l+l'}(b^{q-j}ca^{l'}) &= \sigma_{-1,q-1,l+l'}(b^jca^{l'}) = \sigma_{1,q-1,0}(b^jca^l) = b^{q-j}ca^l.
\end{aligned}$$

This shows that $\sigma_{-1,1,l+l'}(S) = \sigma_{-1,q-1,l+l'}(S) = \sigma_{1,q-1,0}(S) = S$. *Thus,* $Aut(G_3, S)$ *acts transitively on* S. *Apply Theorem 1.1(3) and Theorem 1.1(5) to deduce that* Γ *is normal edge−transitive and normal arc−transitive Cayley graph of degree 4. Therefore,* Γ *is not normal* $\frac{1}{2}$*−arc−transitive.*

Proposition 2.4. *The connected Cayley graph* $\Gamma = Cay(G_3, S)$ *is normal edge−transitive if and only if* $G_3 =< S >$, $|S| > 2$ *is an even integer and* $S \subseteq \{cb^ja^i \mid 0 \leq i \leq p-1 \ \& \ 1 \leq j \leq q-1 \}$, *where* S *is an orbit of* $Aut(G_3, S)$.

Proof. Since $G_3 \cong Z_q \times D_{2p}$, $H = \{ca^i \mid 0 \leq i \leq p-1\}$ is the set of all elements of G_3 of order 2. Thus, $b \notin \langle H \rangle$ and so $G_3 \neq \langle H \rangle$. Apply Theorem 1.1(4) to deduce that $|S|$ is an even integer > 2. Consider the automorphisms $\sigma_{-1,j^{-1}j',l+l'}$, $0 \leq i \leq p-1$ & $1 \leq j \leq q-1$ and $0 \leq l, l' \leq p-1$. Then, one can easily see that $\sigma_{-1,j^{-1}j',l+l'}(S) = S$ and so $Aut(G_3, S)$ acts transitively on S. Therefore, Γ is normal edge− and arc−transitive. □

Table 1. Element Orders of G_3, G_4, G_5 and G_6.

Elements	G_3	G_4	G_5	G_6
a^i	p	p	p	p
c	2	2	2	2
b^ja^k	$\begin{cases} q & k=0 \\ pq & k\neq 0 \end{cases}$	$\begin{cases} q & k=0 \\ pq & k\neq 0 \end{cases}$	q	q
cb^ja^k	2q	$\begin{cases} 2 & k=0 \\ 2p & k\neq 0 \end{cases}$	2q	2q
ca^i	2	$2p$	$2p$	2

Corollary 2.5. *The connected Cayley graph $\Gamma = Cay(G_3, S)$ is not normal $\frac{1}{2}-$arc$-$transitive.*

Example 2.6. *Suppose $S = \{b^jca^i, b^{q-j}ca^i, b^jca^{p-i}, b^{q-j}ca^{p-i}\}, j \neq 0,\ 1 \leq i \leq p-1$ and $\Gamma = Cay(G_4, S)$. Then we can see that, $(b^jca^i)^{-1} = b^{q-j}ca^{p-i}$, $(b^{q-j}ca^i)^{-1} = b^jca^{p-i}$ and S is a generating set for G_4. This shows that $Cay(G_4, S)$ is connected. We now consider the automorphisms $\sigma_{1,j^{-1}(q-j),0}, \sigma_{i^{-1}(p-i),1,0}$ and $\sigma_{i^{-1}(p-i),j^{-1}(q-j),0}$ that introduced in the proof of Lemma 2.2(2). A simple calculation implies that*

$$\begin{aligned}
\sigma_{1,j^{-1}(q-j),0}(b^jca^i) &= \sigma_{i^{-1}(p-i),1,0}(b^{q-j}ca^{p-i}) = \sigma_{i^{-1}(p-i),j^{-1}(q-j),0}(b^jca^{p-i}) = b^{q-j}ca^i,\\
\sigma_{1,j^{-1}(q-j),0}(b^{q-j}ca^i) &= \sigma_{i^{-1}(p-i),1,0}(b^jca^{p-i}) = \sigma_{i^{-1}(p-i),j^{-1}(q-j),0}(b^{q-j}ca^{p-i}) = b^jca^i,\\
\sigma_{1,j^{-1}(q-j),0}(b^jca^{p-i}) &= \sigma_{i^{-1}(p-i),1,0}(b^{q-j}ca^i) = \sigma_{i^{-1}(p-i),j^{-1}(q-j),0}(b^jca^i) = b^{q-j}ca^{p-i},\\
\sigma_{1,j^{-1}(q-j),0}(b^{q-j}ca^{p-i}) &= \sigma_{i^{-1}(p-i),1,0}(b^jca^i) = \sigma_{i^{-1}(p-i),j^{-1}(q-j),0}(b^{q-j}ca^i) = b^jca^{p-i}.
\end{aligned}$$

Therefore, $\sigma_{1,j^{-1}(q-j),0}(S) = \sigma_{i^{-1}(p-i),1,0}(S) = \sigma_{i^{-1}(p-i),j^{-1}(q-j),0}(S) = S$ and $Aut(G_4, S)$ acts transitively on S. We now apply Theorem 1.1(3) and Theorem 1.1(5) to deduce that Γ is normal edge$-$transitive and normal arc$-$transitive Cayley graph of degree 4. Therefore, Γ is not normal $\frac{1}{2}-$arc$-$transitive.

Proposition 2.7. *The connected Cayley graph $\Gamma = Cay(G_4, S)$ is normal edge$-$transitive if and only if $G_4 =< S >$, $|S| > 2$ is an even integer and $S \subseteq \{cb^ja^i \mid 1 \leq i \leq p-1\ \&\ 0 \leq j \leq q-1\ \}$, where S is an orbit of $Aut(G_4, S)$.*

Proof. We first notice that $G_4 \cong Z_p \times D_{2q}$. Then $H = \{cb^j \mid 0 \leq j \leq q-1\}$ is the set of all elements of G_4 of order 2. Since $a \notin \langle H\rangle$, $G_4 \neq \langle H\rangle$. Apply Theorem 1.1(4) to deduce that $|S|$ is an even integer > 2. Consider the automorphisms $\sigma_{i^{-1}i',-1,j+j'}$, where $1 \leq i, i' \leq p-1$ and $0 \leq j, j' \leq q-1$. Then, it is easy to see that $\sigma_{i^{-1}i',-1,j+j'}(S) = S$, for all i and j. So, $Aut(G_4, S)$ acts transitively on S. Therefore, Γ is normal edge$-$ and arc$-$transitive. □

Table 2. The Orbits of $Aut(G_i)$ on G_i under Natural Group Action, $3 \leq i \leq 6$.

$$
\begin{aligned}
\Omega_1^3 &= \Omega_1^4 = \Omega_1^5 = \Omega_1^6 = \{1\},\\
\Omega_2^3 &= \Omega_2^4 = \{a^i \mid 1 \leq i \leq p-1\},\\
\Omega_3^3 &= \{ca^i \mid 0 \leq i \leq p-1\}, \quad \Omega_3^4 = \{cb^j \mid 0 \leq j \leq q-1\},\\
\Omega_4^3 &= \{b^j a^i \mid 1 \leq i \leq p-1, 1 \leq j \leq q-1\}, \ \Omega_4^4 = \Omega_6^3 = \{b^j \mid 1 \leq j \leq q-1\},\\
\Omega_5^3 &= \{cb^j a^i \mid 0 \leq i \leq p-1, 1 \leq j \leq q-1\}, \ \Omega_5^4 = \{cb^j a^k \mid 1 \leq k \leq p-1, 0 \leq j \leq q-1\},\\
\Omega_6^4 &= \{b^j a^k \mid 1 \leq j \leq q-1, 1 \leq k \leq p-1\},\\
\Omega_2^5 &= \{c\}, \quad \Omega_3^5 = \{a^i \mid 1 \leq i \leq p-1\}, \quad \Omega_4^5 = \{ca^i \mid 1 \leq i \leq p-1\},\\
\Omega_5^5 &= \{ba^i \mid 0 \leq i \leq p-1\}, \quad \Omega_6^5 = \{b^2 a^i \mid 0 \leq i \leq p-1\},\\
\vdots \quad & \quad \cdots\\
\Omega_{q+2}^5 &= \{b^{q-2} a^i \mid 0 \leq i \leq p-1\}, \quad \Omega_{q+3}^5 = \{b^{q-1} a^i \mid 0 \leq i \leq p-1\},\\
\Omega_{q+4}^5 &= \{cba^i \mid 0 \leq i \leq p-1\}, \quad \Omega_{q+5}^5 = \{cb^2 a^i \mid 0 \leq i \leq p-1\},\\
\vdots \quad & \quad \cdots\\
\Omega_{2q+1}^5 &= \{cb^{q-2} a^i \mid 0 \leq i \leq p-1\}, \quad \Omega_{2q+2}^5 = \{cb^{q-1} a^i \mid 0 \leq i \leq p-1\},\\
\Omega_2^6 &= \{a^i \mid 1 \leq i \leq p-1\}, \quad \Omega_3^6 = \{ca^i \mid 0 \leq i \leq p-1\},\\
\Omega_4^6 &= \{ba^i \mid 0 \leq i \leq p-1\}, \quad \Omega_5^6 = \{b^2 a^i \mid 0 \leq i \leq p-1\},\\
\vdots \quad & \quad \cdots\\
\Omega_{q+1}^6 &= \{b^{q-2} a^i \mid 0 \leq i \leq p-1\}, \quad \Omega_{q+2}^6 = \{b^{q-1} a^i \mid 0 \leq i \leq p-1\},\\
\Omega_{q+3}^6 &= \{cba^i \mid 0 \leq i \leq p-1\}, \quad \Omega_{q+4}^6 = \{cb^2 a^i \mid 0 \leq i \leq p-1\},\\
\vdots \quad & \quad \cdots\\
\Omega_{2q}^6 &= \{cb^{q-2} a^i \mid 0 \leq i \leq p-1\}, \quad \Omega_{2q+1}^6 = \{cb^{q-1} a^i \mid 0 \leq i \leq p-1\}.
\end{aligned}
$$

Example 2.8. *If $S = \{cba^k, cba^l, (cba^k)^{-1}, (cba^l)^{-1}\}, l \neq k$ and $\Gamma = Cay(G_5, S)$. It is clear that S is a generating set for G_5 and so $Cay(G_5, S)$ is connected. Also, $(cba^k)^{-1} = cb^{q-1}a^{-r^{q-1}k}$ and $(cba^l)^{-1} = cb^{q-1}a^{-r^{q-1}l}$. Set $T = \{cba^l, cba^k\}$. Thus, $T^{-1} = \{cb^{q-1}a^{-r^{q-1}k}, cb^{q-1}a^{-r^{q-1}l}\}$ and $S = T \bigcup T^{-1}$. We now prove that T and T^{-1} are orbits of $Aut(G_5, S)$ under natural action. Suppose $\sigma_{-1,l+k} \in Aut(G_5)$. Then $\sigma_{-1,l+k}(cba^l) = cba^k$, $\sigma_{-1,l+k}(cba^k) = cba^l$, $\sigma_{-1,l+k}(cb^{q-1}a^{-r^{q-1}k}) = cb^{q-1}a^{-r^{q-1}l}$ and $\sigma_{-1,l+k}(cb^{q-1}a^{-r^{q-1}l}) = cb^{q-1}a^{-r^{q-1}k}$. This shows that $\sigma_{-1,l+k} \in Aut(G_5, S)$ and so T and T^{-1} are orbits of $Aut(G_5, S)$. Therefore, $\Gamma = Cay(G_5, S)$ is connected normal edge$-$transitive Cayley graph of degree 4 which is not normal arc$-$transitive. Therefore, by Theorems 1.1(3) and 1.1(5), Γ is normal $\frac{1}{2}-$arc$-$transitive.*

Proposition 2.9. *The connected Cayley graph $\Gamma = Cay(G_5, S)$ is $\frac{1}{2}-$arc$-$transitive if and only if $G_5 =< S >$, $|S| > 2$ is an even integer and $S = T \cup T^{-1}$, where $T \subseteq \{cb^j a^i | 0 \leq i \leq p-1\}$ and $1 \leq j \leq q-1$ is an orbit of $Aut(G_5, S)$.*

Proof. Since c is the unique involution of G_5, $a, b \notin \langle c \rangle$, $|S|$ is even. On the other hand, if $T \subseteq \{cb^j a^i | 0 \leq i \leq p-1\}$ then $T^{-1} \subseteq \{cb^{q-j} a^i | 0 \leq i \leq p-1\}$. Set $S = T \cup T^{-1}$. Then $S = S^{-1}$, $\sigma_{k^{-1}l,0}(cb^{q-j}a^l) = (cb^{q-j}a^k)$ and $\sigma_{k^{-1}l,0}(cb^{q-j}a^k) = (cb^{q-j}a^l)$. Thus, $\sigma_{k^{-1}l,0}(T) = T$ and $\sigma_{k^{-1}l,0}(T^{-1}) = T^{-1}$. Since S is a union of two orbits, Γ is normal $\frac{1}{2}-$arc$-$transitive, proving the result. □

Proposition 2.10. *If $S = \{cba^l, cba^k, cb^{q-1}a^{-r^{q-1}l}, cb^{q-1}a^{-r^{q-1}k}\}$, $l \neq k$, then $Aut(G_5, S) \cong Z_2$.*

Proof. Obviously, $G_5 = \langle S \rangle$ and so $Aut(G_5, S)$ has a faithful action on S. This implies that $Aut(G_5, S)$ is isomorphic to a subgroup of S_4. We first prove that $Aut(G_5, S)$ does not have an element of order 3 and 4. If $\sigma \in Aut(G_5, S)$ has order 3, then the automorphism σ is fixed an element $y \in S$. This implies that y^{-1} is another fixed element of σ, a contradiction. Next, we assume that $\sigma \in Aut(G_5, S)$ has order 4, $x = cba^l$ and $y = cba^k$. Then σ has one of the forms $g = (xyx^{-1}y^{-1})$ or $h = (xy^{-1}x^{-1}y)$.

On the other hand, $\sigma \in Aut(G_5, S) \leq Aut(G_5)$ and so there exist i, j, $1 \leq i \leq p-1$ and $0 \leq j \leq p-1$ such that $\sigma = \sigma_{i,j}$. It is clear that $\sigma_{i,j}(cba^k) = cba^j a^{ki} = cba^{j+ki}$. If $\sigma = g$ then $\sigma(y) = x^{-1}$ which implies that $cba^{j+ki} = cb^{q-1}a^{-r^{q-1}l}$. So, $a^{j+ki} = b^{q-2}a^{-r^{q-1}l}$. If $p | j + ki$ then $a^{j+ki} = e$ and so $b^{q-2}a^{-r^{q-1}l} = e$. On the other hand, $o(b^{q-2}) = 3$, $o(a^{-r^{q-1}l}) = p \neq 3$ and $b^{q-2} = a^{r^{q-1}l}$, which is impossible. If $p \not| j + ki$ then $o(a^{j+ki}) = p$ and $o(b^{q-2}a^{-r^{q-1}l}) = q, p \neq q$, lead to another contradiction. Thus $\sigma \neq g$. A similar argument shows that $\sigma \neq h$. Therefore, $Aut(G_5, S)$ does not have elements of order 3 or 4. Since $Aut(G)$ is isomorphic to a subgroup of the symmetric group S_4 without elements of order 3 and 4, it is enough to prove that $Aut(G)$ has a unique element of order 2. It is easy to see that $o(\sigma_{-1,l+k}) = 2$ and $\sigma_{-1,l+k} \in Aut(G_5, S)$. Suppose $\sigma_{i,j} \in Aut(G, S)$, $\sigma_{i,j}(cba^l) = cba^{j+li} = cba^k$ and $\sigma_{i,j}(cba^k) = cba^{j+ki} = cba^l$. Then

$$j + li \equiv k \ (\ mod \ p) \ \text{ and } \ j + ki \equiv \ l \ (\ mod \ p). \qquad (1)$$

On the other hand,

$$\begin{aligned}
\sigma_{i,j}(cb^{q-1}a^{-r^{q-1}l}) &= cb^{q-1}a^{j(r^{q-2}+\ldots+r+1)-ir^{q-1}l} = cb^{q-1}a^{-r^{q-1}k}, \\
\sigma_{i,j}(cb^{q-1}a^{-r^{q-1}k}) &= cb^{q-1}a^{j(r^{q-2}+\ldots+r+1)-ir^{q-1}k} = cb^{q-1}a^{-r^{q-1}l}.
\end{aligned}$$

Hence, $j(r^{q-2} + ... + r + 1) \equiv (il - k)r^{q-1} \ (\ mod \ p)$ and $j(r^{q-2} + ... + r + 1) \equiv (ik - l)r^{q-1} \ (mod \ p)$. These congruences imply that $(il - k)r^{q-1} \equiv (ik - l)r^{q-1} \ (mod \ p)$ and so $i(l - k) \equiv -(l - k) \ (mod \ p)$. Since $l \neq k$, $i \equiv -1 \ (mod \ p)$ and by Eq (1) $j - k \equiv l \ (mod \ p)$ which implies that $j \equiv l + k \ (mod \ p)$. Therefore, $Aut(G_5, S) = \langle \sigma_{-1,l+k} \rangle$. This completes the proof. □

Example 2.11. *Set $S = \{cb^j a, cb^j a^{p-1}, cb^{q-j}a^{r^{q-j}}, cb^{q-j}a^{r^{q-j}(p-1)}\}$ and $\Gamma = Cay(G_6, S)$. It is clear that S is a generating set for G_6 and so $Cay(G_6, S)$ is connected. Set $T = \{cb^j a, cb^j a^{p-1}\}$. Thus, $T^{-1} = \{cb^{q-j}a^{r^{q-j}}, cb^{q-j}a^{r^{q-j}(p-1)}\}$ and $S = T \bigcup T^{-1}$. To prove T and T^{-1} are orbits of $Aut(G_6, S)$ under natural action, we assume that $\sigma_{-1,0} \in Aut(G_6)$. Then $\sigma_{-1,0}(cb^j a) = cb^j a^{p-1}$, $\sigma_{-1,0}(cb^j a^{p-1}) = cb^j a$, $\sigma_{-1,0}(cb^{q-j}a^{r^{q-j}}) = cb^{q-j}a^{r^{q-j}(p-1)}$ and $\sigma_{-1,0}(cb^{q-j}a^{r^{q-j}(p-1)}) = cb^{q-j}a^{r^{q-j}}$. This shows that*

$\sigma_{-1,0} \in Aut(G_6, S)$ and so T and T^{-1} are orbits of $Aut(G_6, S)$. Therefore, $\Gamma = Cay(G_6, S)$ is a connected normal edge−transitive Cayley graph of degree 4 which is not normal arc−transitive, i.e. by Theorems 1.1(3) and 1.1(5), Γ is normal $\frac{1}{2}$−arc−transitive.

Proposition 2.12. *The connected Cayley graph $\Gamma = Cay(G_6, S)$ is normal $\frac{1}{2}$−arc−transitive if and only if $G_6 =< S >$, $|S| > 2$ is an even integer and $S = T \cup T^{-1}$, where $T \subseteq \{cb^j a^i | 0 \leq i \leq p-1\}$ is an orbit of $Aut(G_6, S)$, $1 \leq j \leq q-1$.*

Proof. The proof is similar to Proposition 2.9, and so it is omitted. □

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