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ON MAGNUS' FREIHEITSSATZ AND FREE POLYNOMIAL ALGEBRAS

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Abstract. The Freiheitssatz of Magnus for one-relator groups is one of the cornerstones of combinatorial group theory. In this short note which is mostly expository we discuss the relationship between the Freiheitssatz and corresponding results in free power series rings over fields. These are related to results of Schneerson not readily available in English. This relationship uses a faithful representation of free groups due to Magnus. Using this method in free polynomial algebras provides a proof of the Freiheitssatz for one-relation monoids. We show how the classical Freiheitssatz depends on a condition on certain ideals in power series rings in noncommuting variables over fields. A proof of this result over fields would provide a completely different proof of the classical Freiheitssatz.

1. Introduction

The Freiheitssatz or independence theorem for one-relator groups is one of the cornerstones of combinatorial group theory. This was originally proved by Magnus in 1930 [7]. In his proof he developed a technique now called the **Magnus method** which has proven to be exceptionally powerful in handling one-relator groups as well as many other classes of infinite groups. Subsequently there have been many variations and extensions of Magnus' original result. Lyndon [5] reproved the Freiheitssatz by recasting it in the guise of combinatorial geometry while Howie [6] recast it in terms of topology and used Papakyriapopolous towers. These proofs, although recast, still use the basic Magnus method. The basic idea of the Freiheitssatz in group theory in general has been extended in many directions. If $A = \langle X; R\rangle$ and $G = \langle X \cup Y; R \cup S\rangle$ with the unions being disjoint unions then we say that A is a Freiheitssatz factor or FHS factor if A injects into G under the identity mapping. In [3] a broad description where FHS factors arise is given.

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In this short note, which is mostly expository, we discuss the relationship between the Freiheitssatz and corresponding results in free power series rings over fields. These relationships are based on a faithful representation of a free group in a ring of formal power series in noncommuting variables over a field together with two versions of the independence theorems within several varieties of algebras originally due to L. M. Shneerson [10, 11, 12, 13]. Shneerson's results are very powerful and are proved in a much different manner than any of the group theoretic proofs. Further they show that the Freiheitssatz is actually a very general idea in algebra. Shneerson's work in this area is basically inaccesible in English and has not received the attention it deserves. We take the opportunity in this note to partially publicize these results. One of Shneerson's results was recently reproved by Dotsenko, Iyudu and Korytin [2]. Based on Schneerson's results we give a sufficient condition involving ideals in formal power series rings to prove the group theoretic Freiheitssatz . An independent proof of the result over fields would provide a completley different proof of the classical Freiheitssatz. Using an analogous technique as for groups we give a proof of the Freiheitssatz for one-relation monoids.

2. The Freiheitssatz and the Magnus Representation

The Freiheitssatz or independence theorem for one-relator groups is the following:

Theorem 2.1. *(Freiheitssatz) Let $G = \langle x_1, \ldots, x_n; R = 1\rangle$ where R is a cyclically reduced word in the free group on $x_1, \ldots, x_n$ which involves all the generators. Then the subgroup generated by $x_1, \ldots, x_{n-1}$ is free on these generators.*

This was proved originally by Magnus [8]. For the proof Magnus developed a general method, now called the Magnus Method, to handle one-relator groups. This involved using group amalgams coupled with induction on the length of the relator. These techniques have become standard in combinatorial group theory.

The Freiheitssatz itself has been generalized in many directions. In a more general context the Freiheitssatz can be described as follows. Let X, Y be disjoint sets of generators and suppose that the group A has the presentation $A = \langle X; \mathrm{Rel}(X)\rangle$ and that the group G has the presentation $G = \langle X, Y; \mathrm{Rel}\ (X), \mathrm{Rel}(X, Y)\rangle$. Then we say that G satisfies a **Freiheitssatz** which we abbreviate by **FHS** {relative to A} if $\langle X\rangle_G \cong A$. In other words the subgroup of G generated by X is isomorphic to A. In simpler language this says that the complete set of relations on X in G is the "obvious" one from the presentation of G. An alternative way to look at this is that A injects into G under the obvious map taking X to X. In this language Magnus' original FHS can be phrased as a one-relator group satisfies a FHS relative to the free group on any proper subset of the generators. In the setting above we say that the group A is a **FHS factor** of G.

From this point of view, for any group amalgam, free product with amalgamation or HNN group, an amalgam factor is a FHS factor. Thus any factor in a free product with amalgamation and the base in an HNN group embed as a FHS factor in the resulting groups. This becomes the basic idea in Magnus' method. The method is to embed the group into an amalgam in such a way that the proposed FHS factor embeds into an amalgam factor which in turn contains the proposed FHS factor as a FHS factor. The result can then be obtained by applying the FHS for amalgams. A detailed explanation and discussion of many aspects of the Freiheitssatz can be found in [3].

Magnus points out that the Freiheitssatz can also be viewed as a noncommutative analog of the following theorem in commutative algebra.

Theorem 2.2. *Let F be a field and $K = F[x_1, \dots, x_n]$ be the free polynomial algebra over F in commuting variables $x_1, \dots, x_n$. Then if $p(x_1, \dots, x_n)$ is any polynomial in K which involves all the variables then if $H = K/\langle p\rangle$ then any polynomial relation in H must involve all the variables.*

Further if $\langle p(x_1, \dots, x_n)\rangle$ is an irreducible ideal then

$$F[x_1, \dots, x_n]/\langle p(x_1, \dots, x_n)\rangle \cong F[y_1, \dots, y_{n-1}].$$

Proof. The proof of this is very direct in the commutative case. Here we say that the ideal $\langle p(x_1, \dots, x_n)\rangle$ is irreducible if the polynomial $p(x_1, \dots, x_n)$ is irreducible. Suppose $p(x_1, \dots, x_n)$ involves all the variables. Let $H = K/\langle p\rangle$ and suppose that $q(x_1, \dots, x_n) = 0$ in H. Then $q \in \langle p\rangle$. However in the commutative case $\langle p\rangle$ is just the K multiples of p and hence $q = ph$ for some polynomial $h \in K$. Since p involves all the variables so does q.

The second part follows easily from the first if the ideal $\langle p(x)\rangle$ is irreducible. □

Our discussion depends on a faithful representation of a free group in a formal power series ring. This is called the **Magnus representation** and we briefly review it.

Let

$$H = F\langle\langle x_1, \dots, x_n\rangle\rangle$$

be the formal power series ring in noncommuting variables $x_1, \dots, x_n$ over a commutative field F and let $U(H)$ be its group of units. Then for each $i = 1, \dots, n$ let $\alpha_i = 1 + x_i$

Notice that in the formal power series ring H we have the well known expansion

$$\frac{1}{1 + x_i} = 1 - x_i + x_i^2 - x_i^3 + \dots\dots$$

Therefore each α_i is invertible within H. Therefore each α_i is in the unit group $U(H)$ of H and therefore the set $\{\alpha_1, \dots, \alpha_n\}$ generates a multiplicative subgroup of $U(H)$.

Magnus' result is the following.

Theorem 2.3. *(The Magnus Representation) The elements*

$$\alpha_1 = 1 + x_1, \dots, \alpha_n = 1 + x_n$$

freely generate a subgroup of $U(H)$. Therefore the map given by

$$y_1 \to \alpha_1, \dots, y_n \to \alpha_n$$

provides a faithful representation of the free group on $y_1, \dots, y_n$ into H.

We sketch the proof. To show that the group generated by the $1 + x_i$ is free, we show that no nontrivial freely reduced word in the $\alpha_i = 1 + x_i$ can be the identity and hence the group they generate must be a free group.

From the binomial expansion we have for any non-zero integer n, positive or negative,

$$(1+x_i)^n = 1 + nx_i + \text{ terms in higher powers }.$$

Now let

$$W(\alpha_1,\ldots,\alpha_n) = \alpha_{i_1}^{n_1}\alpha_{i_2}^{n_2}\cdots\alpha_{i_k}^{n_k}$$

be a freely reduced word in the α_i with each $|n_i| \geq 1$ and $\alpha_{i_j} \neq \alpha_{i_{j+1}}$ for $j = 1,\ldots,k-1$. We call k the **block length**. In the ring H we then have

$$W(\alpha_1,\ldots,\alpha_n) = (1+x_{i_1})^{n_1}\cdots(1+x_{i_k})^{n_k}$$

and hence

$$W(\alpha_1,\ldots,\alpha_n) = (1+n_1x_{i_1} + \text{higher powers in } x_{i_1})\cdots(1+n_kx_{i_k} + \text{higher powers in } x_{i_k})$$

The variables are noncommuting, so that in analyzing this product we see that there is a unique monomial term of maximal block length k where each x_{i_j} appears to the power 1. That is there is a unique monomial term

$$n_1n_2\cdots n_k(x_{i_1}x_{i_2}\cdots x_{i_k}).$$

Notice that this monomial term contains all the variables x_j corresponding to the generators α_i appearing in the word W.

Since each $n_i \neq 0$ this term must appear and therefore $W(\alpha_1,\ldots,\alpha_n) \neq 1$. It follows that the group generated by $\alpha_1,\ldots,\alpha_n$ is freely generated by them.

3. Independence Theorems over Free Algebras

We showed that if $F[x_1,\ldots,x_n]$ is a polynomial ring in commuting variables $x_1,\ldots,x_n$ over a commutative field F and $p(x_1,\ldots,x_n)$ is a polynomial involving all the variables then

$$F[x_1,\ldots,x_n]/\langle p(x_1,\ldots,x_n)\rangle \cong F[y_1,\ldots,y_{n-1}],$$

if $\langle p\rangle = \langle p(x_1,\ldots,x_n)\rangle$ the ideal generated by p is irreducible.

The following two results, orginally due to L. Shneerson, (see [10] and [9]) say that the above is the common situation for free polynomial algebras. In his papers, which are not accesible in English, Shneerson proves a wealth of important material generalizing both Magnus' Freiheitssatz and the above polynomial result (see [10, 11, 12, 13, 14]). We hope this note will partially publicize Shneerson's work in this area. The first result (Theorem 3.1 below) was recently reproved by Dotsenko, Iyudu and Korytin [2]. A further source of material on free polynomial algebras is the book by Mikhalev, Shpilrain and Yu [9].

Theorem 3.1. *(see [10] and [9]) Let F be a field, $X = \{x_1,\ldots,x_n\}$ a set of variables and $K = F[X]$ the free associative algebra over F. Let $p = p(x_1,\ldots,x_n)$ be a polynomial in K which involves all the variables. Let x_i be one of the variables and let $\langle p\rangle$ be the ideal in K generated by p. If the ideal $\langle p\rangle$ is irreducible then $F[X\setminus\{x_i\}]\cap\langle p\rangle = \{0\}$.*

Equivalently this says that $F[X]/\langle p\rangle \cong F[y_1,\dots,y_{n-1}]$ *where* $F[y_1,\dots,y_{n-1}]$ *is the free associative algebra of rank* $n-1$*.*

Theorem 3.2. *(see [10] and [9]) Let* F *be a field,* $X=\{x_1,\dots,x_n\}$ *a set of variables and* $K=F\langle\langle X\rangle\rangle$ *the free formal power series algebra over* F*. Let* $p=p(x_1,\dots,x_n)$ *be a noninvertible polynomial in* K *which involves all the variables. Let* x_i *be one of the variables and let* $\langle p\rangle$ *be the ideal in* K *generated by* p*. Then if the ideal* $\langle p\rangle$ *is irreducible then any power series in* $\langle p\rangle$ *must involve all the variables.*

Equivalently this says that $F\langle\langle X\rangle\rangle/\langle p\rangle \cong F\langle\langle y_1,\dots,y_{n-1}\rangle\rangle$ *where* $F\langle\langle y_1,\dots,y_{n-1}\rangle\rangle$ *is the free power series algebra of rank* $n-1$*.*

Note that a power series $p(x_1,\dots,x_n)$ in $F\langle\langle X\rangle\rangle$ is noninvertible if the constant term is 0.

4. The Freiheitssatz in One-Relation Monoids

There have been several proofs of the corresponding Freiheitssatz for one-relation monoids. Several are based on the Freiheitssatz for groups while one due to Squier and Wrathall [1] is direct and independent of the Magnus method. Shneerson in [10] gives a proof based on free algebras and in [14] provides a very straightforward proof using the classical Freiheitssatz for groups. In [10] Shneerson presented other proofs of the Freiheitssatz for both one-relation and one-relator monoid. Using free associative algebras we give another proof for one-relation monoids.

Theorem 4.1. *(Freiheitssatz for One-Relation Monoids) Let* $G=\langle x_1,\dots,x_n; U=V\rangle$ *be a one relation monoid where* U,V *are words in the free monoid on* $x_1,\dots,x_n$ *such that the union of the words* $\{U,V\}$ *involves all the generators. Then the submonoid of* G *generated by* $x_1,\dots,x_{n-1}$ *is free on these generators.*

Proof. Let $K=F[x_1,\dots,x_n]$ be the free associative algebra over the field F in noncommuting variables $x_1,\dots,x_n$. Since these are noncommuting variables the identity of F together with the multiplicative set generated by the variables $x_1,\dots,x_n$ provides a faithful representation of the free monoid on $x_1,\dots,x_n$ within K. Now let $G=\langle y_1,\dots,y_n; U=V\rangle$ be a one relation monoid where U,V are words in the free monoid on $y_1,\dots,y_n$ such that $\{U,V\}$ involves all the generators.

Consider now the polynomial q in K given by

$$q(x_1,\dots,x_n)=U(x_1,\dots,x_n)-V(x_1,\dots,x_n).$$

If $q=0$ in K then G is already a free monoid, and the statement of Theorem 4.1 holds.

Now let q be nonzero. Since U,V together involve all the generators and U,V are monomials it follows that the polynomial $q(x_1,\dots,x_n)$ contains nontrivially all the variables $x_1,\dots,x_n$.

Let $\langle q\rangle$ be the ideal in K generated by q and let $H=K/\langle q\rangle$. Since the relation $q=0$ implies $U=V$ the mapping $y_i\to x_i, i=1,\dots,n$, provides a representation (perhaps not faithful) of the one-relation monoid G in H.

Consider the submonoid of K generated by $x_1,\dots,x_{n-1}$. Suppose that we have a nontrivial relation

$$U_1(x_1,\dots,x_{n-1})=V_1(x_1,\dots,x_{n-1})$$

Let $p(x_1, \ldots, x_{n-1}) = U_1 - V_1$. Since the relation is nontrivial, we have that p is not 0 in K but equal 0 in H, that is $p(x_1, \ldots, x_{n-1}) = 0$ in H. Therefore $p(x_1, \ldots, x_{n-1}) \in \langle q \rangle$. But this is impossible since p does not involve all the variables while q does. Therefore the submonoid of K generated by $x_1, \ldots, x_{n-1}$ is free on these generators. Hence the preimages in G, $y_1, \ldots, y_{n-1}$, also generate a free monoid completing the proof. □

We note that the the embedding of a monoid G in its semigroup ring $\mathbb{Z}[G]$ was used previously by Freyd [4] to prove that all finitely generated commutative semigroups are finitely presented.

5. The Freiheitssatz in One-Relator Groups

Schneerson's results almost provide another proof of the classical Freiheitssatz that is somewhat independent of group theory. Consider the following, which we will call **Conjecture A**.

Conjecture A Let F be a field and $K = F\langle\langle x_1, \ldots, x_n \rangle\rangle$ be the formal power series ring in the noncommuting variables $x_1, \ldots, x_n$. Let $\alpha_i = 1 + x_i$. Let $R = R(\alpha_1, \ldots, \alpha_n)$ be a cyclically reduced word in the variables $\alpha_1, \ldots, \alpha_n$ which involves all the variables and $p(x_1, \ldots, x_n) = R - 1$ be the corresponding formal power series in K. Then the ideal $\langle p \rangle$ in K is irreducible, that is, p is irreducible.

Proposition 5.1. *Conjecture A implies the classical Freiheitssatz for one-relator groups.*

Proof. Assume Conjecture A holds and let $G = \langle y_1, \ldots, y_n; W = 1 \rangle$ be a one relator group where $W = W(y_1, \ldots, y_n)$ is a cyclically reduced word in the free group on $y_1, \ldots, y_n$ which involves all the generators.

Let F be a field and let $K = F\langle\langle x_1, \ldots, x_n \rangle\rangle$ be the ring of formal power series in noncommuting variables over F.

Let $\alpha_i = 1 + x_i$ so that by the Magnus representation

$$y_i \to \alpha_i$$

provides a faithful representation of the free group on the y_i. Suppose that

$$R(y_1, \ldots, y_n) = y_{i_1}^{n_1} y_{i_2}^{n_2} \ldots y_{i_k}^{n_k}.$$

Consider now the polynomial p in K given by

$$p(x_1, \ldots, x_n) = W(\alpha_1, \ldots, \alpha_n) - 1 = \alpha_{i_1}^{n_1} \alpha_{i_2}^{n_2} \ldots \alpha_{i_k}^{n_k}.$$

As explained in the proof of the Magnus representation since W involves all the generators it follows that the polynomial $p(x_1, \ldots, x_n)$ contains nontrivially all the variables $x_1, \ldots, x_n$ - in fact contains a monomial term with all of them. Further $p(x_1, \ldots, x_n)$ has zero constant term so is noninvertible in K.

Let $\langle p \rangle$ be the ideal in K generated by p and let $H = K/\langle q \rangle$. By assumption then $\langle q \rangle$ is an irreducible ideal. Since the relation $p = 0$ implies $R = 1$ the mapping $y_i \to \alpha_i, i = 1, \ldots, n$ provides a representation (perhaps not faithful) of the one-relator group G in H.

Consider the subgroup of $U(H)$ generated by $\alpha_1, \ldots, \alpha_{n-1}$. Suppose that we have a nontrivial relation

$$V(\alpha_1, \ldots, \alpha_{n-1}) = 1$$

Let $p(x_1, \ldots, x_{n-1}) = V(\alpha_1, \ldots, \alpha_{n-1}) - 1$ so that $p(x_1, \ldots, x_{n-1}) = 0$ in H. Therefore $p(x_1, \ldots, x_{n-1}) \in \langle q \rangle$. But this is impossible from Theorem 3.2 since $\langle p \rangle$ is an irreducible ideal and p does not involve all the variables while q does. Therefore the subgroup of $U(H)$ generated by $\alpha_1, \ldots, \alpha_{n-1}$ is free on these generators. Hence the preimages in G, $y_1, \ldots, y_{n-1}$, also generate a free group completing the proof that Conjecture A implies the Freiheitssatz. $\square$

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