www.theoryofgroups.ir

International Journal of Group Theory
ISSN (print): 2251-7650, ISSN (on-line): 2251-7669
Vol. 01 No.2 (2012), pp. 25-38.

www.ui.ac.ir

GROUPS WITH REALITY AND CONJUGACY CONDITIONS

D. CHILLAG, P. LONGOBARDI AND M. MAJ*

Communicated by Alireza Abdollahi

Abstract. Many results were proved on the structure of finite groups with some restrictions on their real elements and on their conjugacy classes. We generalize a few of these to some classes of infinite groups. We study groups in which real elements are central, groups in which real elements are 2-elements, groups in which all non-trivial classes have the same finite size and FC-groups with two non-trivial conjugacy class sizes.

1. **Introduction**

Let G be a group. A real element of G is an element that is conjugate to its inverse.

A periodic group X in which all elements have odd order, contains no non-identity real element. For if $x \in X \setminus \{1\}$ is real and $g \in X$ is such that $x^g = x^{-1} \neq x$, then $g \in N_X(\langle x \rangle) \setminus C_X(\langle x \rangle)$ and $g^2 \in C_X(\langle x \rangle)$. Then $N_X(\langle x \rangle)/C_X(\langle x \rangle)$ contains an involution, so $N_X(\langle x \rangle)$ contains a 2-element.

We study groups with "few" real elements for several meanings of "few". In [5] it is proved that a finite group G in which all real elements lie in $Z(G)$ is a direct product $T \times O$ where O is a subgroup of odd order and T a Sylow 2-subgroup in which all real elements are central. First we show that if G is a periodic group with the same property, then the situation is close but not the same.

Theorem 1.1. *Let G be a periodic group. Then every real element of G lies in $Z(G)$ if and only if one of the following holds:*

(i) $G = D \times O$, where O is a $2'$-subgroup and D a hypercentral 2-subgroup in which all real elements are central.

MSC(2010): Primary: 20E45; Secondary: 20F24, 20F50.

Keywords: conjugacy classes, real elements, FC-groups.

Received: 18 November 2011, Accepted: 01 March 2012.

∗Corresponding author.

(ii) G is a central product, $G = DH$, where D is a hypercentral 2-subgroup in which all real elements are central and H is a non splitting extension of the abelian 2-subgroup $H \cap D$ by a $2'$-subgroup.

Let E be any abelian group. Then any central non-split extension G of E by a $2'$-group B can serve as an example of groups satisfying conclusion (ii). Indeed, by the remark at the beginning, G/E cannot contain real elements, so all real elements of G are in $E \leq Z(G)$. An explicit construction of a family of such extensions can be found in Section 2.

We will see that if we replace "periodic" by "locally finite" in the hypothesis of Theorem 1 then conclusion (i) holds (Corollary 2.7). The same is true if we add "residually finite" to the assumption (Corollary 2.8). A consequence for groups in which all conjugacy classes are finite (FC-groups) is included as well.

A special case of groups in which all real elements are central are the groups in which the square of every conjugacy class is a conjugacy class. Such finite groups are discussed in [5]. We study in Section 2 infinite groups of this type.

The property "all real elements are central" is equivalent to "no two distinct real elements are conjugate". On the other extreme are the groups G in which all non-trivial real elements are conjugate. Iwasaki ([15]) classified such finite groups. We extend this result to periodic FC-groups (Theorem 19). Relaxing the assumption to locally finite groups G in which "all non-trivial real elements are 2-elements", we still can show that G has a normal Sylow 2-subgroup (Theorem 18). These results can be found in Section 3.

In Section 4 we consider groups in which all non-central conjugacy classes have the same finite size. Such groups are necessarily FC-groups. Ito ([13]) proved that such finite groups are direct products $A \times P$, where A is abelian and P a p-group for some prime p. Verardi ([19]) proved that the same is true (with a finite P), for finitely generated groups. Ishikawa proved in [12] that the nilpotence class of a finite p-group with this property is at most 3. Our result is the following:

Theorem 1.2. *Let G be a periodic group in which all non-central conjugacy classes have the same finite size n. Then n is a power of a prime p, and $G = A \times P$, where A is an abelian group and P a nilpotent p-group in which all non-central conjugacy classes have the same finite size n.*

More generally:

Corollary 1.3. *Let G be a group. Assume that all non-central conjugacy classes have the same finite size n. Then n is a power of a prime p and G can be embedded in a direct product $A \times P$, where A is an abelian group and P a nilpotent p-group in which all non-central conjugacy classes have the same finite size n.*

Finally in Section 5 we study FC-groups with two non-trivial finite class sizes. Ito ([14]) proved that finite groups with this property are soluble. More precise results on the structure of such groups have been obtained by Camina ([3]) and Dolfi and Jabara ([8]). We prove the following result:

Theorem 1.4. *Let G be a periodic FC-group with two non-trivial conjugacy class sizes. Then $G = NM$, where N is normal in G and N, M are nilpotent. In particular, G is soluble.*

More generally:

Corollary 1.5. *Let G be an FC-group with two non-trivial conjugacy class sizes. Then G is soluble.*

Most of our notation is standard, taken mainly from [17] (see also [9]). The terms of the upper central series of the group G are denoted by $Z_i(G)$. The conjugacy class of x in the group G is denoted by $cl_G(x)$.

2. Groups in which real elements are central

We denote the class of groups G in which every real element lies in $Z(G)$ by $\mathcal{P}$.

We start by a simple lemma that will be needed.

Lemma 2.1. *Let G be a group and $x, y \in G$.*

(i) If $\left[x^2, y\right] = 1$, then $[x, y]$ is real.

(ii) If $\langle x, y\rangle$ is nilpotent and x and y have relatively prime (finite) orders, then $[x, y] = 1$.

Proof. (i) From $\left[x^2, y\right] = 1$ we get $[x, y]^x [x, y] = 1$ which implies $[x, y]^x = [x, y]^{-1}$.

(ii) This is trivial. □

Proposition 2.2. *Let $G \in \mathcal{P}$ and $x \in G \setminus \{1\}$. Then x is real if and only if x is an involution.*

Proof. Clearly, every involution is real. Conversely, if $x \in G \setminus \{1\}$ is real, then $x \in Z(G)$ and $x^g = x^{-1}$ for some $g \in G$. Thus $x = x^g = x^{-1}$. □

Clearly, every subgroup of a group in $\mathcal{P}$ is also in $\mathcal{P}$. Next we show that if $G \in \mathcal{P}$ then so is $G/Z(G)$.

Proposition 2.3. *Let $G \in \mathcal{P}$. Then $G/Z(G) \in \mathcal{P}$.*

Proof. Let $xZ(G)$ be a real element of $G/Z(G)$. Then $x^g Z(G) = x^{-1}Z(G)$ for some $g \in G$ and consequently $x^g = x^{-1}c$ for some $c \in Z(G)$. Then $x^{g^2} = \left(x^{-1}c\right)^g = \left(x^{-1}\right)^g c = c^{-1}xc = x$ so that $\left[x, g^2\right] = 1$. By Lemma 2.1, $[x, g]$ is real. By definition of $\mathcal{P}$, $[x, g] \in Z(G)$. But $[x, g] = x^{-1}x^g = x^{-2}c$, so $x^2 \in Z(G)$. It follows that $\left[x^2, y\right] = 1$ for all $y \in G$. Again, Lemma 2.1 implies that $[x, y]$ is real. Hence $[x, y] \in Z(G)$ for all $y \in G$, which is the same as $xZ(G) \in Z(G/Z(G))$. □

Let $G \in \mathcal{P}$, we define $E_1(G) = \{x \in G \mid x \text{ is real}\}$. By Proposition 2.2, $E_1(G) = \left\{x \in G \mid x^2 = 1\right\}$ and $G \in \mathcal{P}$ implies $E_1(G) \leq Z(G)$.

Next corollary and its proof are the same as in the finite case (see [5]).

Corollary 2.4. *Let $G \in \mathcal{P}$. Then $G/E_1(G) \in \mathcal{P}$.*

Proof. Let $xE_1(G)$ be a real element of $G/E_1(G)$. Since $E_1(G) \leq Z(G)$ we get that $xZ(G)$ is real in $G/Z(G)$, and Proposition 2.3 implies that $xZ(G) \in Z(G/Z(G))$. It follows by Propositions 2.2 and 2.3 that $x^2 Z(G) = Z(G)$ and $\left[x^2, y\right] = 1$ for all $y \in G$. Lemma 2.1 now implies that $[x, y]$ is real, that is $[x, y] \in E_1(G)$ for all $y \in G$. Thus $xE_1(G) \in Z(G/E_1(G))$. □

Definition 2.5. *Let $G \in \mathcal{P}$. Define $E_2(G)$ by*

$$E_2(G)/E_1(G) = E_1(G/E_1(G))$$

and $E_{i+1}(G)$ by

$$E_{i+1}(G)/E_i(G) = E_1(G/E_i(G))$$

for all $i \in \mathbb{N}$. Let

$$D(G) = \bigcup_{i \in \mathbb{N}} E_i(G).$$

Proposition 2.6. *Let $G \in \mathcal{P}$, $E_i = E_i(G)$ and $D = D(G)$. Then:*

(i) *$G/E_i \in \mathcal{P}$ for all $i \in \mathbb{N}$.*

(ii) *$E_i \leq Z_i(G)$ for all $i \in \mathbb{N}$. So $D \leq Z_\infty(G)$ (the hypercenter of G).*

(iii) *$E_{i+1}/E_i = \{xE_i \in G/E_i \mid \; xE_i$ is real$\} =$*
$= \left\{xE_i \in G/E_i \mid \; (xE_i)^2 = E_i\right\}.$

(iv) *D is a hypercentral 2-subgroup.*

(v) *G/D contains no nontrivial 2-element.*

Proof. (i) By the previous Corollary the result is true for $i = 1$. Assume by induction that $G/E_i \in \mathcal{P}$, then Corollary 2.4 implies that $(G/E_i)/(E_1(G/E_i)) \in \mathcal{P}$.

However $(G/E_i)/(E_1(G/E_i)) = (G/E_i)/(E_{i+1}/E_i) \cong G/E_{i+1}$.

(ii) The result for $i = 1$ follows from the definition of $\mathcal{P}$. Assume by induction that $E_i \leq Z_i(G)$. Since $G/E_i \in \mathcal{P}$ for all $i \in \mathbb{N}$ we get $E_{i+1}/E_i = E_1(G/E_i) \leq Z(G/E_i)$. Let $e_{i+1} \in E_{i+1}$, then $[e_{i+1}, y] \in E_i \leq Z_i(G)$ for all $y \in G$. Thus $e_{i+1}Z_i(G) \in Z(G/Z_i(G)) = Z_{i+1}(G)/Z_i(G)$, so $e_{i+1} \in Z_{i+1}(G)$.

(iii) and (iv). Follow from the definitions and from (ii).

(v) If G/D contains a nontrivial 2-element, it contains an involution, say xD. Then $x^2 \in D$, so $x^2 \in E_i$ for some $i \in \mathbb{N}$ and by definition $x \in E_{i+1} \leq D$. This contradiction proves (v). □

Proof of Theorem 1.1. Suppose $G \in \mathcal{P}$. We will use the notation of the previous proposition. Let $E_i = E_i(G)$ and $D = D(G) \leq Z_\infty(G)$. Since G is periodic, G/D is a $2'$-group. In particular, every 2-element of G lies in D.

Let O be the set of all elements of G of odd order. We claim that $O \subseteq C_G(D)$. To see that, let $y \in O$ and $d \in D$. Then $d \in E_i$ for some $i \in \mathbb{N}$, so $d \in Z_i(G)$. Therefore $\langle y, d\rangle \leq \langle y\rangle Z_i(G)$. Since $\langle y\rangle Z_i(G)$ is nilpotent, so is $\langle y, d\rangle$. By Lemma 2.1 $[y, d] = 1$.

Set $H = \langle O \rangle$, then $G = HD$ and $[H, D] = 1$.

If O is a subgroup then $G = DO = D \times O$, so assume that O is not a subgroup. Thus $H \cap D \le Z(G) \cap H \le Z(H)$ and $H/(H \cap D)$ is a $2'$-group. If H does not split over $H \cap D$, then conclusion (ii) holds. If H does split over $H \cap D$ then $H = (H \cap D) \times X$ for some subgroup X consisting of elements of odd order. So $G = D \times X$ and $O = X$ is a subgroup, a contradiction.

For the converse, assume that G satisfies the conclusions. Then $G = KD$ with $D \in \mathcal{P}$ a 2-group, $[K, D] = 1$ and G/D a $2'$-group ($K = O$ in (i) and $K = H$ in (ii)). Let $x \in G$ be real and $g \in G$ be such that $x^g = x^{-1}$. By the second paragraph of this paper, $x \in D$. Write $g = kd$, where $k \in K$ and $d \in D$. Then $x^g = x^d = x^{-1}$ and hence x is real in D. Since $D \in \mathcal{P}$, $x \in Z(D) \le Z(G)$ as required. □

Corollary 2.7. *Let G be a locally finite group. Then every real element lies in $Z(G)$ if and only if $G = D \times O$, where O is a $2'$-subgroup and D a hypercentral 2-subgroup in which every real element is in $Z(D)$.*

Proof. Assume that every real element of G lie in $Z(G)$. In the notation of the proof of Theorem 1, we need to show that O is a subgroup. Let $x, y \in G$ be of odd order. Then $A = \langle x, y \rangle$ is finite and $A \le \langle O \rangle \le C_G(D)$. Since D contains all 2-elements of G, $D \cap A$ is a Sylow 2-subgroup of A. By the Schur Zassenhaus Theorem, $A = (D \cap A)V$ is a semidirect product, where V is a subgroup of odd order. Consequently $A = (D \cap A) \times V$. Hence V contains all the odd order elements of A, in particular $x, y \in V$. So $xy \in V$ and $xy \in O$ as required.

For the converse, we argue as in the proof of Theorem 1.1. □

Corollary 2.8. *Let G be a periodic residually finite group. Then every real element lies in $Z(G)$ if and only if $G = D \times O$, where O is a $2'$-subgroup and D a hypercentral 2-subgroup in which every real element is in $Z(D)$.*

Proof. Assume that all real elements of G lie in $Z(G)$. Again, using the notation of the proof of Theorem 1, we take $x, y \in O$ and set $A = \langle x, y \rangle$. Then $A \le \langle O \rangle \le C_G(D)$. Write $u = xy$.

Let $u = u_2 \times u_o$ be the factorization of u into its 2-part u_2 and its $2'$-part u_o. We need to show that $u_2 = 1$. Assume the contrary, that $u_2 \neq 1$. Then G contains a normal subgroup N of finite index such that $\overline{x} \neq 1$, $\overline{y} \neq 1$ and $\overline{u_2} \neq 1$, here we denote images modulo N by bars. Clearly $\overline{u} = \overline{u_2} \times \overline{u_o} = \overline{xy}$. It follows that $\overline{u}$ is not of odd order. Since D contains all 2-elements of G, $\overline{D}$ is a normal Sylow 2-subgroup of the finite group $\overline{G}$, and $\overline{D} \cap \overline{A}$ is a normal Sylow 2-subgroup of $\overline{A}$. Moreover, $\overline{A} \le C_{\overline{G}}\left(\overline{D}\right)$. As in the previous proof $\overline{A} = \left(\overline{D \cap A}\right) \times \overline{V}$, where $\overline{V}$ is a subgroup of odd order, containing all odd order elements of $\overline{A}$. In particular $\overline{x}, \overline{y} \in \overline{V}$. Then $\overline{u} \in V$, so $\overline{u}$ has odd order, a contradiction.

For the converse, we argue as in the proof of Theorem 1.1. □

Corollary 2.9. *Let G be an FC-group. Then every real element of G belongs to $Z(G)$ if and only if G is a subgroup of a direct product $O \times D \times A$, where D is an FC 2-group, hypercentral and such that every real element of D is in $Z(D)$, O is a $2'$-group and A is an abelian torsion-free group.*

Proof. Suppose that all real elements of G lie in $Z(G)$. First assume that G is periodic. Then, by Theorem 1, there exists a normal 2-subgroup D of G such that G/D is a $2'$-group. Then, by a Theorem of Černikov ([18], Theorem 5.25, p. 100), D has a complement in G, and G has the structure in (i) of Theorem 1.

Now assume that G is any FC-group in $\mathcal{P}$. Let H be a maximal torsion-free subgroup of $Z(G)$. Then G/H is periodic and $G' \cap H = 1$ (see [18], p. 4). Consider the subgroup $K = H^2$. Then H/K is periodic, thus G/K is periodic. Moreover, K is torsion-free, thus $K \cap G' = 1$. Hence G is isomorphic to a subgroup of $G/G' \times G/K$. Once we prove that G/K is in $\mathcal{P}$, then G will have the required structure.

In fact, let aK be a real element in G/K. Then $a^x K = a^{-1}K$ for some $x \in G$. Then $a^x = a^{-1}c$, with $c \in K = H^2$, and $c = d^2$, with $d \in H$. Then we have $(ad^{-1})^x = a^x d^{-1} = a^{-1}d^2 d^{-1} = a^{-1}d = (ad^{-1})^{-1}$. Thus ad^{-1} is a real element in G, then $ad^{-1} \in Z(G)$, therefore $a \in Z(G)$ and $aK \in Z(G/K)$, as required.

Conversely, suppose that G is a subgroup of a direct product $O \times D \times A$, where O, D and A satisfy the hypothesis of the theorem. Let a be a real element of G. Then $a = a_1a_2a_3$, with $a_1 \in O$, $a_2 \in D$, $a_3 \in A$. There exists $x \in G$ such that $a^x = a^{-1}$ and again we have $x = x_1x_2x_3$, with $x_1 \in O$, $x_2 \in D$, $x_3 \in A$. Then we have $a^{-1} = a_1^{-1}a_2^{-1}a_3^{-1} = a^x = a_1^{x_1}a_2^{x_2}a_3^{x_3}$, thus $a_1^{x_1} = a_1^{-1}$, $a_2^{x_2} = a_2^{-1}$, $a_3^{x_3} = a_3^{-1}$, then $a_1 = 1$, since O has no real elements, $a_3 = 1$, since A is an abelian torsion-free group, and $a_2 \in Z(D)$. Therefore $a = a_2 \in Z(G)$, as required. □

Next is an example of a group $G \in \mathcal{P}$ satisfying (ii) of the Theorem 1 (that is H does not split over $D \cap H$).

Example. Let $B = B\,(m,n)$ be the free Burnside group with m generators and odd exponent $n > 10^{10}$. By Corollary 1 of [2], the Schur multiplier $M\,(B)$ is a free abelian group of infinite rank. Let E be an abelian 2-group and consider E as a trivial B-module. Then $Hom\,(M\,(B)\,,E) \neq 0$. By the Universal Coefficients Theorem (see e.g. [17] , 11.4.18, p. 336), we get $H^2\,(B,E) \neq 0$. Thus there is a central non splitting extension G of E by B. By the comment after the statement of Theorem 1, $G \in \mathcal{P}$.

The investigation of groups in $\mathcal{P}$ with the above finiteness conditions "reduces" the problem to the study of hypercentral 2-groups in $\mathcal{P}$. Such finite 2-groups are described in proposition 4.7 of [5]. An easy corollary is that the same statement holds for any hypercentral 2-group. Namely:

Corollary 2.10. *Let G be a hypercentral 2-group. Then the following conditions are equivalent:*

(i) $x^2 = y^2$ *holds if and only if* $(xy^{-1})^2 = 1$ *holds.*

(ii) *all involutions are central, and G contains no subgroup isomorphic to one of the groups*

$$< x, y \mid \; x^4 = y^{2^n} = 1, y^{-1}xy = x^{-1} >, n \geqslant 2,$$

or the quaternion group of order 8.

(iii) *every real element of* G *is central.*

Proof. Since G is hypercentral, it is locally nilpotent. Let H be a finitely generated subgroup of G. Then H is nilpotent, it is also periodic, thus H is finite. Hence G is locally finite. We claim that each of three conditions are local, namely, they hold in G if and only if they holds in any finitely generated subgroup of G. Once we show that, the corollary follows from 4.7 of [5], since all finitely generated subgroups are finite.

Clearly, each of the three conditions is inherited by finitely generated subgroups. Condition (i) and the second phrase of condition (ii), are stated within the finitely generated subgroup $\langle x, y\rangle$. So they are local. Suppose that condition (iii) holds in every finitely generated subgroup of G. Let $x \in G$ be real, and let $g \in G$ be such that $x^g = x^{-1}$. Then x is real in $\langle x, g\rangle$ and so x is central in $\langle x, g\rangle$ forcing $[x, g] = 1$ and $x = x^{-1}$. Therefore x is real in $\langle x, u\rangle$ for all $u \in G$. Hence x is central in $\langle x, u\rangle$, implying $[x, u] = 1$. Thus $x \in Z(G)$. So all three conditions are local. $\square$

As we have seen, if G is a periodic group in $\mathcal{P}$, then every odd order element commutes with every 2-element. When G is not periodic it is possible that a torsion-free element does not commute with all 2-elements.

Example. Let $< a_n >$ be a cyclic group of order 2^n generated by a_n, and $E = Dr_{n\in\mathbb{N}} < a_n >$. Let $x \in Aut\ E$ defined by $a_1^x = a_1$, $a_n^x = a_n a_{n-1}$, for any $n \in \mathbb{N} \setminus \{1\}$. It follows that $(a_n)^{x^{2^{n-1}}} = a_n$ and $(a_n)^{x^{2^{n-2}}} \neq a_n$. Let $G = E \rtimes < x >$. Then $G \in \mathcal{P}$, G is metabelian and no power of x centralizes E.

Example. For any $t \geqslant 3$ consider the 2-adic integer $y_t = (k_{t,n})_{n\geqslant 1}$, defined as follows:

$$k_{t,1} = \cdots = k_{t,t-1} = 1, k_{t,j} = k_{t,j-1} + 2^{j-1}, \forall j \geqslant t.$$

The group $G = \mathbb{Z}(2^\propto) \rtimes < y_t \mid t \geqslant 3 >$ is in $\mathcal{P}$ and $\mathbb{Z}(2^\propto) = \mathbb{Z}_\propto(G)$ is the set of all periodic elements of G.

For a subset A of a group, we define

$$A^2 = \{ab | a, b \in A\}.$$

A special case of groups in $\mathcal{P}$ are the groups in which the square of every conjugacy class is a conjugacy class.

By [5] (remarks 5.4 and 5.5), if G is such a finite group, then G is nilpotent with an abelian Sylow 2-subgroup. Using similar methods, we show next:

Theorem 2.11. *Let* G *be an FC-group in which the square of any conjugacy class is a conjugacy class. Then* G *is hypercentral. Furthermore, if* G *is also periodic, then* $G = O \times D$*, where* O *is a* $2'$*-group and* D *an abelian* 2*-group.*

The proof follows from the next Lemma.

Lemma 2.12. *Let G be a group in which the square of any conjugacy class is a conjugacy class. Then:*

(i) *The assumption is equivalent to : $(cl_G(x))^2 = cl_G(x^2)$ for all $x \in G$.*

(ii) *Every real element lies in $Z(G)$.*

(iii) *Every 2-element lies in $Z(G)$.*

(iv) *If X is a homomorphic image of G, then the square of any conjugacy class of X is a conjugacy class.*

(v) *Assume that G is an FC-group and $x \in G$ is of odd order. Set $C = cl_G(x)$, then $x^{-1}C$ is a finite normal subgroup of G.*

Proof. (i) This is true as $cl_G(x^2)$ is always contained in $(cl_G(x))^2$.

(ii) Let x be real in G and $g \in G$ be such that $x^g = x^{-1}$. Put $C = cl_G(x)$. Then $xx^g = 1 \in C^2$. Since C^2 is a conjugacy class, $C^2 = \{1\}$. As x is real, $x^{-1} \in C$. Then $x^{-1}x^h \in C^2 = \{1\}$ and $x^h = x$ for all $h \in G$. Thus $x \in Z(G)$.

(iii) Let $x \in G$ be of order 2^n and set $C = cl_G(x)$. We use induction on n. If $n = 1$, then $x \in Z(G)$, by (ii). So $n > 1$ and by induction $x^2 \in Z(G)$. Then $C^2 = cl_G(x^2)$, since C^2 is a conjugacy class. So $C^2 = \{x^2\}$ and hence $xx^y = x^2$ for all $y \in G$. Hence $x = x^y$ for all $y \in G$ as required.

(iv) Let N be a normal subgroup of G with $X = G/N$. Let $xN \in X$ and $x^g x^h N \in (cl_X(xN))^2$, where $g, h \in G$. By (i), $x^g x^h = (x^2)^u$ for some $u \in G$ and so $x^g x^h N = (x^2)^u N \in cl_X(x^2N)$. Thus $(cl_X(xN))^2 \leq cl_X(x^2N)$ and the equality follows.

(v) By (i), $C^2 = cl_G(x^2)$ and as $o(x)$ is odd, $C_G(x) = C_G(x^2)$. Therefore

$$|C| = |G : C_G(x)| = |G : C_G(x^2)| = |C^2|$$

So $C^2 = xC$. Write $H = x^{-1}C$, then

$$HH = x^{-1}Cx^{-1}C = x^{-2}C^{x^{-1}}C = x^{-2}C^2 = x^{-2}xC = x^{-1}C = H.$$

Therefore H is as subgroup of G. Let $g, g_1 \in G$. Then $[x, g_1g] = [x, g][x, g_1]^g$, thus $(x^{-1}x^{g_1})^g \in H$. Therefore $H^g \subseteq H$ for any $g \in G$ and $H \lhd G$. Clearly $|H| = |C|$ is finite. $\square$

Proof of Theorem 16. By (iii) of Lemma 2.12, G has a normal central (abelian) Sylow 2-subgroup D containing all 2-elements of G. So if G is also periodic then $G = O \times D$, where O is a $2'$-group, by a Theorem of Černikov ([18], Theorem 5.25, p. 100). So we need to show that G is hypercentral. It suffices to show that $U = G/Z(G)$ is hypercentral. We know that U is periodic ([18], Theorem 1.4, p. 4) and as $D \leq Z(G)$, U is a $2'$-group. By (iv) of Lemma 2.12, we conclude that U is a periodic FC $2'$-group in which the square of every conjugacy class is a conjugacy class. The same is true for any non-trivial homomorphic image X of U. We claim that $Z(X) \neq 1$. Once we show this, U will be hypercentral (see [7], Lemma 1.2.3, p. 9).

Let $x \in X \setminus \{1\}$ and $C = cl_X(x)$. As x is of odd order, (v) of Lemma 2.12 implies that $H = x^{-1}C$ is a finite normal subgroup of X. Let N be a minimal normal subgroup of X contained in H, and consider $a \in N \setminus \{1\}$. Then again, $a^{-1} \cdot cl_X(a)$ is a normal subgroup of X contained in N. So $a^{-1} \cdot cl_X(a)$ is either N or $\{1\}$. In the first case $a^{-2} \in N = a^{-1} \cdot cl_X(a)$, so $a^{-2} = a^{-1}a^g$ for some $g \in X$. Hence a is real and so $a \in Z(X)$ (lemma 2.12, (ii)). If $a^{-1} \cdot cl_X(a) = \{1\}$, then $a = a^c$ for all $c \in X$. Again, $a \in Z(X)$, as needed. □

3. Groups in which real elements are 2-elements

As mentioned in the Introduction, the other extreme of "all real elements are central" is "all non-trivial real elements are conjugate". Iwasaki ([15]) proved that if G is such a finite group, then G has a normal Sylow subgroup that is either homocyclic or a Suzuki 2-group $A(n,\theta)$, where $A(n,\theta) = \{(a,b) \mid a,b \in F_{2^n}\}, (a,b)(a',b') = (a+a', b+b'+aa'^{\theta})$, where F_{2^n} is the field with 2^n elements and θ is an automorphism of F_{2^n} of odd order .

If G is a periodic group with all non-trivial real elements conjugate, then obviously all non-trivial real elements are involutions. In this section we consider such groups and a somewhat more general case, of groups in which real elements are 2-elements.

Theorem 3.1. *Let G be a locally finite group in which all real elements are 2-elements. Then G has a normal Sylow 2-subgroup.*

Proof. Assume first that G is finite. Let X be a subgroup of G of odd order. We claim that $(N_G(X))/(C_G(X))$ has odd order. If not, then there exists $x \in N_G(X) \setminus C_G(X)$ such that $x^2 \in C_G(X)$. Then $[a, x^2] = 1$ for all $a \in X$. Then (i) of Lemma 2.1 implies that $[a,x]$ is real, so by assumption $[a,x]$ is a 2-element. However, $x \in N_G(X)$ so $[a,x] \in X$, forcing $[a,x] = 1$ for all $a \in X$. Thus $x \in C_G(X)$, a contradiction. So $|(N_G(X))/(C_G(X))|$ is odd, and G is a so-called $2'$-homogeneous group. By a theorem of Arad ([1], Lemma 2.6, p. 2) G is 2-closed, that is G has a normal Sylow 2-subgroup.

Now let G be any locally finite group. If a,b are 2-elements of G, consider the subgroup $H = \langle a,b \rangle$. Then H is finite and obviously all non-trivial real elements of H are 2-elements. Hence H has a normal Sylow 2-subgroup D. Thus $a,b \in D, ab \in D$, and ab is a 2-element. Therefore all 2-elements of G form a subgroup, as required. □

Theorem 3.2. *Let G be a periodic FC-group in which all non-trivial real elements are conjugate. Then $G = SO$, where O is a $2'$-subgroup and S is a normal Sylow 2-subgroup, which is either a finite homocyclic 2-group, or a finite Suzuki group, or the direct product of finitely many Prüfer 2-groups.*

Proof. All non-trivial real elements of G are involutions. Since G is a periodic FC-group, it is locally finite, so it has a normal Sylow 2-subgroup S, by the previous theorem. Then $G = SO$, where O is a $2'$-subgroup, by a Theorem of Černikov ([18], Theorem 5.25, p. 100). Assume $S \neq \{1\}$.

Now we show that S has the required structure.

Let $t \in G$ be an involution, and write $cl_G(t) = \{t, t^{g_1}, t^{g_2}, \ldots, t^{g_k}\}$ for some $g_1, g_2, \ldots, g_k \in G$. Let $N = cl_G(t) \cup \{1\}$. If $a,b \in N$, then ab is real and so $ab \in N$. It follows that N is a normal elementary abelian subgroup of G. Moreover, N contains all the involutions of G. Set $H = N\langle g_1, g_2, \ldots, g_k\rangle$ and let X be any finite 2-subgroup of G. Then $\langle H, X\rangle$ is a finite group in which all non-identity real elements are conjugate. By [15], a Sylow 2-subgroup of $\langle H, X\rangle$ is either abelian and homocyclic, or it is a Suzuki 2-group. If the Sylow 2-subgroup S of G is finite, then we can take $X = S$, thus S is a Sylow 2-subgroup of $\langle H, X\rangle$, and we have the result.

Now assume S infinite. First we prove that S is abelian. To see this, we recall that the order of a Suzuki 2-group T is less or equal to $|\{x \in T \mid x^2 = 1\}|^3$ ([11], theorem 7.9, p. 313). Let X be a non-abelian finitely generated subgroup of S, then a Sylow 2-subgroup Y of $\langle H, X\rangle$ is a Suzuki 2-group, thus $|X| \leq |Y| \leq |N|^3$; thus every finitely generated subgroup of S containing X and of order $> |N|^3$ is abelian, a contradiction.

Hence S is an infinite abelian 2-group. Moreover, every finitely generated subgroup of S has at most $|N|$ elements of order 2, then it has rank $\leq |N|$. Hence S is an abelian 2-group of rank $\leq |N|$. Therefore S is a direct product of finitely many (at most $|N|$) cyclic or quasicyclic 2-groups. Finally all the direct factors of S are quasicyclic, since every Sylow 2-subgroup of a finitely generated subgroup of G is homocyclic. □

4. **Groups in which all non-central conjugacy classes have the same finite size**

Proof of Theorem 1.2. Clearly G is a BFC-group, since the conjugacy classes are of bounded sizes. A theorem of B.H Neumann ([16]) states that G' is finite. Now a theorem of P.Hall ([10]) implies that $G/Z_2(G)$ is finite.

Assume first that $G/Z(G)$ is finite. Write $G/Z(G) = \bigcup_{i=1}^{k} g_i Z(G)$, for some $g_i \in G$ and $k \in \mathbb{N}$. Let $H = \langle g_1, g_2, \ldots, g_k\rangle$. Then H is finite and $G = Z(G)H$. Hence $cl_H(h) = cl_G(h)$, and so all non-central conjugacy classes of H have the same finite size n. By [13], $H = B \times P$ where B is an abelian subgroup and P a (finite) p-group, and n is a power of p. Then $G = Z(G)(B \times P)$ and $B \leq Z(G)$. Thus $G = Z(G)P$. Let Q be the set of all p-elements of $Z(G)$, then QP is a normal nilpotent Sylow p-subgroup of G. The conclusion now follows from a theorem of Černikov ([18], Theorem 5.25, p. 100).

Thus we assume that $G/Z(G)$ is not finite. Therefore $Z(G) < Z_2(G)$. Let $x \in Z_2(G) \setminus Z(G)$ be an element of minimal order. Then $x^p \in Z(G)$ for some prime p. Moreover, $G' \leq C_G(x)$, so that $C_G(x) \lhd G$. For every $y \in G$, we have that $[x, y] \in Z(G)$, which implies $1 = [x^p, y] = [x, y]^p = [x, y^p]$. Thus $y^p \in C_G(x)$.

Suppose that $(o(y), p) = 1$, then $y \in C_G(x)$. Consequently $G/C_G(x)$ is a p-group and so $n = |G : C_G(x)|$ is a power of p. Moreover, we claim that $y \in Z(C_G(x))$. This is clear if $xy \in Z(G)$. So assume that $xy \notin Z(G)$. Then $x = (xy)^{o(y)m}$ for some positive integer m. It follows that $C_G(xy) \leq C_G(x)$. Since $|G : C_G(xy)| = |G : C_G(x)|$, we get $C_G(xy) = C_G(x)$. Hence $C_G(x) \leq C_G(y)$, as claimed. Let S be the set of all p'-elements of $Z(C_G(x))$. Then S is an abelian normal subgroup of G. As $G/C_G(x)$ is a finite p-group, S is a Hall p'-subgroup of G.

Let $a \in G \setminus Z(G)$. Then $|G : C_G(a)|$ is a power of p, therefore the finite number $|C_G(a)\, S : C_G(a)| = |S : S \cap C_G(a)|$ is a power of p. Since $S/(S \cap C_G(a))$ is a finite p'-group, we get that $S \leq C_G(a)$ and consequently $S \leq Z(G)$. Now Černikov's theorem implies that $G = P \times S$, where P is a p-group. Since $(PZ_2(G))/Z_2(G)$ is a finite p-group, P is nilpotent. □

Proof of Corollary 1.3. Assume that all non-central conjugacy classes have the same finite size n. Let H be a maximal torsion-free subgroup of $Z(G)$. Then G/H is periodic and $G' \cap H = 1$ (See [18], p. 4). Hence G is isomorphic to a subgroup of $G/G' \times G/H$.

For any $x \in G$, we have $H \leq C_G(x)$, so $C_G(x)/H \leq C_{G/H}(xH)$. Conversely, if $yH \in C_{G/H}(xH)$, then $[y, x] \in G' \cap H = 1$. Therefore $yH \in C_G(x)/H$. Thus $C_{G/H}(xH) = C_G(x)/H$ and so $\left|cl_{G/H}(xH)\right| = |G/H : C_G(x)/H| = |G : C_G(x)|$. It follows that G/H is a periodic FC-group in which all non-central conjugacy classes have the same finite size. By Theorem 1.2, $G/H = A \times P$ where A is abelian and P a nilpotent p-group. The corollary follows. □

5. Groups with two non-trivial conjugacy class sizes

Proof of Theorem 1.4. Let $n_1, n_2 > 1$ be the two non-trivial sizes. Obviously G is a BFC-group, thus, arguing as in the proof of Theorem 1.2, we obtain that $G/Z_2(G)$ is finite.

If $G/Z(G)$ is finite, then, arguing as in the proof of Theorem 1.2 we can write $G = Z(G)H$, where H is a finite group with two non-trivial conjugacy class sizes. Then the result follows from a result of Dolfi and Jabara ([8]).

Then we can assume that $G/Z(G)$ is infinite.

If G is abelian by finite, then there exists a normal abelian subgroup A of G, with $G/A = \{y_1A, \cdots, y_nA\}$, then the subgroup $A \cap C_G(y_1) \cap \cdots \cap C_G(y_n)$ is contained in $Z(G)$ and it is of finite index in G. Then we have $G/Z(G)$ finite, a contradiction.

Hence we can assume that G is not abelian by finite.

If $Z(G) = Z_2(G)$, then $G/Z(G)$ is finite, a contradiction. Thus $Z(G) \subset Z_2(G)$. Let $a \in Z_2(G) \setminus Z(G)$ such that $a^p \in Z(G)$, where p is a prime. Then $1 = [a^p, x] = [a, x]^p = [a, x^p]$, for any $x \in G$. In particular $[a, b] = 1$, for any p'-element b. Moreover $G' \subseteq C_G(a)$, since $a \in Z_2(G)$, then $C_G(a)$ is a normal subgroup of G. Hence $G/C_G(a)$ is a finite p-group, write $|G/C_G(a)| = p^\alpha$.

First we show that $Z_2(G)/Z(G)$ is a p-group.

Assume not, let $c \in Z_2(G) \setminus Z(G)$ with $c^q \in Z(G)$, where q is a prime different from p. Then, arguing as before, we obtain that every p-element of G commutes with c, moreover $C_G(c)$ is normal in G and $G/C_G(c)$ is a finite q-group, write $|G/C_G(c)| = q^\beta$. Consider the element ac, then $(ac)^p = c^p z$, where $z \in Z(G)$, thus $C_G(ac) \subseteq C_G(c^p) = C_G(c)$, similarly, from $(ac)^q = a^q s$, where $s \in Z(G)$, we obtain that $C_G(ac) \subseteq C_G(a)$. Then $|G : C_G(ac)| = |G : C_G(a)||C_G(a) : C_G(ac)|$, thus p divides $|G : C_G(ac)|$. Similarly $|G : C_G(ac)| = |G : C_G(c)||C_G(c) : C_G(ac)|$, thus q divides $|G : C_G(ac)|$. Then $ac \in Z(G)$, therefore $G = C_G(ac) \subseteq C_G(a)$ and $a \in Z(G)$ a contradiction.

Therefore $Z_2(G)/Z(G)$ is a p-group.

Hence, for any $b \in Z_2(G), x \in G$, we have $[b^{p^\gamma}, x] = 1$, for some γ. Then $1 = [b,x]^{p^\gamma} = [b, x^{p^\gamma}]$, thus every element of $Z_2(G)$ commutes with every p'-element.

If G is a p-group, then $G/Z_2(G)$ is a finite p-group, then it is nilpotent and G is nilpotent. So we may assume that $G/Z_2(G)$ is not a p-group.

If all p'-elements commute, then they form an abelian normal subgroup S, obviously S is normal in G and G/S is a p-group, thus $(G/S)/(Z_2(G)S/S)$ is a finite p-group, then it is nilpotent and then G/S is nilpotent. Moreover $G = S \rtimes P$ for some subgroup P, by a theorem of Černikov, and we have the result.

Then we may assume that there exist in G two p' elements y and y_1 which not commute.

If $C_G(a_1y) = C_G(y)$ for any $a_1 \in Z_2(G)$, then we have $Z_2(G) \subseteq C_G(y) = C_G(a_1y)$ and then $Z_2(G) \subseteq C_G(a_1)$, for every $a_1 \in Z_2(G)$, thus $Z_2(G)$ is abelian, and G is abelian by finite, a contradiction.

Therefore there exists $d \in Z_2(G)$ such that $C_G(dy) \subset C_G(y)$. If $C_G(dy) = C_G(d)$, then we have $y_1 \in C_G(d) = C_G(dy)$, and y_1 commutes with y, a contradiction.

Thus we have that $C_G(dy) \subset C_G(y), C_G(dy) \subset C_G(d)$.

Then $d, y \notin Z(G)$, and $|G : C_G(dy)| \neq |G : C_G(d)|, |G : C_G(dy)| \neq |G : C_G(y)|$, then $|G : C_G(d)| = |G : C_G(y)|$ and we can assume $n_1 = |G : C_G(d)| = |G : C_G(y)|, n_2 = |G : C_G(dy)|$.

Now we prove that n_1 and n_2 are both power of p.

We have that $d \in Z_2(G)$ is a p-element modulo $Z(G)$, y is a p'-element, and d commutes with all p'-elements of G. This implies that $C_G(dy) = C_G(d) \cap C_G(y)$ and that $G' \leq C_G(d)$. In particular $C_G(d)$ is normal in G. Moreover, $|G : C_G(y)| = |G : C_G(d)| = n_1$ is a p-power, and therefore so is $|C_G(y) : C_G(d) \cap C_G(y)| = |C_G(d)C_G(y) : C_G(d)|$. It follows that $n_2 = |G : C_G(dy)| = |G : C_G(d) \cap C_G(y)|$ is a power of p as well.

Then we have that both n_1 and n_2 are power of p. Then $|G : C_G(g)|$ is a power of p, for any $g \in G$.

Now we can show that G is nilpotent, and the result will follows.

In fact, since $G/Z_2(G)$ is finite, we can write $G = Z_2(G)H$, where H is a finite normal subgroup of G (see for example Lemma 1.3 of [18]). Then, arguing as in Lemma 1.1 of [6], $|cl_H(x)|$ divides $|cl_G(x)|$, for any $x \in H$. Thus H is a finite group with all class sizes that are power of p, then H is nilpotent (see Lemma 1 of [4], or Proposition 4 of [6]) and G is nilpotent, as required. □

Proof of Corollary 1.5. Let H be a maximal torsion-free subgroup of $Z(G)$. Then G/H is periodic and $G' \cap H = 1$ (see [18], p. 4). Hence G is isomorphic to a subgroup of $G/G' \times G/H$.

For any $x \in G$, we have $H \leq C_G(x)$, so $C_G(x)/H \leq C_{G/H}(xH)$. Conversely, if $yH \in C_{G/H}(xH)$, then $[y,x] \in G' \cap H = 1$. Therefore $yH \in C_G(x)/H$. Thus $C_{G/H}(xH) = C_G(x)/H$ and so $\left|cl_{G/H}(xH)\right| = |G/H : C_G(x)/H| = |G : C_G(x)|$. It follows that G/H is a periodic FC-group with two non-trivial conjugacy class sizes. Thus G/H is soluble, by Theorem 1.4, and the result follows. □

6. **Final remarks**

It was shown in [5] (theorem 1.5, p. 2043), that if G is a finite group in which all non-real elements are central, then either G is abelian, or a real group, or a 2-group. Using similar arguments it can be shown that the same holds for periodic groups and FC-groups.

Real groups can have complicated structure. Theorems of P. Hall, S.V. Ivanov and G. Higman, B.H. Neumann and H. Neumann, show that many families of groups can be embedded in groups in which every two elements of the same order are conjugate.

Acknowledgments

The first author wishes to thank the University of Salerno and the INdAM - GNSAGA for the hospitality and support.

References

[1] Z. Arad, π-homogeneity and π'-closure of finite groups, *Pacific J. Math.* **51**(1974), 1-9.

[2] I.S. Ashmanov and A.Yu. Ol'shanskij, On abelian and central extensions of aspherical groups, *Sov. Math.* **29** (1985), 65-82.

[3] A.R. Camina, Finite groups of conjugate rank 2, *Nagoya Math. J.* **53** (1974), 47-57.

[4] A.R. Camina, Arithmetical conditions on the conjugacy class numbers of a finite group, *J. London Math. Soc.* (2) **5** (1972), 127-132.

[5] D. Chillag and A. Mann, Nearly odd-order and nearly real finite groups, *Comm. Algebra* **26** (1998), 2041-2064.

[6] D. Chillag and M. Herzog, On the Length of the Conjugacy Classes of finite Groups, *J. Algebra* **131** (1990), 110-125.

[7] M.R. Dixon, *Sylow Theory, Formations and Fitting Classes in Locally Finite Groups*, World Scientific, Singapore, 1994.

[8] S. Dolfi and E. Jabara, The structure of finite groups of conjugate rank 2, *Bull. London Math. Soc.* **41** (5) (2009), 916-926.

[9] D. Gorenstein, *Finite Groups*, Harper & Row, New York, 1968.

[10] P. Hall, Finite-by-nilpotent groups, *Proc. Cambridge Philos. Soc.* **52** (1956), 611-616.

[11] B. Huppert and N. Blackburn, *Finite Groups II*, Springer, Berlin, 1982.

[12] K. Ishikawa, On finite p-groups which only two conjugacy lengths, *Isr. J. Math.* **129** (2002), 119-123.

[13] N. Ito, On finite groups with given conjugate types I, *Nagoya Math. J.* **6** (1953), 17-28.

[14] N. Ito, On finite groups with given conjugate types II, *Osaka J. Math.* **7** (1970), 231-251.

[15] S. Iwasaki, On finite groups with exactly two real conjugate classes, *Arch. Math. (Basel)* **33** (1979/80), 512-517.

[16] B.H. Neumann, Groups with finite classes of conjugate subgroups, *Math. Z.* **63** (1955), 76-96.

[17] D.J.S. Robinson, *A course in Group Theory*, Springer, Berlin, 1982.

[18] M.J. Tomkinson, *FC-groups*, Pitman, London, 1984

[19] L. Verardi, On groups whose noncentral elements have the same finite number of conjugates, *Boll. Un. Mat. Ital. A* (7) **2** (1988), 391-400.

David Chillag

Patrizia Longobardi
Dipartimento di Matematica, Università di Salerno, Via Ponte don Melillo 84084, Fisciano (SA), Italy
Email: plongobardi@unisa.it

Mercede Maj
Dipartimento di Matematica, Università di Salerno, Via Ponte don Melillo 84084, Fisciano (SA), Italy
Email: mmaj@unisa.it