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A NOTE ON THE AFFINE SUBGROUP OF THE SYMPLECTIC GROUP

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Abstract. We examine the symplectic group $Sp_{2m}(q)$ and its corresponding affine subgroup. We construct the affine subgroup and show that it is a split extension. As an illustration of the above we study the affine subgroup $2^5{:}Sp_4(2)$ of the group $Sp_6(2)$.

1. Introduction

An affine subgroup of a classical group acting on a vector space is the stabilizer of a nonzero vector. Gow in [4] used Clifford theory to establish the existence of complex irreducible characters of certain degrees of affine subgroups of the general linear, symplectic and unitary groups. It is proved in particular that these characters remain irreducible as modular characters modulo any prime distinct from the characteristic of the defining field. Using the underlying theory described in [4], here we prove the following theorem:

Theorem 3.10. *Let q be a power of an odd prime p. Then $A(m)$ is a split extension of $P(m)$ by H where $H \cong G(m-1) \cong Sp_{2m-2}(q)$, i.e., $A(m) = P(m){:}H = P(m){:}Sp_{2m-2}(q)$.*

The proof of the theorem will follow from a series of lemmas proved in Section 3. In Section 4 we construct the group $A(3)$, the affine subgroup of $Sp_6(2)$ as an illustrative example of the theory outlined in Section 3 and in Section 5 using GAP we construct its conjugacy classes.

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2. Background and terminology

For a finite dimensional vector space V over a field $\mathbb{F}$, a function f from the set $V \times V$ of ordered pairs in V to $\mathbb{F}$ is called a **bilinear form** on V if for each $v \in V$, the functions $f(v,\ )$ and $f(\ ,v)$ are linear functionals on V. In this case we say that (V, f) is an **inner product space.**

A bilinear form f on V such that for each non-zero $x \in V$, there exists $y \in V$ for which $f(x, y) \neq 0$, then f is said to be **non-degenerate.** A bilinear form f is called **alternating** or **symplectic** on V if $f(x, x) = 0$ for all $x \in V$. If $char(\mathbb{F}) \neq 2$ then we obtain that for all $x, y \in V,\ f(x, x) = 0$. If a bilinear form $f : V \times V \longrightarrow \mathbb{F}$ on V satisfies:
(i) $f(x, x) = 0, \quad \forall\, x \in V$
(ii) $f(x, y) = -f(y, x), \quad \forall\, x, y \in V$ we say that (V, f) is a **symplectic space** over the field $\mathbb{F}$. If $char(\mathbb{F}) \neq 2$, then the properties (i) and (ii) are equivalent.

Let (V, f) and (U, g) be symplectic spaces over $\mathbb{F}$, then we say that $V \cong U$ if there exists an isomorphism $T \in L(V, U)$ such that $\forall\, x, y \in V$ we have $f(x, y) = g(T(x), T(y))$.

If W is a subspace of V then the **orthogonal complement** of W is defined by $W^{\perp} = \{y \in V \mid f(x, y) = 0,\ \forall\ x \in W\}$. If $\{v_1, v_2, \ldots, v_m\}$ is an ordered basis of V, then the **inner product matrix** of f relative to this basis is given by an $m \times m$ matrix $A = [f(v_i, v_j)]_{m\times m}$. Clearly f is completely determined by an inner product matrix, for if $u = \sum \alpha_i v_i$ and $v = \sum \beta_j v_j$ then $f(u, v) = \sum_{i,j} \alpha_i \beta_j f(v_i, v_j)$.

An **isometry** of a non-degenerate space (V, f) is a linear transformation $T : V \longrightarrow V$, such that $f(T(x), T(y)) = f(x, y)$ for all $x, y \in V$. If (V, f) is a non-degenerate space, then every isometry is non-singular, and so the set of all isometries denoted by $\mathbf{Isom}(V, f)$ forms a subgroup of $GL(V)$.

A non-degenerate symplectic space (V, f) has even dimension $2m$ over a field $\mathbb{F}$, and the group $\mathbf{Isom}(V, f)$ is denoted by $Sp_{2m}(\mathbb{F})$, and is called the **symplectic group** of degree $2m$ over $\mathbb{F}$. If $\mathbb{F} = GF(q)$ is a Galois field of q elements, where $q = p^k$ for some k with p a prime, we denote $Sp_{2m}(\mathbb{F})$ by $Sp_{2m}(q)$. We further obtain that $\mathbf{Isom}(V, f) = Sp_{2m}(\mathbb{F}) \leq GL_{2m}(\mathbb{F})$. Within isomorphism this group is independent of the choice of f.

If V is a non-degenerate symplectic space over a field $\mathbb{F}$, and $Sp_{2m}(\mathbb{F})$ is the symplectic group of isometries of V, then the centre $Z(Sp_{2m}(\mathbb{F}))$ of $Sp_{2m}(\mathbb{F})$ consists of the transformations $T = kI$, where $k = \pm 1$. The factor group $Sp_{2m}(\mathbb{F})/Z(Sp_{2m}(\mathbb{F}))$ is called the *projective symplectic group* and is denoted by $PSp_{2m}(\mathbb{F})$. The projective symplectic groups are simple except for $PSp_2(2) = PSL_2(2)$, $PSp_2(3) = PSL_2(3)$ and $PSp_4(2)$. If $\mathbb{F} = GF(q)$, is the Galois field of q elements, then $Sp_{2m}(q)$ and $PSp_{2m}(\mathbb{F})$ are denoted by $Sp_{2m}(q)$ and $PSp_{2m}(q)$ respectively, and we have $Z(Sp_{2m}(q)) = \{I\}\quad if \quad char(\mathbb{F}) = 2\ \ and\ \ Z(Sp_{2m}(q)) = \{I, -I\} \quad if \quad char(\mathbb{F}) \neq 2$. We also have $|PSp_{2m}(q)| = \frac{1}{(2,q-1)} \times |Sp_{2m}(q)| = \frac{q^{m^2}}{(2,q-1)} \prod_{i=1}^{m} (q^{2i} - 1)$.

Let (V, f) be a non-degenerate symplectic space. Let $B = \{x_1, y_1, x_2, y_2, \ldots, x_m, y_m\}$ be a basis for V such that $\{x_i, y_i\}$ is a hyperbolic pair for all $1 \leq i \leq m$ with $f(x_i, x_j) = 0 = f(y_i, y_j)$ for all i, j *and* $f(x_i, y_j) = \delta_{ij} = -f(y_j, x_i)$, where $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ if $i \neq j$. Then we say that the set B is a **hyperbolic basis** for V. It is clear that all inner product are 0 except $f(x_i, y_i) = 1 = -f(y_i, x_i)$ for all i.

The following results which could be found in [9] are relevant in the proof of the main theorem,

Lemma 2.1. [9] *If (V, f) is a non-degenerate symplectic space, then the following occur,*
(i) V has a hyperbolic basis $\{x_1, y_1, x_2, y_2, \ldots, x_m, y_m\}$.
(ii) The inner product matrix A of f relative to this ordered basis is the matrix J which is the direct sum of 2×2 blocks $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.
(iii) If $u = \sum_i(\alpha_i x_i + \beta_i y_i)$ and $v = \sum_i(\lambda_i x_i + \mu_i y_i)$, then $f(u, v) = \sum_i(\alpha_i\mu_i - \beta_i\lambda_i)$.
(iv) All non-degenerate symplectic forms on V are equivalent and the symplectic group $\mathbf{Isom}(V, f)$, up to isomorphism, do not depend on the choice of f.

Remark 2.2. Property (iii) in Lemma 2.1 implies that the symplectic forms are sums of 2×2 determinants. If a hyperbolic basis is reordered so that all the x_i precede all the y_i, then the matrix J is congruent to $\begin{pmatrix} 0 & I_m \\ -I_m & 0 \end{pmatrix}$. In this case we say that $\{x_1, x_2, \ldots, x_m, y_1, y_2, \ldots, y_m\}$ is a **reordered hyperbolic** basis of V.

Lemma 2.3. [9] *Let (V, f) be a non-degenerate symplectic space.*
Let $T \in GL(V)$, and let $\{x_1, x_2, \ldots, x_m, y_1, y_2, \ldots, y_m\}$ be a reordered hyperbolic basis of V. If the matrix Q of T with respect to this basis is decomposed into $m \times m$ blocks $Q = \begin{pmatrix} A & C \\ B & D \end{pmatrix}$ then T^* exists and has a matrix $Q^* = \begin{pmatrix} D^t & -C^t \\ -B^t & A^t \end{pmatrix}$, moreover, $Q \in Sp_{2m}(q)$ if and only if $Q^*Q = I_{2m}$.

Remark 2.4. For a p-group G we have that the derived subgroup $G' \leq \Phi(G)$. G is said to be a **special** p-group if $Z(G) = G' = \Phi(G)$ is elementary abelian.

3. The affine subgroup of the symplectic group

Since $Sp_{2m}(q)$ is transitive on the non-zero vectors of V, we consider the subgroup of $Sp_{2m}(q)$ which is a stabilizer of a non-zero vector of V and study its structure. Following a series of lemmas we prove Theorem 3.7.

Let (V, f) be a non-degenerate symplectic space over $\mathbb{F} = GF(q)$, where q is a power of a prime. Let $B = \{e_1, e_2, \ldots, e_{2m}\}$ be a basis for V and $f : V \times V \longrightarrow \mathbb{F}$ defined by $f(e_i, e_j) = \delta(i, 2m + 1 - j)$, where $i \leq j$. Let T be an isometry of (V, f) and

$$G(m) = Sp_{2m}(q) = \{T \mid f(T(x), T(y)) = f(x, y) \ \forall x, y \in V\}.$$

Since $G(m)$ acts transitively on $V - \{0_V\}$, let $\alpha \in V - \{0_V\}$ and $A(m)$ be the stabilizer of α in $G(m)$ then we have that $A(m) = \{T \in G(m) \mid T(\alpha) = \alpha\}$. Thus $A(m) \leq G(m)$. The group $A(m)$ is called the **affine subgroup**.

Note 3.1. *Since $A(m)$ is the stabilizer of $\alpha \in V - \{0_V\}$ in $G(m)$, we have that $[G(m){:}A(m)] = |V - \{0_V\}| = q^{2m} - 1$. Now since $G(m)$ acts transitively on $V - \{0_V\}$, then we let $A(m)$ to be the stabilizer in $G(m)$ of the non-zero vector e_1 of $V - \{0_V\}$. That is $A(m) = \{T \in G(m) \mid T(e_1) = e_1\}$.*

Lemma 3.2. *Let $a \in GF(q)$, with $char(GF(q)) = p$ and $(n,p) = 1$, $n \in \mathbb{N}$. Then there exists $b \in GF(q)$ such that $nb = a$.*

Proof. If $a = 0$, then let $b = 0$. Suppose $a \neq 0$. Let $(GF(q))^* = \langle x \rangle$ where $o(x) = q - 1$. Then $a \in (GF(q))^*$ implies that $a = x^{r'}, 1 \leq r' \leq q-1$. Since $(n,p) = 1, nx \neq 0$. Thus $nx \in (GF(q))^*$ and hence $nx = x^r$ where $1 \leq r \leq q-1$. Let $b = x^{r'-r+1}$. Then $b \in (GF(q))^*$ and $nb = nx^{r'-r+1} = (nx)x^{r'-r} = x^r.x^{r'-r} = x^{r'} = a$. □

Remark 3.3. Now we define $P(m)$ to be the subgroup of $A(m)$ consisting of elements $T \in G(m)$, such that

$$T(e_1) = e_1$$

$$T(e_i) = \alpha_i e_1 + e_i, \quad 2 \leq i \leq 2m-1$$

and

$$T(e_{2m}) = \sum_{i=1}^{2m} \beta_i e_i$$

with $\beta_{2m} = 1$ and

$$\alpha_j = \begin{cases} -\beta_{2m+1-j} & 2 \leq j \leq m \\ \beta_{2m+1-j} & m < j \leq 2m-1. \end{cases}$$

If $T \in P(m)$, then T is represented by the following matrix, with respect to the basis B given above:

$$\begin{pmatrix} 1 & -\beta_{2m-1} & -\beta_{2m-2} & \cdots & \beta_2 & \beta_1 \\ 0 & 1 & 0 & \cdots & 0 & \beta_2 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & \beta_{2m-1} \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix}.$$

It is convenient to describe $P(m)$ as an abstract group P in the following manner: Let (V,f) be a non-degenerate symplectic space of dimension $2m-2$ over $GF(q)$ and consider the pairs $[v,a]$, where $v \in V$ and $a \in GF(q)$. Define a multiplication on such pairs by $[v,a][u,b] = [u+v, a+b+f(v,u)]$. It is clear that $|P| = q^{2m-2} \times q = q^{2m-1}$.

Lemma 3.4. [4] *If $char(\mathbb{F}) = p$ where p is an odd prime, then the group P is a non-abelian special p-group of order q^{2m-1} isomorphic to $P(m)$.*

Proof. It is not difficult to see that P is a non-abelian group of order q^{2m-1} under the multiplication defined above, with $[0_V,\ 0] = 1_P$ and $[v,\ a]^{-1} = [-v\ , -a]$.

(i) P is non-abelian: Since

$$[v,a][u,b] = [u+v, a+b+f(v,u)] \text{ and } [u,b][v,a] = [u+v, a+b+f(u,v)],$$

it suffices to show that $f(u,v) \neq f(v,u)$. Now, since f is symplectic we have that $f(u,v) = -f(v,u) \neq f(v,u)$, thus P is non-abelian.

(ii) P is special: We need to show that $Z(P) = P' = \Phi(P)$. Now

$$\begin{aligned}
[v,a] \in Z(P) &\Leftrightarrow [v,a][u,b] = [u,b][v,a] \quad \forall u \in V \ \textit{and} \quad \forall a,b \in GF(q) \\
&\Leftrightarrow f(u,v) = f(v,u) \quad \forall u \in V \\
&\Leftrightarrow f(u,v) = f(v,u) = -f(u,v) \quad \forall u \in V \\
&\Leftrightarrow 2f(u,v) = 0 \quad \forall u \in V \\
&\Leftrightarrow f(u,v) = 0 \quad \forall u \in V \\
&\Leftrightarrow v = 0_V.
\end{aligned}$$

Thus $Z(P) = \{[0_V, a] \mid a \in GF(q)\} \cong GF(q)$ and $|Z(P)| = |GF(q)| = q$. Now

$$\begin{aligned}
P' &= \langle\, [v,a],[u,b] \mid u,v \in V, \quad a,b \in GF(q)\,\rangle \\
&= \{[v,a][u,b][v,a]^{-1}[u,b]^{-1} \mid u,v \in V,\ a,b \in GF(q)\} \\
&= \{[v,a][u,b]([u,b][v,a])^{-1} \mid u,v \in V, a,b \in GF(q)\} \\
&= \{[0_V,\ \ 2f(u,v)] \mid u,v \in V, a,b \in GF(q)\}.
\end{aligned}$$

Hence $P' \subseteq Z(P)$. Conversely we need to show that $Z(P) \subseteq P'$. By Lemma 3.2, if $[0_V, a] \in Z(P)$, with $a \in GF(q)$ we can find $b \in GF(q)$ such that $2b = a$. Now let $v = e_1$ and $u = be_{2m}$, then $f(v,u) = f(e_1, be_{2m}) = b\delta_{11} = b$, so $2f(v,u) = 2b = a$. Hence $[0_V, a] = [0_V,\ 2f(v,u)] \in P'$, which implies that $Z(P) \subseteq P'$. Thus $P' = Z(P)$. Therefore $P' = \{[0_V, a] \mid a \in GF(q)\}$. Since $P = \{[v,a] \mid v \in V,\ a \in GF(q)\}$, we have

$$P^p \;=\; \langle\, T^p \mid T \in P\,\rangle \;=\; \langle\, [v,a]^p \ \mid v \in V,\ \ a \in GF(q)\,\rangle.$$

But

$$\begin{aligned}
[v,a]^p &= [\ pv,\ pa + \underbrace{f(v,v) + \cdots + f(v,v)}_{(p-1)-times}\ ] \\
&= [\ 0_V,\ \underbrace{f(v,v) + \cdots + f(v,v)}_{(p-1)-times}\ ] = [0_V, c] \in P',\ \text{where } c = \underbrace{f(v,v) + \cdots + f(v,v)}_{(p-1)-times},
\end{aligned}$$

implies that $[v,a]^p \in P'$ and hence $P^p \subseteq P'$. Since $\Phi(P) = P'P^p$, we have that $\Phi(P) = P'P^p = P'$. Thus P is a special p-group.

(iii) $P(m)$ is isomorphic to P : Define $\phi : P(m) \longrightarrow P$ given by $\phi(T) = [v,a]$, where $T \in P(m)$ has the form

$$T \;=\; \begin{pmatrix}
1 & -\beta_{2m-1} & -\beta_{2m-2} & \cdots & \beta_2 & \beta_1 \\
0 & 1 & 0 & \cdots & 0 & \beta_2 \\
\vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 1 & \beta_{2m-1} \\
0 & 0 & 0 & \cdots & 0 & 1
\end{pmatrix},$$

and $v = (-\beta_{2m-1}, -\beta_{2m-2}, \cdots, \beta_3, \beta_2)$ with $a = \beta_1$. We need to show that ϕ is a one to one and onto homomorphism.

ϕ is a homomorphism: Let $T_1, T_2 \in P(m)$ such that

$$T_1 = \begin{pmatrix} 1 & -\beta_{2m-1} & -\beta_{2m-2} & \cdots & \beta_2 & \beta_1 \\ 0 & 1 & 0 & \cdots & 0 & \beta_2 \\ \vdots & \vdots & \vdots & \ldots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & \beta_{2m-1} \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix}$$

and

$$T_2 = \begin{pmatrix} 1 & -\beta'_{2m-1} & -\beta'_{2m-2} & \cdots & \beta'_2 & \beta'_1 \\ 0 & 1 & 0 & \cdots & 0 & \beta'_2 \\ \vdots & \vdots & \vdots & \ldots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & \beta'_{2m-1} \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix}.$$

If $v = (-\beta_{2m-1}, -\beta_{2m-2}, \cdots, \beta_3, \beta_2)$ and $v' = (-\beta'_{2m-1}, -\beta'_{2m-2}, \cdots, \beta'_3, \beta'_2)$ then we have that $\phi(T_1, T_2) = [\, v + v',\ \beta_1 - \beta_{2m-1}\beta'_2 - \beta_{2m-2}\beta'_3 - \cdots + \beta_3\beta'_{2m-2} + \beta_2\beta'_{2m-1} + \beta'_1\,]$. Since $\phi(T_1) = [v,\ \beta_1]$ and $\phi(T_2) = [v',\ \beta'_1]$, we have $\phi(T_1)\phi(y) = [v + v', \beta_1 + \beta'_1 + f(v, v')]$. So it suffices to show that $-\beta_{2m-1}\beta'_2 - \beta_{2m-2}\beta'_3 - \cdots + \beta_3\beta'_{2m-2} + \beta_2\beta'_{2m-1} = f(v, v')$. Now direct calculations show that $f(v, v') = -\beta_{2m-1}\beta'_2 - \beta_{2m-2}\beta'_3 - \cdots + \beta_3\beta'_{2m-2} + \beta_2\beta'_{2m-1}$, and hence we have that $\phi(T_1, T_2) = \phi(T_1)\phi(T_2)$.

ϕ is one-to-one:

$$\begin{aligned} \phi(T_1) = \phi(T_2) \ &\Rightarrow\ [v,\ \beta_1] = [v',\ \beta'_1] \\ &\Rightarrow\ v = v' \ \textit{and}\ \beta_1 = \beta'_1 \\ &\Rightarrow\ \beta_i = \beta'_i \ \textit{and}\ \beta_1 = \beta'_1 \quad 2 \le i \le 2m-1 \\ &\Rightarrow\ \beta_i = \beta'_i \quad \forall i \quad 1 \le i \le 2m-1 \\ &\Rightarrow\ T_1 = T_2. \end{aligned}$$

Now

$$\begin{aligned} Im(\phi) = \{\phi(T) \mid T \in P(m)\ \} &= \{[v,\ \beta_1] \mid v \in V,\ \beta_1 = a \in GF(q)\ \} \\ &= \{[v,\ a] \mid v \in V,\ a \in GF(q)\} = P. \end{aligned}$$

Hence ϕ is onto, so $P(m) \cong P$.

We know that

$$P(m) = \left\{ \begin{pmatrix} 1 & -\beta_{2m-1} & -\beta_{2m-2} & \cdots & \beta_2 & \beta_1 \\ 0 & 1 & 0 & \cdots & 0 & \beta_2 \\ \vdots & \vdots & \vdots & \ldots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & \beta_{2m-1} \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix},\quad \beta_i \in GF(q), \quad i = 1, 2 \cdots, 2m-1 \right\}.$$

Since the entries above the main diagonal are arbitrary elements of $GF(q)$ and there are exactly $2m-1$ places above the diagonal, we have that $|P(m)| = q^{2m-1}$. □

Note 3.5. *If $p = 2$, then $P(m)$ is an elementary abelian 2-group.* Let $T \in P$, then $T = [v, a]$ where $v \in V$ and $a \in GF(q)$. Since $p = 2$ we have that $T^2 = [v, a]^2 = [2v, 2a] = [0_V, 0] = 1_P$. Thus P is an elementary abelian 2-group. Since $P(m) \cong P$, then $P(m)$ is also an elementary abelian 2-group.

Lemma 3.6. [8] *Let H be the subgroup of $A(m)$ which fixes e_{2m}. Then H fixes both e_1 and e_{2m} and acts on $W = \langle e_2, e_3, \ldots, e_{2m-1}\rangle$ as $G(m-1)$. Moreover $H \cong G(m-1) \cong Sp_{2m-2}(q)$.*

Theorem 3.7. $P(m) \cap H = \{I_{A(m)}\}$.

Proof. Let $T \in P(m)$ then $T(e_{2m}) = \beta_1 e_1 + \beta_2 e_2 + \cdots + \beta_{2m-1}e_{2m-1} + e_{2m}$. If $T \in H$, then $T(e_{2m}) = e_{2m}$ and hence we must have $\beta_i = 0 \quad 1 \leq i \leq 2m-1$ and $\beta_{2m} = 1$. Hence $\alpha_i = 0, \quad 2 \leq i \leq 2m-1$. Thus $T(e_i) = e_i \quad \forall i$, so that $T = I_{A(m)}$. □

Theorem 3.8. $P(m) \cdot H = A(m)$.

Proof. Since $G(m)$ acts transitively on $V - \{0_V\}$, we have that $|G(m)| = |V^*||A(m)|$.

So

$$\begin{aligned} |A(m)| &= \frac{q^{m^2}(q^{2m-2}-1)(q^{2m-4}-1)\cdots(q^4-1)(q^2-1)(q^{2m}-1)}{q^{2m}-1} \\ &= q^{m^2}(q^{2m-2}-1)(q^{2m-4}-1)\cdots(q^4-1)(q^2-1). \end{aligned}$$

Now $|P(m) \cdot H| = \frac{|P(m)||H|}{|P(m)\cap H|} = |P(m)||H|$. Since $H \cong SP(2m-2, q)$, we have that

$$|H| = q^{(m-1)^2}(q^{2m-2}-1)(q^{2m-4}-1)\cdots(q^4-1)(q^2-1).$$

Thus

$$\begin{aligned} |P(m)||H| &= q^{2m-1}q^{(m-1)^2}(q^{2m-2}-1)\cdots(q^4-1)(q^2-1) \\ &= q^{m^2}(q^{2m-2}-1)\cdots(q^4-1)(q^2-1) \\ &= |A(m)|. \end{aligned}$$

Since $P(m) \cdot H \subseteq A(m)$, we must have $P(m) \cdot H = A(m)$. □

Lemma 3.9. $P(m) \trianglelefteq A(m)$.

Proof. If $A \in A(m)$, then $A = P_0T_0$ where $P_0 \in P(m)$ and $T_0 \in H$. To show that $P(m) \trianglelefteq A(m)$, let $P \in P(m)$. Then $P_0T_0P(P_0T_0)^{-1} = P_0T_0PT_0^{-1}P_0^{-1} = P_0(T_0PT_0^{-1})P_0{}^{-1}$. Hence it suffices to show that $TPT^{-1} \in P(m), \quad \forall T \in H, \forall P \in P(m)$.

If $B' = \{e_1, e_2, \cdots, e_m, e_{2m}, e_{2m-1}, e_{2m-2}, \cdots, e_{m+2}, e_{m+1}\}$ is the ordered basis obtained from B, then the action of $T \in H$ on the vectors of B' is described by

$$T(e_1) = e_1, \quad T(e_{2m}) = e_{2m}$$

and

$$T(e_i) = \sum_{j=1}^{m} \lambda_{ji} e_j + \sum_{j=m+1}^{2m} \lambda_{ji} e_{3m+1-j}, \quad 2 \leq i \leq m,$$

$$T(e_i)=\sum_{j=1}^{m}\lambda_{j\ 3m+1-i}e_j+\sum_{j=m+1}^{2m}\lambda_{j\ 3m+1-i}e_{3m+1-j},\quad m<i\leq 2m-1.$$

Hence T can be represented by the following matrix, with respect to B'

$$\begin{pmatrix} 1 & \lambda_{1\ 2} & \cdots & \lambda_{1\ m} & 0 & \lambda_{1\ m+2} & \cdots & \lambda_{1\ 2m} \\ 0 & \lambda_{2\ 2} & \cdots & \lambda_{2\ m} & 0 & \lambda_{2\ m+2} & \cdots & \lambda_{2\ 2m} \\ \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & \lambda_{m\ 2} & \cdots & \lambda_{m\ m} & 0 & \lambda_{m\ m+2} & \cdots & \lambda_{m\ 2m} \\ 0 & \lambda_{m+1\ 2} & \cdots & \lambda_{m+1\ m} & 1 & \lambda_{m+1\ m+2} & \cdots & \lambda_{m+1\ 2m} \\ \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & \lambda_{2m\ 2} & \cdots & \lambda_{2m\ m} & 0 & \lambda_{2m\ m+2} & \cdots & \lambda_{2m\ 2m} \end{pmatrix}=(\lambda_{ij})_{2m\times 2m}.$$

Since T is an isometry, then the adjoint of T exists and its action on the vectors of B' is given by

$$T^*(e_1)=e_1,\ \ T^*(e_{2m})=e_{2m}$$

and

$$T^*(e_i)=\sum_{j=1}^{m}\mu_{ji}e_j+\sum_{j=m+1}^{2m}\mu_{ji}e_{3m+1-j},\quad 2\leq i\leq m,$$

$$T^*(e_i)=\sum_{j=1}^{m}\mu_{j\ 3m+1-i}e_j+\sum_{j=m+1}^{2m}\mu_{j\ 3m+1-i}e_{3m+1-j},\quad m<i\leq 2m-1.$$

Using the definition of adjoint of T, we find the entries of the first and the $(m+1)$-th rows of the matrix of T^* with respect to B'. For example, entries in the first row of the matrix of T^* are obtained as follows:

$$\begin{aligned} f(T(e_{2m}),e_1) &= f(e_{2m},T^*(e_1))=f(e_{2m},e_1)=1,\ \text{so}\ \mu_{1\ 1}=\lambda_{m+1\ m+1}=1 \\ f(T(e_{2m}),e_2) &= f(e_{2m},T^*(e_2)),\ \text{so}\ \mu_{1\ 2}=\lambda_{m+2\ m+1}=0 \\ &\vdots \\ f(T(e_{2m}),e_m) &= f(e_{2m},T^*(e_m)),\ \text{so}\ \mu_{1\ m}=\lambda_{2m\ m+1}=0 \\ f(T(e_{2m}),e_{2m}) &= f(e_{2m},T^*(e_{2m}))=f(e_{2m},e_{2m})=0,\ \text{so}\ \mu_{1\ m+1}=\lambda_{1\ m+1}=0 \\ &\vdots \\ f(T(e_{2m}),e_{m+1}) &= f(e_{2m},T^*(e_{m+1})),\ \text{so}\ \mu_{1\ 2m}=\lambda_{m\ m+1}=0. \end{aligned}$$

Similar calculations give the entries of the $(m+1)$-th row of the matrix of T^*. Now by using Lemma 2.3, we can easily see that T and T^* have the following matrix representations (with respect to B') respectively

$$\left(\begin{array}{cccc|cccc} 1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & \lambda_{2\ 2} & \cdots & \lambda_{2\ m} & 0 & \lambda_{2\ m+2} & \cdots & \lambda_{2\ 2m} \\ \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & \lambda_{m\ 2} & \cdots & \lambda_{m\ m} & 0 & \lambda_{m\ m+2} & \cdots & \lambda_{m\ 2m} \\ \hline 0 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\ 0 & \lambda_{m+2\ 2} & \cdots & \lambda_{m+2\ m} & 0 & \lambda_{m+2\ m+2} & \cdots & \lambda_{m+2\ 2m} \\ \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & \lambda_{2m\ 2} & \cdots & \lambda_{2m\ m} & 0 & \lambda_{2m\ m+2} & \cdots & \lambda_{2m\ 2m} \end{array}\right) = \begin{pmatrix} A & C \\ B & D \end{pmatrix}$$

and

$$\left(\begin{array}{cccc|cccc} 1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & \lambda_{m+2\ m+2} & \cdots & \lambda_{2m\ m+2} & 0 & -\lambda_{2\ m+2} & \cdots & -\lambda_{m\ m+2} \\ \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & \lambda_{m+2\ 2m} & \cdots & \lambda_{2m\ 2m} & 0 & -\lambda_{2\ 2m} & \cdots & -\lambda_{m\ 2m} \\ \hline 0 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\ 0 & -\lambda_{m+2\ 2} & \cdots & -\lambda_{2m\ 2} & 0 & \lambda_{2\ 2} & \cdots & \lambda_{m\ 2} \\ \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & -\lambda_{m+2\ m} & \cdots & -\lambda_{2m\ m} & 0 & \lambda_{2\ m} & \cdots & \lambda_{m\ m} \end{array}\right) = \begin{pmatrix} D^t & -C^t \\ -B^t & A^t \end{pmatrix} .$$

Since $T \in H \subseteq Sp_{2m}(q)$, by Lemma 2.3 we have that $T^*T = I_{2m}$. Using this fact we obtain the following relations $\sum_{k=2}^{m}(\lambda_{m+k\ m+i}\lambda_{kj} - \lambda_{k\ m+i}\lambda_{m+k\ j}),\quad 2 \leq i \leq m,\quad 2 \leq j \leq 2m$, and $\sum_{k=0}^{m-2}(-\lambda_{i+k\ i-m}\lambda_{lj} + \lambda_{l\ i-m}\lambda_{i+k\ j}),\quad m+2 \leq i \leq 2m,\quad 2 \leq j \leq 2m,\quad 2 \leq l \leq m$.

Now if $P \in P(m)$, then P can be represented (with respect to B') by the following matrix

$$\begin{pmatrix} 1 & -\beta_{2m-1} & -\beta_{2m-2} & \cdots & -\beta_{m+1} & \beta_1 & \cdots & \beta_m \\ 0 & 1 & 0 & \cdots & 0 & \beta_2 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 & \beta_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & \beta_m & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 & \beta_{2m-1} & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & \beta_{m+1} & \cdots & 1 \end{pmatrix} .$$

Using the above relations obtained for the entries of T we deduce that $T^{-1}PT$ has the following matrix representation

$$\begin{pmatrix} 1 & -\beta'_{2m-1} & -\beta'_{2m-2} & \cdots & -\beta'_{m+1} & \beta'_1 & \cdots & \beta'_m \\ 0 & 1 & 0 & \cdots & 0 & \beta'_2 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 & \beta'_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & \beta'_m & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 & \beta'_{2m-1} & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & \beta'_{m+1} & \cdots & 1 \end{pmatrix} .$$

Thus $T^{-1}PT \in P(m)$, so that $P(m) \trianglelefteq A(m)$. □

Theorem 3.10. *Let q be a power of an odd prime p. Then $A(m)$ is a split extension of $P(m)$ by H where $H \cong G(m-1) \cong Sp_{2m-2}(q)$, i.e., $A(m) = P(m){:}H = P(m){:}Sp_{2m-2}(q)$.*

Proof. Follows from Lemma 3.4 to Lemma 3.9 above. □

4. **The group** $A(3) = 2^5{:}Sp_4(2)$

Here as an illustrative example of the theory described in Section 3 we construct the group $A(3)$, the affine subgroup of $Sp_6(2)$. It is the subgroup of $Sp_6(2)$ fixing the non-zero vector e_1 in $V_6(2)$, where $V_6(2)$, is the symplectic vector space of dimension six over $GF(2)$. From Theorem 3.8 we have that

$$A(3) = [Sp_6(2)]_{e_1} = P(3) : H = 2^5 : Sp_4(2)$$

where $H = [Sp_6(2)]_{[e_1,e_6]} \cong Sp_4(2)$, by Lemma 3.6. We constructed $H \cong Sp_4(2)$ and $P(3)$ inside $Sp_6(2)$. Using the construction of $P(m)$ as outlined in Remark 3.3 with $m = 3$ we have that

$$\begin{aligned} T(e_1) &= e_1, \\ T(e_2) &= \alpha_2 e_1 + e_2 = -\beta_5 e_1 + e_2, \\ T(e_3) &= \alpha_3 e_1 + e_3 = -\beta_4 e_1 + e_3, \\ T(e_4) &= \alpha_4 e_1 + e_4 = \beta_3 e_1 + e_4, \\ T(e_5) &= \alpha_5 e_1 + e_5 = \beta_2 e_1 + e_5, \\ T(e_6) &= \beta_1 e_1 + \beta_2 e_2 + \beta_3 e_3 + \beta_4 e_4 + \beta_5 e_5 + e_6. \end{aligned}$$

So an element T of $P(3)$ has the following matrix form $T = \begin{pmatrix} 1 & -\beta_5 & -\beta_4 & \beta_3 & \beta_2 & \beta_1 \\ 0 & 1 & 0 & 0 & 0 & \beta_2 \\ 0 & 0 & 1 & 0 & 0 & \beta_3 \\ 0 & 0 & 0 & 1 & 0 & \beta_4 \\ 0 & 0 & 0 & 0 & 1 & \beta_5 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$ where $\beta_i \in \{0,1\}$.

Now by giving appropriate values to the $\beta_i's$ we determined all the 32 elements of $P(3)$. We also observed that the group $P(3)$ is generated by P_1, P_2, P_3, P_4, P_5, where

$$P_1 = \begin{pmatrix} 1&1&1&1&1&1 \\ 0&1&0&0&0&1 \\ 0&0&1&0&0&1 \\ 0&0&0&1&0&1 \\ 0&0&0&0&1&1 \\ 0&0&0&0&0&1 \end{pmatrix}, \quad P_2 = \begin{pmatrix} 1&0&1&1&1&1 \\ 0&1&0&0&0&1 \\ 0&0&1&0&0&1 \\ 0&0&0&1&0&1 \\ 0&0&0&0&1&0 \\ 0&0&0&0&0&1 \end{pmatrix}, \quad P_3 = \begin{pmatrix} 1&0&0&1&1&1 \\ 0&1&0&0&0&1 \\ 0&0&1&0&0&1 \\ 0&0&0&1&0&0 \\ 0&0&0&0&1&0 \\ 0&0&0&0&0&1 \end{pmatrix},$$

$$P_4 = \begin{pmatrix} 1&0&0&0&1&1 \\ 0&1&0&0&0&1 \\ 0&0&1&0&0&0 \\ 0&0&0&1&0&0 \\ 0&0&0&0&1&0 \\ 0&0&0&0&0&1 \end{pmatrix}, \quad P_5 = \begin{pmatrix} 1&0&0&0&0&1 \\ 0&1&0&0&0&0 \\ 0&0&1&0&0&0 \\ 0&0&0&1&0&0 \\ 0&0&0&0&1&0 \\ 0&0&0&0&0&1 \end{pmatrix}$$

with ${P_i}^2 = I_6$, for $1 \le i \le 5$. We also have $H = \langle \alpha, \beta, \gamma \rangle$ where

$$\alpha = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad \beta = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad \gamma = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

with $o(\alpha) = 2, o(\beta) = 4$ and $o(\gamma) = 2$.

Clearly $A(3) = \langle P_1, P_2, P_3, P_4, P_5, \alpha, \beta, \gamma \rangle$. Now using GAP we can generate $A(3)$ by five 6×6 matrices over $GF(2)$, which we list below.

$$x_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad x_2 = \begin{pmatrix} 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad x_3 = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$x_4 = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad x_5 = \begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

where $o(x_i) = 2$ for $1 \le i \le 4$ and $o(x_5) = 4$.

5. **The conjugacy classes of** $A(3) = 2^5{:}Sp_4(2)$

We have used GAP4 [5] to compute the conjugacy classes of the group $A(3)$, and we found that $A(3)$ has 37 conjugacy classes. Following Atlas [3] notation, in Table 1 we list class representatives for each $g \in A(3)$ in terms of 6×6 matrices over $GF(2)$, where $[g]_{A(3)}$ is the class containing g and M is the matrix that represents that particular conjugacy class. Note that $P(3) = [1A] \cup [2A] \cup [2B] \cup [2C]$ where $|[2A]| = 1, |[2B]| = |[2C]| = 15$.

Table 1: The conjugacy classes of elements of $2^5{:}Sp_4(2)$

$[g]_{A(3)}$	M	$\lvert[g]_{A(3)}\rvert$	$[g]_{A(3)}$	M	$\lvert[g]_{A(3)}\rvert$
$1A$	$\begin{pmatrix} 1&0&0&0&0&0\\ 0&1&0&0&0&0\\ 0&0&1&0&0&0\\ 0&0&0&1&0&0\\ 0&0&0&0&1&0\\ 0&0&0&0&0&1 \end{pmatrix}$	1	$2A$	$\begin{pmatrix} 1&0&0&0&0&1\\ 0&1&0&0&0&0\\ 0&0&1&0&0&0\\ 0&0&0&1&0&0\\ 0&0&0&0&1&0\\ 0&0&0&0&0&1 \end{pmatrix}$	1
$2B$	$\begin{pmatrix} 1&0&0&0&1&0\\ 0&1&0&0&0&1\\ 0&0&1&0&0&0\\ 0&0&0&1&0&0\\ 0&0&0&0&1&0\\ 0&0&0&0&0&1 \end{pmatrix}$	15	$2C$	$\begin{pmatrix} 1&0&0&0&1&1\\ 0&1&0&0&0&1\\ 0&0&1&0&0&0\\ 0&0&0&1&0&0\\ 0&0&0&0&1&0\\ 0&0&0&0&0&1 \end{pmatrix}$	15
$2D$	$\begin{pmatrix} 1&0&0&0&1&0\\ 0&1&0&0&1&1\\ 0&0&1&0&0&0\\ 0&0&0&1&0&0\\ 0&0&0&0&1&0\\ 0&0&0&0&0&1 \end{pmatrix}$	30	$2E$	$\begin{pmatrix} 1&0&0&0&1&1\\ 0&1&0&0&1&1\\ 0&0&1&0&0&0\\ 0&0&0&1&0&0\\ 0&0&0&0&1&0\\ 0&0&0&0&0&1 \end{pmatrix}$	30
$2F$	$\begin{pmatrix} 1&0&0&1&1&1\\ 0&1&0&0&1&1\\ 0&0&1&0&0&1\\ 0&0&0&1&0&0\\ 0&0&0&0&1&0\\ 0&0&0&0&0&1 \end{pmatrix}$	180	$2G$	$\begin{pmatrix} 1&0&0&0&1&1\\ 0&1&0&1&0&1\\ 0&0&1&0&1&0\\ 0&0&0&1&0&0\\ 0&0&0&0&1&0\\ 0&0&0&0&0&1 \end{pmatrix}$	60
$2H$	$\begin{pmatrix} 1&0&0&0&1&0\\ 0&1&0&1&0&1\\ 0&0&1&0&1&0\\ 0&0&0&1&0&0\\ 0&0&0&0&1&0\\ 0&0&0&0&0&1 \end{pmatrix}$	60	$2I$	$\begin{pmatrix} 1&0&0&0&0&1\\ 0&1&0&1&1&0\\ 0&0&1&0&1&0\\ 0&0&0&1&0&0\\ 0&0&0&0&1&0\\ 0&0&0&0&0&1 \end{pmatrix}$	180
$2J$	$\begin{pmatrix} 1&0&0&0&0&0\\ 0&1&0&1&1&0\\ 0&0&1&0&1&0\\ 0&0&0&1&0&0\\ 0&0&0&0&1&0\\ 0&0&0&0&0&1 \end{pmatrix}$	180	$3A$	$\begin{pmatrix} 1&0&0&0&0&0\\ 0&1&0&0&0&0\\ 0&0&1&1&0&0\\ 0&0&1&0&0&0\\ 0&0&0&0&1&0\\ 0&0&0&0&0&1 \end{pmatrix}$	160

Table 1: The conjugacy classes of elements of $2^5{:}Sp_4(2)$ (continued)

$[g]_{A(3)}$	M	$\vert[g]_{A(3)}\vert$	$[g]_{A(3)}$	M	$\vert[g]_{A(3)}\vert$
$3B$	$\begin{pmatrix} 1&0&1&0&1&0\\ 0&1&1&0&0&1\\ 0&1&0&0&0&1\\ 0&0&0&1&1&1\\ 0&0&0&1&0&1\\ 0&0&0&0&0&1 \end{pmatrix}$	640	$4A$	$\begin{pmatrix} 1&1&0&0&1&0\\ 0&1&0&0&1&0\\ 0&0&1&0&0&0\\ 0&0&0&1&0&0\\ 0&0&0&0&1&1\\ 0&0&0&0&0&1 \end{pmatrix}$	120
$4B$	$\begin{pmatrix} 1&1&0&0&1&1\\ 0&1&0&0&1&0\\ 0&0&1&0&0&0\\ 0&0&0&1&0&0\\ 0&0&0&0&1&1\\ 0&0&0&0&0&1 \end{pmatrix}$	120	$4C$	$\begin{pmatrix} 1&0&1&0&1&0\\ 0&1&0&1&0&0\\ 0&0&1&0&1&0\\ 0&0&0&1&0&1\\ 0&0&0&0&1&0\\ 0&0&0&0&0&1 \end{pmatrix}$	360
$4D$	$\begin{pmatrix} 1&0&1&0&0&0\\ 0&1&0&1&1&1\\ 0&0&1&0&1&0\\ 0&0&0&1&0&1\\ 0&0&0&0&1&0\\ 0&0&0&0&0&1 \end{pmatrix}$	180	$4E$	$\begin{pmatrix} 1&0&1&0&0&1\\ 0&1&0&1&1&1\\ 0&0&1&0&1&0\\ 0&0&0&1&0&1\\ 0&0&0&0&1&0\\ 0&0&0&0&0&1 \end{pmatrix}$	180
$4F$	$\begin{pmatrix} 1&1&1&0&0&1\\ 0&1&0&1&1&0\\ 0&0&1&0&1&1\\ 0&0&0&1&0&1\\ 0&0&0&0&1&1\\ 0&0&0&0&0&1 \end{pmatrix}$	720	$4G$	$\begin{pmatrix} 1&0&0&1&1&0\\ 0&1&1&1&0&0\\ 0&0&1&1&0&1\\ 0&0&0&1&1&0\\ 0&0&0&0&1&0\\ 0&0&0&0&0&1 \end{pmatrix}$	720
$4H$	$\begin{pmatrix} 1&0&0&1&1&1\\ 0&1&1&1&0&0\\ 0&0&1&1&0&1\\ 0&0&0&1&1&0\\ 0&0&0&0&1&0\\ 0&0&0&0&0&1 \end{pmatrix}$	720	$4I$	$\begin{pmatrix} 1&0&0&1&0&0\\ 0&1&1&1&1&1\\ 0&0&1&1&0&1\\ 0&0&0&1&1&0\\ 0&0&0&0&1&0\\ 0&0&0&0&0&1 \end{pmatrix}$	720
$4J$	$\begin{pmatrix} 1&0&0&1&0&1\\ 0&1&1&1&1&1\\ 0&0&1&1&0&1\\ 0&0&0&1&1&0\\ 0&0&0&0&1&0\\ 0&0&0&0&0&1 \end{pmatrix}$	720	$5A$	$\begin{pmatrix} 1&0&0&1&1&0\\ 0&1&1&1&0&0\\ 0&0&0&1&1&0\\ 0&1&0&0&0&1\\ 0&0&0&1&0&0\\ 0&0&0&0&0&1 \end{pmatrix}$	2304
$6A$	$\begin{pmatrix} 1&1&0&0&1&0\\ 0&1&0&0&0&1\\ 0&0&1&1&0&0\\ 0&0&1&0&0&0\\ 0&0&0&0&1&1\\ 0&0&0&0&0&1 \end{pmatrix}$	480	$6B$	$\begin{pmatrix} 1&1&0&0&1&1\\ 0&1&0&0&0&1\\ 0&0&1&1&0&0\\ 0&0&1&0&0&0\\ 0&0&0&0&1&1\\ 0&0&0&0&0&1 \end{pmatrix}$	480

Table 1: The conjugacy classes of elements of $2^5{:}Sp_4(2)$ (continued)

$[g]_{A(3)}$	M	$\lvert[g]_{A(3)}\rvert$	$[g]_{A(3)}$	M	$\lvert[g]_{A(3)}\rvert$
$6C$	$\begin{pmatrix} 1&0&0&0&0&1\\ 0&1&0&0&0&0\\ 0&0&1&1&0&0\\ 0&0&1&0&0&0\\ 0&0&0&0&1&0\\ 0&0&0&0&0&1 \end{pmatrix}$	160	$6D$	$\begin{pmatrix} 1&0&0&0&0&0\\ 0&1&0&0&1&0\\ 0&0&1&1&0&0\\ 0&0&1&0&0&0\\ 0&0&0&0&1&0\\ 0&0&0&0&0&1 \end{pmatrix}$	960
$6E$	$\begin{pmatrix} 1&0&0&0&0&1\\ 0&1&0&0&1&0\\ 0&0&1&1&0&0\\ 0&0&1&0&0&0\\ 0&0&0&0&1&0\\ 0&0&0&0&0&1 \end{pmatrix}$	960	$6F$	$\begin{pmatrix} 1&0&1&0&1&1\\ 0&1&1&0&0&1\\ 0&1&0&0&0&1\\ 0&0&0&1&1&1\\ 0&0&0&1&0&1\\ 0&0&0&0&0&1 \end{pmatrix}$	640
$6G$	$\begin{pmatrix} 1&0&1&1&1&1\\ 0&1&1&1&0&1\\ 0&1&0&0&0&1\\ 0&0&0&1&1&1\\ 0&0&0&1&0&1\\ 0&0&0&0&0&1 \end{pmatrix}$	1920	$6H$	$\begin{pmatrix} 1&0&1&1&1&0\\ 0&1&1&1&0&1\\ 0&1&0&0&0&1\\ 0&0&0&1&1&1\\ 0&0&0&1&0&1\\ 0&0&0&0&0&1 \end{pmatrix}$	1920
$8A$	$\begin{pmatrix} 1&1&0&1&1&0\\ 0&1&1&1&0&0\\ 0&0&1&1&0&1\\ 0&0&0&1&1&1\\ 0&0&0&0&1&1\\ 0&0&0&0&0&1 \end{pmatrix}$	1440	$8B$	$\begin{pmatrix} 1&1&0&1&0&1\\ 0&1&1&1&1&0\\ 0&0&1&1&0&1\\ 0&0&0&1&1&1\\ 0&0&0&0&1&1\\ 0&0&0&0&0&1 \end{pmatrix}$	1440
$10A$	$\begin{pmatrix} 1&0&0&1&1&1\\ 0&1&1&1&0&0\\ 0&0&0&1&1&0\\ 0&1&0&0&0&1\\ 0&0&0&1&0&0\\ 0&0&0&0&0&1 \end{pmatrix}$	2304	$12A$	$\begin{pmatrix} 1&1&0&0&0&1\\ 0&1&0&0&1&1\\ 0&0&1&1&0&0\\ 0&0&1&0&0&0\\ 0&0&0&0&1&1\\ 0&0&0&0&0&1 \end{pmatrix}$	960
$12B$	$\begin{pmatrix} 1&1&0&0&0&0\\ 0&1&0&0&1&1\\ 0&0&1&1&0&0\\ 0&0&1&0&0&0\\ 0&0&0&0&1&1\\ 0&0&0&0&0&1 \end{pmatrix}$	960			

Using the technique of coset analysis, Mpono in [7] has determined the conjugacy classes of $A(3)$ given in terms of 5×5 matrices and using the theory of Fischer-Clifford matrices constructed its character table. In a similar fashion Ali in [1] has constructed the group $A(4) = 2^7 : Sp_6(2)$ which sits maximally in $Sp_8(2)$ in terms of 7×7 matrices and found its character table. Recent work of Basheer and Moori [2], and Moori and Seretlo [6] make considerable use of the principles outlined in this article.

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