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HOMOGENOUS FINITARY SYMMETRIC GROUPS

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ABSTRACT. We characterize strictly diagonal type of embeddings of finitary symmetric groups in terms of cardinality and the characteristic. Namely, we prove the following. Let κ be an infinite cardinal. If $G=\bigcup_{i=1}^{\infty} G_i$, where $G_i \cong FSym(\kappa n_i)$, ($H=\bigcup_{i=1}^{\infty} H_i$, where $H_i \cong Alt(\kappa n_i)$), is a group of strictly diagonal type and $\xi=(p_1,p_2,\dots)$ is an infinite sequence of primes, then G is isomorphic to the homogenous finitary symmetric group $FSym(\kappa)(\xi)$ (H is isomorphic to the homogenous alternating group $Alt(\kappa)(\xi)$), where $n_0=1$, $n_i=p_1p_2\cdots p_i$.

1. Introduction

Let κ be an arbitrary infinite cardinal number. Let $FSym(\kappa)$ denote the finitary symmetric group and $Alt(\kappa)$ denote the alternating group on the set κ. Let Π be a set of sequences of prime numbers and $\xi \in \Pi$. Then ξ is a sequence of not necessarily distinct primes. Let $\alpha \in FSym(\kappa)$, $(Alt(\kappa))$. For a natural number $p \in \mathbb{N}$, a permutation $d^p(\alpha) \in FSym(\kappa p)$ defined by $(\kappa s+i)^{d^p(\alpha)}=\kappa s+i^{\alpha}$, $\quad i \in \kappa$ and $0 \le s \le p-1$, is called a *homogeneous p-spreading* of the permutation α. We divide the ordinal κp into p equal parts and on each part we repeat the permutation diagonally as in the finite case. So if $\alpha=\begin{pmatrix} 1\dots\ n \\ i_1\dots i_n\end{pmatrix} \in FSym(\kappa)$, then the homogeneous $p-$spreading of the permutation α is

$$d^p(\alpha)=\left(\begin{array}{ccc|ccc|c|ccc} 1 & \dots & n & \kappa+1 & \dots & \kappa+n & \dots & \kappa(p-1)+1 & \dots & \kappa(p-1)+n \\ i_1 & \dots & i_n & \kappa+i_1 & \dots & \kappa+i_n & \dots & \kappa(p-1)+i_1 & \dots & \kappa(p-1)+i_n \end{array}\right)$$

with the assumption that the elements in $\kappa p \setminus supp(d^p(\alpha))$ are fixed.

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We continue to take the embeddings using homogeneous p-spreadings with respect to the given sequence of primes in ξ. From the given sequence of embeddings, we have direct systems and hence direct limit groups $FSym(\kappa)(\xi)$, $(Alt(\kappa)(\xi))$. Observe that $FSym(\kappa)(\xi)$ and $Alt(\kappa)(\xi)$ are subgroups of $Sym(\kappa\omega)$. For the basic properties of groups $FSym(\kappa)(\xi)$, $Alt(\kappa)(\xi)$, definitions and the notations, see [3].

An embedding d of the transitive permutation group (G, X) into the permutation group (H, Y) is called *diagonal* if the restriction of $d(G)$ on every orbit of length more than 1 is permutation isomorphic to (G, X). A diagonal embedding is called a *strictly diagonal embedding* if the length of the every orbit of the image $d(G)$ on the set Y is greater than 1.

A group G is said to be of *(strictly) diagonal type* if G is a direct limit of an ascending chain of finitary symmetric groups $FSym(\kappa n_i)$ or alternating groups $Alt(\kappa n_i)$, $i \in \mathbb{N}$ where all inclusions $FSym(\kappa n_1) \leq FSym(\kappa n_2) \leq FSym(\kappa n_3)\ldots$, respectively $Alt(\kappa n_1) \leq Alt(\kappa n_2) \leq Alt(\kappa n_3)\ldots$ are (strictly) diagonal. Observe that the groups $FSym(\kappa)(\xi)$ and $Alt(\kappa)(\xi)$ constructed above are groups of strictly diagonal type.

For a detailed survey on diagonal type groups, see [8]. Automorphisms of the groups $S(\xi)$ is classified in [6].

Lemma 1.1. *Let f^r be a strictly diagonal embedding of $FSym(\kappa)$ in $FSym(\kappa r)$. Then the subgroups $f^r(FSym(\kappa))$ and $d^r(FSym(\kappa))$ are conjugate in $Sym(\kappa r)$. Similarly $f^r(Alt(\kappa))$ and $d^r(Alt(\kappa))$ are conjugate in $Sym(\kappa r)$.*

Proof. We prove the lemma for finitary symmetric groups. The alternating case can be proved similarly. Let $\mathcal{O}_1, \mathcal{O}_2, \ldots \mathcal{O}_r$ be the orbits of $f^r(FSym(\kappa))$ on the set κr. Then for each m, the groups $(FSym(\kappa), \kappa)$ and $(f^r(FSym(\kappa)|_{\mathcal{O}_m}), \mathcal{O}_m)$ are permutational isomorphic pairs, where $1 \leq m \leq r$. It follows that there exist bijective maps $\phi_1, \ldots, \phi_r$, where $\phi_m : \kappa \to \mathcal{O}_m$ satisfying $f^r(\alpha)\phi_m(i) = \phi_m(\alpha(i))$ for each $1 \leq i < \kappa$ and for each $\alpha \in FSym(\kappa)$. Then there exists a bijection $\pi \in Sym(\kappa r)$ such that

$$(\kappa(m-1)+i)^\pi = \phi_m(i) \quad (1 \leq m \leq r,\ 1 \leq i < \kappa).$$

Then the permutation π satisfies the following:

$$\pi^{-1} f^r(FSym(\kappa))\pi = d^r(FSym(\kappa))$$

Hence $f^r(FSym(\kappa))$ and $d^r(FSym(\kappa))$ are conjugate in $Sym(\kappa r)$. □

Definition. The formal product $n = 2^{r_2}3^{r_3}5^{r_5}\ldots$ of prime powers with $0 \leq r_k \leq \infty$ for all primes k is called a **Steinitz number** (supernatural number). The set of Steinitz numbers forms a partially ordered set with respect to division, namely if $\alpha = 2^{r_2}3^{r_3}5^{r_5}\ldots$ and $\beta = 2^{s_2}3^{s_3}5^{s_5}\ldots$ are two Steinitz numbers, then $\alpha|\beta$ if and only if $r_i \leq s_i$ for all primes i. Moreover they form a lattice if we define meet and join as $\alpha \wedge \beta = 2^{min\{r_2,s_2\}}3^{min\{r_3,s_3\}}5^{min\{r_5,s_5\}}\ldots$ and $\alpha \vee \beta = 2^{max\{r_2,s_2\}}3^{max\{r_3,s_3\}}5^{max\{r_5,s_5\}}\ldots$.

For each sequence ξ we define $Char(\xi) = p_1^{r_{p_1}} p_2^{r_{p_2}} \ldots$ where r_{p_i} is the number of times that prime p_i repeat in ξ. If it repeats infinitely often, then we write p_i^{∞}. Therefore for each $\xi \in \Pi$, there corresponds a Steinitz number $Char(\xi)$. For the group $FSym(\kappa)(\xi)$ obtained from the sequence ξ we define $Char(FSym(\kappa)(\xi)) = Char(\xi)$.

An immediate consequence of the following lemma is an answer to the following question: when do two homogenous finitary symmetric groups coincide

Lemma 1.2. [3, Lemma 14] *For any ξ_1, ξ_2 in Π, the group $FSym(\kappa)(\xi_1)$ is a subgroup of $FSym(\kappa)(\xi_2)$ if and only if $Char(\xi_1)$ divides $Char(\xi_2)$.*

In fact, for an infinite $\lambda \leq \kappa$, we have

$$S(\xi_1) \lesssim FSym(\lambda)(\xi_1) \lesssim FSym(\kappa)(\xi_1) \leq FSym(\kappa)(\xi_2).$$

In the above the notation $\lesssim$ is used with meaning "isomorphic to a subgroup".

Lemma 1.3. *Two homogenous finitary symmetric groups of the same cardinality coincide if and only if their characteristics are equal.*

Proof. This follows immediately by Lemma 1.2, as $FSym(\kappa)(\xi_1) \leq FSym(\kappa)(\xi_2)$ if and only if $Char(\xi_1)$ divides $Char(\xi_2)$ and the set of Steinitz numbers form a partially ordered set with respect to division of two Steinitz numbers. □

We now prove that two strictly diagonal type groups of the same cardinality and characteristic are isomorphic.

Theorem 1.4. *Let κ be an infinite cardinal. If $G = \bigcup_{i=1}^{\infty} G_i$, where $G_i \cong FSym(\kappa n_i)$, ($H = \bigcup_{i=1}^{\infty} H_i$, where $H_i \cong Alt(\kappa n_i)$), is a group of strictly diagonal type and $\xi = (p_1, p_2, \ldots)$, then G is isomorphic to the homogenous finitary symmetric group $FSym(\kappa)(\xi)$ (H is isomorphic to homogenous alternating group $Alt(\kappa)(\xi)$), where $n_0 = 1$, $n_i = p_1 p_2 \ldots p_i$, $i \in \mathbb{N}$.*

Proof. We prove the theorem by using the same technique as in [5, Theorem 3] for homogenous symmetric groups. Let $G = \bigcup_{i=1}^{\infty} FSym(\kappa n_i)$ be arbitrary group of strictly diagonal type and $\xi = (p_1, p_2, \ldots)$. Denote by f^{p_i} the strictly diagonal embedding $FSym(\kappa n_{i-1}) \to FSym(\kappa n_i)$ where p_i is the i^{th} prime in the sequence and $n_0 = 1$, $n_i = p_1 p_2 \ldots p_i$. The group G is a direct limit of the direct system

$$FSym(\kappa) \overset{f^{p_1}}{\to} FSym(\kappa n_1) \overset{f^{p_2}}{\to} FSym(\kappa n_2) \overset{f^{p_3}}{\to} FSym(\kappa n_3) \ldots$$

Suppose that $FSym(\kappa) \overset{d^{p_1}}{\to} FSym(\kappa n_1) \overset{d^{p_2}}{\to} FSym(\kappa n_2) \overset{d^{p_3}}{\to} FSym(\kappa n_3) \ldots$ is the ξ spectrum of the finitary symmetric groups $FSym(\kappa n_1)$, $FSym(\kappa n_2)$, $FSym(\kappa n_3)$ ... as in the construction. By Lemma 1.1 for every $i \in \mathbb{N}$, the images $f^{p_{i+1}}(FSym(\kappa n_i))$ and $d^{p_{i+1}}(FSym(\kappa n_i))$ are conjugate in $Sym(\kappa n_{i+1})$. Therefore there exists a system of automorphisms

$$i_{h_j} : FSym(\kappa n_j) \to FSym(\kappa n_j)$$

such that in the diagram

$$\begin{array}{ccccccccc} FSym(\kappa) & \stackrel{f^{p_1}}{\to} & FSym(\kappa n_1) & \stackrel{f^{p_2}}{\to} & FSym(\kappa n_2) & \stackrel{f^{p_3}}{\to} & FSym(\kappa n_3) & \stackrel{f^{p_4}}{\to} \dots \\ \downarrow i_{h_0} & & \downarrow i_{h_1} & & \downarrow i_{h_2} & & \downarrow i_{h_3} & \\ FSym(\kappa) & \stackrel{d^{p_1}}{\to} & FSym(\kappa n_1) & \stackrel{d^{p_2}}{\to} & FSym(\kappa n_2) & \stackrel{d^{p_3}}{\to} & FSym(\kappa n_3) & \stackrel{d^{p_4}}{\to} \dots \end{array}$$

we have $f^{p_s}(FSym(\kappa n_{s-1}))^{i_{h_s}} = d^{p_s}(FSym(\kappa n_{s-1}))$, where i_{h_s} is conjugation by an element h_s in $Sym(\kappa n_s)$.

Observe that the above diagram is a commutative diagram. As the maps i_{h_j} are uniquely defined automorphisms they are compatible and well defined. Then by [2, Lemma 2.3] the map $h : G \to FSym(\kappa)(\xi)$ is an isomorphism from G to the homogenous finitary symmetric group $FSym(\kappa)(\xi)$, where $h = \lim_{k\to\infty} i_{h_k}$. □

By Theorem 1.4, infinite homogenous finitary symmetric groups are isomorphic if and only if their orders and characteristics are equal. Hence the isomorphism classes of strictly diagonal type can be parameterized by the cardinality of the group and the corresponding Steinitz number and we obtain the complete classification up to isomorphism of such groups.

We use Theorem 1.6 in the proof of Lemma 1.7 . This theorem is stated in [5, Theorem 3 (ii)]. The statement of the theorem is valid, but the given proof there, must be modified according to the corrected structure of centralizers in [3].

We need the following Lemma of Brumberg, see [1] for the proof of the Theorem. See also, Meldrum [7, Corollary 2.7].

Lemma 1.5. [5, Lemma 4.1] *Let (G, X) and (H, Y) be two non-trivial permutation groups. Then the wreath product $G \wr H$ can not be decomposed as a direct product.*

For an element $g \in S(\xi)$ we define the principal beginning as g_0 where $g_0 \in S_{n_i}$ and n_i is the smallest in the chain (changing the order in the sequence of primes ξ is allowed to take the minimum) and such n_i is called the *characteristic of g*.

Theorem 1.6. *If $S(\xi_1)$ $(A(\xi_1))$ and $S(\xi_2)$ $(A(\xi_2))$ are two different homogenous symmetric (alternating) subgroups of $S(\mathbb{N})$, then they are non-isomorphic.*

Proof. Let $S(\xi_1)$ and $S(\xi_2)$ be two distinct homogenous symmetric groups. Then by [5, Theorem (i)] the characteristics are different. Let $Char(\xi_1) = p_1^{r_1} p_2^{r_2} \dots p_i^{r_i} \dots$ and $Char(\xi_2) = p_1^{s_1} p_2^{s_2} \dots p_i^{s_i} \dots$. Then there exists an $i \in \mathbb{N}$ such that $r_i \neq s_i$, say $r_i > s_i$. Then by changing the order if necessary in the sequences, we may take an element $u \in S_{p_i^{s_i+1}}$ with principal beginning a cycle of length $p_i^{s_i+1}$ in $S(\xi_1)$. Then by [3, Theorem 3]

$$C_{S(\xi_1)}(u) \cong C_{p_i^{s_i+1}}(C_{p_i^{s_i+1}} \bar{\wr}\, S(\xi_1'))$$

where $Char(\xi_1') = \frac{Char(\xi_1)}{p_i^{s_i+1}}$. But any element $v \in S(\xi_2)$ of cycle length $p_i^{s_i+1}$ has some fixed points as $p_i^{s_i+1}$ does not divide $Char(\xi_2)$. Then the cycle type will be different from the cycle type of u and

$$C_{S(\xi_2)}(v) \cong C_{p_i^{s_i+1}}(C_{p_i^{s_i+1}} \bar{\wr}\ S(\xi_2')) \times S(\xi_3)$$

$Char(\xi_2') = \frac{Char(\xi_2)}{Char(v)}$ and $Char(\xi_3) = \frac{Char(\xi_2)}{Char(v)} t_1$, where t_1 is the number of fixed points of v in the principal beginning of v.

If $Char(\xi_1')$ is infinite, then $Z(C_{p_i^{s_i+1}} \bar{\wr}\ S(\xi')) = 1$ and in this case $C_{p_i^{s_i+1}} \cap (C_{p_i^{s_i+1}} \bar{\wr}\ S(\xi_1')) = 1$ as the element u is not contained in the restricted wreath product of $(C_{p_i^{s_i+1}} \bar{\wr}\ S(\xi_1'))$ and the product $C_{p_i^{s_i+1}}(C_{p_i^{s_i+1}} \bar{\wr}\ S(\xi_1'))$ is in fact a direct product.

Now the centralizers are non-isomorphic as restricted wreath product is directly indecomposable by Lemma 1.5 and the number of direct factors in $C_{S(\xi_1)}(u)$ and $C_{S(\xi_2)}(v)$ are different. □

Now as in the case of homogenous symmetric groups, we prove the same result for homogenous finitary symmetric groups. If characteristics are different, then by Lemma 1.3 the groups $FSym(\kappa)(\xi_1)$ and $FSym(\kappa)(\xi_2)$ are different. Is it true that if the characteristics are different, then the groups are non-isomorphic We may show this by taking the centralizer of the same subgroup in both groups. Clearly if the centralizers of subgroups are non-isomorphic, then the groups are non-isomorphic. For this we may consider the centralizer of the subgroup $FSym(\kappa)$ in $FSym(\kappa)(\xi_1)$ and $FSym(\kappa)(\xi_2)$.

Lemma 1.7. *If the characteristics of two homogenous finitary symmetric groups are different (homogenous infinite alternating groups), then the groups are non-isomorphic.*

Proof. For simplicity we prove the lemma for homogenous finitary symmetric groups. Clearly we may assume that the cardinalities of the two groups are equal. Let $\xi_1 = (q_1, q_2, \ldots)$ and $\xi_2 = (q_1', q_2', \ldots)$ be two sequences where $Char(\xi_1) = p_1^{r_1} p_2^{r_2} \ldots p_i^{r_i} \ldots$ and $Char(\xi_2) = p_1^{s_1} p_2^{s_2} \ldots p_i^{s_i} \ldots$. Assume that $Char(\xi_1) \neq Char(\xi_2)$. Then there exists $i \in \mathbb{N}$ such that $r_i \neq s_i$. Therefore one of r_i or s_i is finite. We may write $FSym(\kappa)(\xi_1) = \bigcup_{i=1}^{\infty} FSym(\kappa n_i)$ and $FSym(\kappa)(\xi_2) = \bigcup_{j=1}^{\infty} FSym(\kappa m_j)$ where $n_0 = 1$, $n_i = q_1 q_2 \ldots q_i$, $m_0 = 1$, $m_j = q_1' q_2' \ldots q_j'$. Then $C_{FSym(\kappa)(\xi_1)}(FSym(\kappa)) = \bigcup_{i=1}^{\infty} C_{FSym(\kappa n_i)}(FSym(\kappa))$ and $C_{FSym(\kappa)(\xi_2)}(FSym(\kappa)) = \bigcup_{j=1}^{\infty} C_{FSym(\kappa m_j)}(FSym(\kappa))$.

As $Z(FSym(\kappa)) = 1$ and the action of $FSym(\kappa)$ on its orbits in κn_i are all equivalent, by [4, 4.2.5 p. 109] we have $C_{FSym(\kappa n_i)} FSym(\kappa) \cong Sym(n_i)$.

Observe that by a strictly diagonal embedding of $FSym(\kappa n_i)$ into $FSym(\kappa n_{i+1})$ we have strictly diagonal embeddings of the centralizers, namely $C_{FSym(\kappa n_i)}(FSym(\kappa))$ is embedded into

$$C_{FSym(\kappa n_{i+1})}(FSym(\kappa))$$

by a strictly diagonal embedding inherited from the embedding of the main groups. Now by the construction of the groups $S(\xi)$ and the classification of strictly diagonal embeddings of groups of type $S(\xi)$

in [5, Theorem 3 (iii)] we have $C_{FSym(\kappa)(\xi_1)}(FSym(\kappa)) \cong S(\xi_1)$. Similarly $C_{FSym(\kappa)(\xi_2)}(FSym(\kappa)) \cong S(\xi_2)$. But by Theorem 1.6 the groups $S(\xi_1) \cong S(\xi_2)$ if and only if $Char(\xi_1) = Char(\xi_2)$. Hence we have $FSym(\kappa)(\xi_1) \cong FSym(\kappa)(\xi_2)$ if and only if $Char(\xi_1) = Char(\xi_2)$.

□

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